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ORTHIC AXIS, LEMOINE LINE AND LONGCHAMPS' LINE OF THE TRIANGLE IN I_2

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Abstract. The concepts of the orthic axis, Lemoine line and Longchamps' line of the triangle in an isotropic plane are defined. Some relationships between the introduced concepts and other elements of the triangle in an isotropic plane are also studied.

Let ABC be any allowable triangle in an isotropic plane, i.e. such a triangle that none of the three lines BC, CA, AB is isotropic. With the suitable choice of the coordinate system it can be achieved that the circumscribed circle of the triangle ABC has the equation $y = x^2$ and the vertices of the triangle are of the form

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1}$$

where $a+b+c=0$ (see [4]). Then we shall say that the triangle ABC is in the *standard position*. Then the sides BC, CA, AB of the triangle have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab. \tag{2}$$

The altitudes of the triangle ABC are the isotropic lines through the points A, B, C. They have in sequence these equations $x = a$, $x = b$, $x = c$ and they intersect the corresponding sides (2) in the points with the ordinates $-a^2 - bc$, $-b^2 - ca$, $-c^2 - ab$. If we introduce the abbreviations

$$q = bc + ca + ab, \quad p = abc,$$

then because of $a+b+c=0$ these equalities

$$q = bc - a^2, \quad q = ca - b^2, \quad q = ab - c^2 \tag{3}$$

follow, so the previous ordinates can be written in the form $q - 2bc$, $q - 2ca$, $q - 2ab$. We get the *feet* of the altitudes

$$A_h = (a, q - 2bc), \quad B_h = (b, q - 2ca), \quad C_h = (c, q - 2ab), \tag{4}$$

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which form the triangle $A_hB_hC_h$, the so called *orthic triangle* of the triangle ABC.

In [4] it is shown that the corresponding sides of the triangle ABC and its orthic triangle $A_hB_hC_h$ intersect in three points on the line $\mathscr{H}$ with the equation $y = -\frac{q}{3}$ the so called *orthic axis* of the triangle ABC (*Figure 1*). In [1] it is shown that the orthic axis of the triangle is the potential axis of one pencil of circles, in which there are some important circles of that triangle.

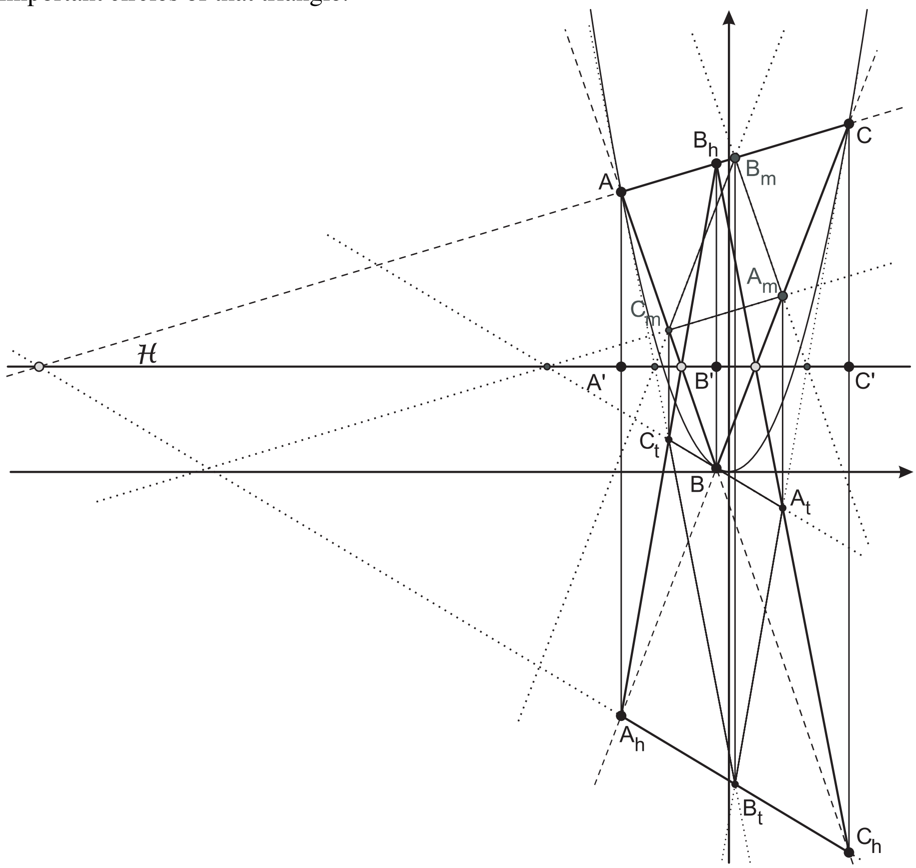

Figure 1.

