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CERTAIN CLASSES OF POLYGONS IN R^2 AND AREAS OF POLYGONS

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In this article we consider certain classes of polygons in R^2 and areas of polygons. The classes are connected with definitions like the following.

Let $A_1 \dots A_n$ be a polygon in R^2 and let k be a positive integer such that $1 < k < n$. A polygon $P_1 \dots P_n$ (if such exists) will be called k-outscribed polygon to the polygon $A_1 \dots A_n$ if it holds

$$P_i + \dots + P_{i+k-1} = kA_i, \quad i = 1, \dots, n.$$

First we prove the following theorem.

Theorem 1. *Let h, k, n be positive integers such that*

$$hk = n - 2, \quad GCD(k, n) = 2 \tag{1}$$

and let $A_1 \dots A_n$ be any given polygon in R^2 such that (1) *is satisfied and that*

$$\sum_{i=1}^{n} (-1)^i A_i = 0. \tag{2}$$

Then for every point $P_1 \in R^2$ there are points $P_2, \dots, P_n$ in R^2 such that

$$P_i + \dots + P_{i+k-1} = kA_i, \quad i = 1, \dots, n \tag{3}$$

and that area of the polygon $P_1 \dots P_n$ is a constant, that is, does not depend of P_1 and is given by

$$\begin{aligned} 2 \text{ area of } P_1 \dots P_n = & |U_1, \sum_{i=2}^{n} (-1)^i U_i| + |U_2, \sum_{i=3}^{n} (-1)^{i+1} U_i| + \\ & |U_3, \sum_{i=4}^{n} (-1)^i U_i| + |U_4, \sum_{i=5}^{n} (-1)^{1+i} U_i| + \dots + |U_{n-1}, U_n| \end{aligned} \tag{4}$$

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where $U_i = P_i + P_{i+1}$ *and* $P_i + P_{i+1}$ *for each* $i = 1, \ldots, n$ *is given by*

$$\begin{aligned} P_i + P_{i+1} = 2A_i - (k-2)A_{i+2} + 2A_{i+4} + 2A_{i+6} + \ldots + 2A_{i+k} - \\ (k-2)A_{i+2+k} + 2A_{i+4+k} + 2A_{i+6+k} + \ldots + 2A_{i+2k} - \\ (k-2)A_{i+2+2k} + 2A_{i+4+2k} + 2A_{i+6+2k} + \ldots + 2A_{i+3k} - \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (k-2)A_{i+2+(h-1)k} + 2A_{i+4+(h-1)k} + 2A_{i+6+(h-1)k} + \ldots + 2A_{i+hk}. \end{aligned} \tag{5}$$

Here let us remark that by $|K, L|$ *is denoted determinant* $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, *where* $K(x_1, y_1)$, $L(x_2, y_2)$.

Proof. The proof that holds (5) is as follows. First let $n = 14$ and $k = 6$. Thus, in this case, $h = 2$. Supposing that

$$P_i + \ldots + P_{i+5} = 6A_i, \quad i = 1, \ldots, 14 \tag{6}$$

we can write the following equations

$$P_i - P_{i+6} = 6(A_i - A_{i+1}), \quad i = 1, \ldots, 14 \tag{7}$$

from which follows

$$\begin{aligned} P_7 &= P_1 - 6(A_1 - A_2), \\ P_{13} &= P_1 - 6(A_1 - A_2) - 6(A_7 - A_8), \\ P_5 &= P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}), \\ P_{11} &= P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}) - 6(A_5 - A_6), \\ P_3 &= P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}) - 6(A_5 - A_6) - 6(A_{11} - A_{12}), \\ P_9 &= P_1 - 6(A_1 - A_2) - 6(A_7 - A_8) - 6(A_{13} - A_{14}) \\ &\quad - 6(A_5 - A_6) - 6(A_{11} - A_{12}) - 6(A_3 - A_4). \end{aligned}$$

Here let us remark that $P_{13} = P_{7+6}$, $P_5 = P_{13+6} = P_{7+2\cdot 6}$ and so on.
Analogously we have

$$P_8 = P_2 - 6(A_2 - A_3), \quad P_{14} = P_2 - 6(A_2 - A_3) - 6(A_8 - A_9)$$

and so on.

Let us remark that in expressions of $P_8, P_{14}, \ldots, P_{10}$, in relation to expressions of $P_7, P_{13}, \ldots, P_9$, each index grow up to 1.