The altitude AA_h intersects the orthic axis $\mathscr{H}$ in the point $A' = (a, -\frac{q}{3})$. Because of the first equality (3) we have the equality $2a^2 + q - 2bc = -q$, which proves that for the point A' and the points A and A_h from (1) and (4) the equality $2A + A_h = 3A'$, i.e.

$$\overrightarrow{A'A_h} = A_h - A' = -2(A - A') = -2\overrightarrow{A'A}$$

is valid. Because of that the dilatation χ with the axis $\mathscr{H}$ and the coefficient -2 in isotropic direction maps the point A to the point A_h, and in the same way it maps the points B and C to the points B_h and C_h. That dilatation χ will be called *orthodilatation* of the triangle ABC. So, we have

TEOREM 1. *The orthodilatation of a triangle maps that triangle to its orthic triangle.*

The midpoints A_m, B_m, C_m of the sides BC, CA, AB of the triangle ABC make its *complementary triangle* $A_mB_mC_m$. In [4] it is shown that its vertices have the form

$$A_m = \left(-\frac{a}{2}, -\frac{q}{2} - \frac{bc}{2}\right), \quad B_m = \left(-\frac{b}{2}, -\frac{q}{2} - \frac{ca}{2}\right), \quad C_m = \left(-\frac{c}{2}, -\frac{q}{2} - \frac{ab}{2}\right). \quad (5)$$

The tangents of the circumscribed circle of the triangle ABC in its vertices A, B, C form the so called *tangential triangle* $A_tB_tC_t$ of the triangle ABC, for whose vertices there is, by [1], this presentation

$$A_t = \left(-\frac{a}{2}, bc\right), \quad B_t = \left(-\frac{b}{2}, ca\right), \quad C_t = \left(-\frac{c}{2}, ab\right). \quad (6)$$

Because of

$$\frac{1}{2}(b+c) = -\frac{a}{2}, \quad \frac{1}{2}(q-2ca+q-2ab) = \frac{1}{2}\cdot 2bc = bc$$

the points B_h and C_h from (4) have the midpoint A_t from (6), and similarly the points B_t and C_t are the midpoints of the segments C_hA_h and A_hB_h. So the following theorem is valid.

TEOREM 2. *The tangential triangle of the given triangle is the complementary triangle of its orthic triangle.*

From Theorems 1 and 2 it follows that orthodilatation of the triangle maps its complementary triangle to its tangential triangle. Since the associated lines of the dilatation intersect on its axis we have the following theorem.

TEOREM 3. *The intersections of the corresponding sides of the complementary and the tangential triangle of the given triangle lie on its orthic axis (Figure 1).*

In [6] Neuberg gives the statement of Theorem 3 in Euclidean geometry.

In [1] (Theorem 2 and its proof) it is shown that the corresponding sides of the triangles ABC and $A_tB_tC_t$ intersect in three points D, E, F which lie on one line $\mathscr{L}$, *Lemoine line* of the triangle ABC (*Figure* 2). For the triangle ABC in the standard position the line $\mathscr{L}$ has the equation

$$y = \frac{3p}{q}x + \frac{q}{3} \quad (7)$$

and we have for example

$$D = \left(-\frac{q}{3a}, \frac{q}{3} - bc\right). \quad (8)$$

The relationship between the Lemoine line and the orthic axis of the triangle is given by the following theorem.

TEOREM 4. *If the Lemoine line of the triangle ABC intersects the lines BC, CA, AB in the points D, E, F then the midpoints of the segments $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ lie on the orthic axis of that triangle (Figure 2).* (In Euclidean plane the statement is given by FARJON [2]).