From (6) we see that $P_9 - P_1 = 6(A_9 - A_{10})$, and from the expression of P_9 we see that

$$P_9 - P_1 = -6(A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + A_7 - A_8 + A_{11} - A_{12} + A_{13} - A_{14}).$$

Thus, the system (6) will be consistent iff $\sum_{i=1}^{14} (-1)^i A_i = 0$.

In the same way can be seen that the system (3) will be consistent iff holds (2).

Using, for example, the equation $P_1+P_2+P_3+P_4+P_5+P_6=6A_1$ and expressions of P_3,P_4,P_5,P_6 we find that

$$P_1+P_2=2A_1-4A_3+2A_5+2A_7-4A_9+2A_{11}+2A_{13}.$$

Since each equation of the system (6) can be used, it holds

$$P_i+P_{i+1}=2A_i-4A_{i+2}+2A_{i+4}+2A_{i+6}-4A_{i+8}+2A_{i+10}+2A_{i+12}, \quad i=1,\dots,14.$$

Now, in relation to (5), let us remark that on the right side of (5) there are h negative terms and that between two negative terms there are $\frac{k-2}{2}$ positive terms. Thus, it holds

$$1+h+h\cdot\frac{k-2}{2}=\frac{n}{2} \quad \text{or} \quad hk+2=n.$$

If right side of (5) is denoted by U_i, then

$$\begin{aligned} &P_2=-P_1+U_1,\\ &P_3=-P_2+U_2=P_1-U_1+U_2,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &P_n=-P_{n-1}+U_{n-1}=-P_1+U_1-U_2+\ldots+U_{n-1}, \end{aligned} \tag{8}$$

where P_1 can be taken arbitrary. Hence

$$\begin{aligned} &P_1+\ldots+P_k=U_1+U_3+\ldots+U_{k-1},\\ &P_2+\ldots+P_{k+1}=U_2+U_4+\ldots+U_k,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &P_n+\ldots+P_{k-1}=U_n+U_2+\ldots+U_{k-2}. \end{aligned}$$

Thus, we have to prove that

$$U_i+U_{i+2}+\ldots+U_{i+k-2}=kA_i, \quad i=1,\dots,n. \tag{9}$$

First let us consider the case where $n=14$ and $k=6$. In this case we have

$$\begin{aligned} P_1+P_2=U_1&=2A_1-4A_3+2A_5+2A_7-4A_9+2A_{11}+2A_{13},\\ P_3+P_4=U_3&=2A_3-4A_5+2A_7+2A_9-4A_{11}+2A_{13}+2A_1,\\ P_5+P_6=U_5&=2A_5-4A_7+2A_9+2A_{11}-4A_{13}+2A_1+2A_3, \end{aligned} \tag{10}$$

from which, by adding, we get

$$P_1+P_2+P_3+P_4+P_5+P_6=U_1+U_3+U_5=6A_1.$$

Since analogously holds for $i=2,\dots,14$, also we have

$$P_2+P_3+P_4+P_5+P_6+P_7=U_2+U_4+U_6=6A_2 \quad \text{and so on.}$$

It is not difficult to see that analogously holds generally for the case where positive integers h,k,n are such that $n-2=hk$. To see this, it is important to see that

$$\begin{array}{c} -(k-2)A_i \\ 2A_i-(k-2)A_{i+2} \\ \iddots \qquad \vdots \\ 2A_i+\ldots-(k-2)A_{i+k-2} \end{array}$$

and $-(k-2)A_i+\frac{k-2}{2}\cdot 2A_i=0$.

Concerning area of the polygon $P_1\ldots P_n$, first let us remark that, as it is known, area of a polygon $A_1\ldots A_n$ in R^2 is given by

$$2 \text{ area of } A_1\ldots A_n=\sum_{i=1}^{n}|A_i,A_{i+1}|.$$

Using expressions for $P_2,\ldots,P_n$ given by (8) it can be found that holds (4).

This completes the proof of Theorem 1.

In the following theorem will be shown that area of the polygon $P_1\ldots P_n$ can be written in a much simpler and interesting form. For this purpose will be used determinant of rectangular matrix. In short about this.

In [1] the following definition of a determinant of rectangular matrix is given: The determinant of a $m\times n$ matrix A with columns $A_1,\ldots,A_n$ and $m\leqslant n$, is the sum

$$\sum_{1\leqslant j_1<j_2<\ldots<j_m\leqslant n}(-1)^{r+s}|A_{j_1},\ldots,A_{j_m}|,$$

where $r=1+\ldots+m$, $s=j_1+\ldots+j_m$.