Proof. The points $A(a, bc-q)$ and D from (8) have the midpoint with the ordinate $-\frac{q}{3}$. □

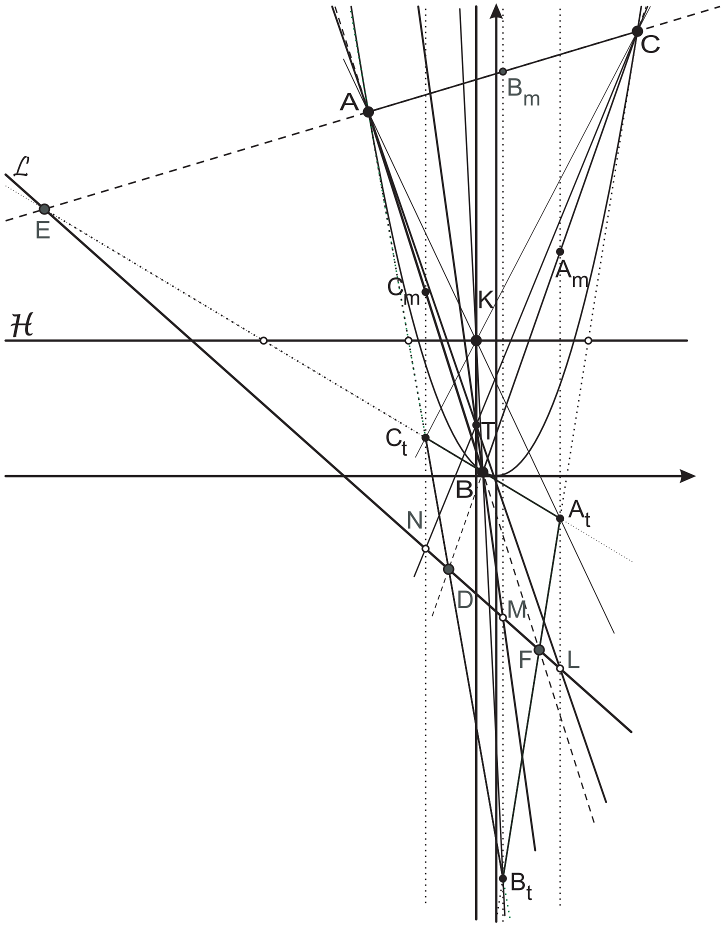

Figure 2.

The fact from Theorem 4 can be stated in this way too.

COROLLARY 1. *The orthic axis of the triangle is Newton–Gauss line of the quadrilateral, which is formed by the sides and Lemoine line of that triangle.* (GOORMAGHTIGH [3] has the Euclidean version).

TEOREM 5. *For the points D,E,F from Theorem 4 the following equalities*

$$AD = \frac{BC\cdot CA\cdot AB}{AB^2-CA^2}, \quad BE = \frac{BC\cdot CA\cdot AB}{BC^2-AB^2}, \quad CF = \frac{BC\cdot CA\cdot AB}{CA^2-BC^2}, \tag{9}$$

$$AD\cdot EF = BE\cdot FD = CF\cdot DE$$

are valid.

Proof. For the point D from (8) we get

$$\begin{aligned} AD &= -\frac{q}{3a}-a = -\frac{1}{3a}(q+3a^2) = -\frac{1}{3a}(3bc-2q) \\ &= \frac{1}{3a}(c-a)(a-b) = \frac{1}{3a}\cdot CA\cdot AB, \end{aligned} \tag{10}$$

because of

$$(c-a)(a-b) = -a^2-bc+ca+ab = -(bc-q)-2bc+q = 2q-3bc,$$

and as we have

$$AB^2-CA^2 = (a-b)^2-(c-a)^2 = (2a-b-c)(c-b) = 3a\cdot BC,$$

the first equality (9) follows. The points E and F which are analogous to the point D from (8) have the abscissas $-\frac{q}{3b}$ and $-\frac{q}{3c}$, and therefore

$$EF = -\frac{q}{3c}+\frac{q}{3b} = \frac{q}{3bc}\cdot(c-b) = \frac{q}{3bc}\cdot BC$$

which together with (10) gives

$$AD\cdot EF = \frac{q}{9p}BC\cdot CA\cdot AB.$$

COROLLARY 2. *With the previous labels the equality*

$$\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF} = 0.$$

is also valid.