This determinant is a skew-symmetric multilinear functional with respect to the rows and therefore has many well known standard properties, for example, the general Laplace's expansion along rows.

In particular, if $m=2$, then

$$|A_1,\ldots,A_n|=\sum_{1\leqslant i<j\leqslant n}(-1)^{3+i+j}|A_i,A_j|. \tag{11}$$

In [2] the following theorem (Theorem 3) is proved.

Let $A_1\ldots A_n$ be any given polygon in R^2. Then

$$2 \text{ area of } A_1\ldots A_n=|A_1+A_2,A_2+A_3,\ldots,A_n+A_1|. \tag{12}$$

It is easy to see that, according to (11), relation (4) can be written as

$$\sum_{i=1}^{n}|P_i,P_{i+1}|=\sum_{1\leqslant i<j\leqslant n}(-1)^{3+i+j}|U_i,U_j|.$$

In the following theorem we shall use the following two theorems given in [2].

Theorem 7. Let $A_1 \ldots A_n$ be a polygon in R^2 with even n and let $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then for every point $X \in R^2$ it holds

$$|A_1 + X, \ldots, A_n + X| = |A_1, \ldots, A_n|. \tag{13}$$

Theorem 8. Let $A_1 \ldots A_n$ be as in Theorem 7. Then for each $i = 1, \ldots, n$ it holds

$$|A_{i+1}, \ldots, A_n, A_1, \ldots, A_i| = |A_1, A_2, \ldots, A_n|. \tag{14}$$

Now we can prove the following theorem.

THEOREM 2. *Let $P_1 \ldots P_n$ be polygon as in Theorem* 1. *If $k = 2$, then*

$$2 \text{ area of } P_1 \ldots P_n = 4|A_1, \ldots, A_n|, \tag{15}$$

and if $k > 2$, then

$$2 \text{ area of } P_1 \ldots P_n = k^2 |V_1, \ldots, V_n|, \tag{16}$$

where

$$V_i = A_i + A_{i+k} + \ldots + A_{i+(h-1)k}, \quad i = 1, \ldots, n. \tag{17}$$

Proof. If $k = 2$, then from (5) can be seen that $P_i + P_{i+1} = 2A_i$. Since determinant has two rows, it holds

$$|2A_1, \ldots, 2A_n| = 4|A_1, \ldots, A_n|.$$

The proof that holds (16) if $k > 2$ is as follows. Since $U_i = P_i + P_{i+1}$, it is easy to see that $\sum_{i=1}^{n}(-1)^i U_i = 0$. Thus, we can use Theorem 7 given in [2] and take

$$X = -2(A_1 + A_3 + \ldots + A_{n-1}) \quad \text{or} \quad X = -2(A_2 + A_4 + \ldots + A_n).$$

Here let us remark that from (2) follows $A_1 + A_3 + \ldots + A_{n-1} = A_2 + A_4 + \ldots + A_n$. Thus, relation

$$2 \text{ area of } P_1 \ldots P_n = |U_1, \ldots, U_n|$$

can be written as

$$2 \text{ area of } P_1 \ldots P_n = |U_1 - X, \ldots, U_n - X|$$

or

$$2 \text{ area of } P_1 \ldots P_n = |-kV_1 + 2S, \ldots, -kV_n + 2S|, \tag{18}$$

where $S = \sum_{i=1}^{n} A_i$. Now, according to the properties expressed by (13) and (14), the relation (18) can be written as (16).

For example, let $n = 14$ and $k = 6$. Then, as can be seen from the considered example (see (10)), it holds

$$P_1 + P_2 - 2S = -6A_3 - 6A_9, \quad P_2 + P_3 - 2S = -6A_4 - 6A_{10}, \ldots,$$

$$P_{13} + P_{14} - 2S = -6A_1 - 6A_7, \; P_{14} + P_1 - 2S = -6A_2 - 6A_8,$$

$$|-6A_3 - 6A_9, -6A_4 - 6A_{10}, \ldots, -6A_1 - 6A_7, -6A_2 - 6A_8| = 6^2|A_1 + A_7, \ldots, A_{14} + A_6|.$$

This proves Theorem 2.

The proof that holds (16) seems to be far from to be easy without using properties of determinant given by (11).

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