THIE [7] has the statement of Theorem 5 and Corollary 2 in Euclidean plane.

TEOREM 6. *If the Lemoine line of the triangle ABC intersects the perpendicular bisectors of the segments BC, CA, AB in the points L, M, N, then the lines AL, BM, CN pass through one point T, which lies on Brocard's diameter of the triangle ABC (Figure 2)* (MINEUR [5] has the Euclidean version)*, i.e. it is parallel to the symmedian center of that triangle.*

Proof. With $x = -\frac{a}{2}$ from (7)

$$y = -\frac{3ap}{2q} + \frac{q}{3} = \frac{2q^2 - 9ap}{6q},$$

follows, so we get

$$L = \left(-\frac{a}{2},\ \frac{1}{6q}(2q^2 - 9ap)\right).$$

The line having the equation

$$9aqy = (9ap + 6bcq - 8q^2)x + 3pq - 9a^2p - aq^2$$

passes through the point $A = (a, a^2)$ and through the point L that can be seen from

$$(9ap + 6bcq - 8q^2)a + 3pq - 9a^2p - aq^2 = 9pq - 9aq^2 = 9aq(bc - q) = 9aq \cdot a^2,$$

$$(9ap+6bcq-8q^2)\left(-\frac{a}{2}\right)+3pq-9a^2p-aq^2 = -\frac{27}{2}a^2p+3aq^2 = 9aq\cdot\frac{1}{6q}(2q^2-9ap).$$

That line passes also through the point

$$T = \left(\frac{3p}{2q},\ \frac{1}{18q^2}(27p^2 - 2q^3)\right) \tag{11}$$

because of

$$(9ap + 6bcq - 8q^2)\frac{3p}{2q} + 3pq - 9(bc - q)p - aq^2 = \frac{27ap^2}{2q} - aq^2 = \frac{a}{2q}(27p^2 - 2q^3).$$

Analogously lines BM and CN pass through the point T. In [1] it is shown that the symmedian center K of the triangle ABC has the form

$$K = \left(\frac{3p}{2q},\ -\frac{q}{3}\right).$$

It is parallel to the point T from (11), i.e. both points lie on the line with the equation $x = \frac{3p}{2q}$, according to [1] Brocard's diameter of the triangle ABC.

TEOREM 7. *The spans of the point T of Theorem 6 from the lines BC, CA, AB are proportional to BC^3, CA^3, AB^3.*

Proof. The span of the point T of (11) from the line BC of (2) is equal to

$$\begin{aligned}
\frac{1}{18q^2}(27p^2 - 2q^3) + a\cdot\frac{3p}{2q} + bc &= \frac{1}{18q^2}(27p^2 - 2q^3 + 27apq + 18bcq^2) \\
&= \frac{1}{18q^2}[27b^2c^2(bc - q) - 2q^3 + 27bcq(bc - q) + 18bcq^2] \\
&= \frac{1}{18q^2}(27b^3c^3 - 9bcq^2 - 2q^3) = \frac{1}{18q^2}(3bc + q)^2(3bc - 2q) \\
&= -\frac{1}{18q^2}(b - c)^4(c - a)(a - b) \\
&= BC^3 \cdot \frac{1}{18q^2}(b - c)(c - a)(a - b)
\end{aligned}$$

because of $3bc-2q=-(c-a)(a-b)$, according to the proof of Theorem 5. Apart from that we have

$$(b-c)^2=(b+c)^2-4bc=a^2-4bc=bc-q-4bc=-(q+3bc). \qquad \square$$

If G is the centroid of the triangle ABC, then the homothecy $(G,-\frac{1}{2})$ maps each points to its *complementary point* and each line to its *complementary line* with respect to the triangle ABC. Conversely, the homothecy $(G,-2)$ maps each point to its *anticomplementary point*, and each line to its *anticomplementary line* with respect to the triangle ABC.

To the orthic axis with the equation $y=-\frac{q}{3}$ the line with the equation $y=-\frac{4}{3}q$ is anticomplementary line, which can be called, by the analogy with the Euclidean case, *Longchamps' line* of the triangle ABC in the standard position.

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