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ABOUT ONE RELATION CONCERNING TWO CIRCLES

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Abstract. This article can be considered as an appendix to the article [1]. Here the article [1] is extended to the cases when one circle is outside of the other and when circles are intersecting.

1. Preliminaries

In [1] the following theorem (Theorem 1) is proved:

Let C_1 and C_2 be any given two circles such that C_1 is inside of the C_2 and let A_1, A_2, A_3 be any given three different points on C_2 such that there are points T_1 and T_2 on C_1 with properties

$$|A_1A_2| = t_1 + t_2, \quad |A_2A_3| = t_2 + t_3, \tag{1}$$

where $t_1 = |A_1T_1|$, $t_2 = |T_1A_2|$, $t_3 = |T_2A_3|$. Then

$$|A_1A_3| = (t_1 + t_3)\frac{2rR}{R^2 - d^2}, \tag{2}$$

where $r =$ radius of C_1, $R =$ radius of C_2, $d = |IO|$, I is the center of C_1, O is center of C_2. (See Figure 1.)

In short about the proof of this theorem. First the following lemma is proved.

It t_1 is given then t_2 can be calculated using the expression

$$(t_2)_{1,2} = \frac{t_1(R^2 - d^2) \pm \sqrt{D_1}}{r^2 + t_1^2} \tag{3a}$$

where

$$D_1 = t_1^2(R^2 - d^2)^2 + (r^2 + t_1^2)\left[4R^2d^2 - r^2t_1^2 - (R^2 + d^2 - r^2)^2\right]. \tag{3b}$$

The values $(t_2)_{1,2}$ given by (3) are solutions of the equation

$$(r^2 + t_1^2)t_2^2 - 2t_1t_2(R^2 - d^2) + r^2t_1^2 - 4R^2d^2 + (R^2 + d^2 - r^2)^2 = 0. \tag{4}$$

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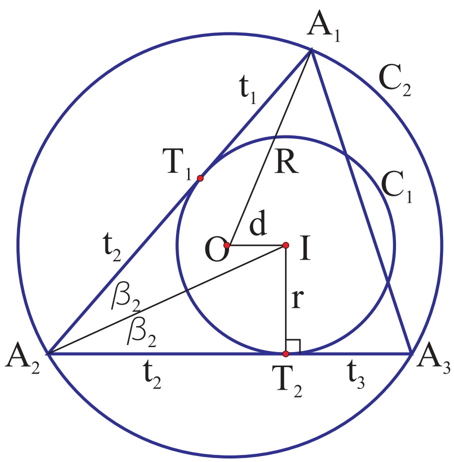

Figure 1

This equation is obtained from the equations

$$(t_1+t_2)^2=(x_1-x_2)^2+(y_1-y_2)^2, \tag{5a}$$

$$t_1^2=(x_1-d)^2+y_1^2-r^2, \quad t_2^2=(x_2-d)^2+y_2^2-r^2 \tag{5b}$$

using relations $x_1^2+y_1^2=x_2^2+y_2^2=R^2$. (See Figure 2.) The length $(t_2)_1$ in Figure 2 is denoted by t_2.

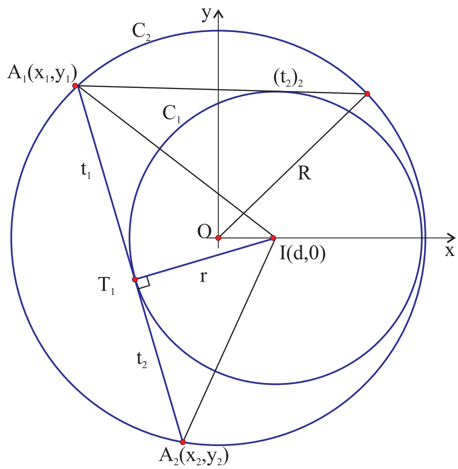

Figure 2

Now from Figure 2 we see that

$$|A_1A_3|^2 = (t_1+t_2)^2 + (t_2+t_3)^2 - 2(t_1+t_2)(t_2+t_3)\frac{t_2^2-r^2}{t_2^2+r^2}, \tag{6}$$

since

$$\cos 2\beta_2 = \frac{1-\tan^2\beta_2}{1+\tan^2\beta 2} = \frac{1-\left(\frac{r}{t_2}\right)^2}{1+\left(\frac{r}{t_2}\right)^2} = \frac{t_2^2-r^2}{t_2^2+r^2}. \tag{7}$$

The tangent length $t_2 = (t_2)_1$ is given by (3) and tangent length t_3 can be written as

$$t_3 = \frac{t_2(R^2-d^2)+\sqrt{D_2}}{r^2+t_2^2} \tag{8a}$$

where

$$D_2 = t_2^2(R^2-d^2)^2 + (r^2+t_2^2)\left[4R^2d^2 - r^2t_2^2 - (R^2+d^2-r^2)^2\right]. \tag{8b}$$

Using computer, it can be found that

$$k(t_1+t_3) - |A_1A_3| = 0 \iff d^4k^2 - 2d^2R^2k^2 - 4r^2R^2 + k^2R^4 = 0. \tag{9}$$

But $d^4k^2 - 2d^2R^2k^2 - 4r^2R^2 + k^2R^4 = 0$ if $k = \frac{2rR}{R^2-d^2}$.

As will be seen in the following sections, analogously holds for the cases when one circle is not inside of the other. Only both circles have to be in the same plane.

2. The case when one circle is outside of the other

The following theorem will be proved.

THEOREM 1. *Let C_1 and C_2 be any given two circles in the same plane such that C_1 is outside of C_2. Let $r =$ radius of C_1, $R =$ radius of C_2, $d = |IO|$, where I is the center of the C_1 and O is the center of C_2 Let $A_1A_2A_3$ be any given triangle such that C_2 is its circumcircle and that lines A_1A_2 and A_2A_3 be tangent lines to C_1. Their tangent points let be denoted by T_1 and T_2 respectively. Then*

$$|A_1A_3| = (t_1+t_3)\frac{2rR}{d^2-R^2}, \tag{10}$$

where $t_1 = |A_1T_1|$, $t_3 = |A_3T_2|$. (See Figure 3)

Proof. First we prove how $t_2 = |A_2T_1| = |A_1T_2|$ and t_3 can be calculated if t_1 is given. For this purpose we prove the following lemma.

LEMMA 1. *If t_1 is given then t_2 can be calculated using expression*

$$t_2 = \frac{t_1(d^2-R^2)+\sqrt{D_1}}{r^2+t_1^2} \tag{11a}$$

where

$$D_1 = t_1^2(d^2-R^2)^2 + (r^2+t_1^2)\left[4d^2R^2 - r^2t_1^2 - (d^2+R^2-r^2)^2\right]. \tag{11b}$$

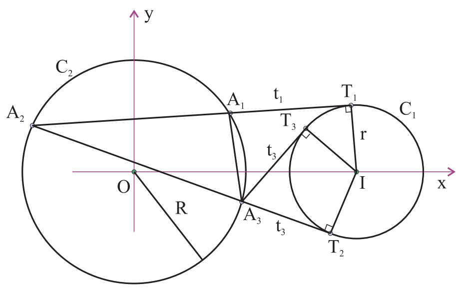

Figure 3

Proof. From triangles A_1IT_1 and A_2IT_2, where $A_1(x_1,y_1)$, $A_2(x_2,y_2)$, $I(d,0)$, we see that

$$t_1^2=(x_1-d)^2+y_1^2-r^2,\quad t_2^2=(x_2-d)^2+y_2^2-r^2$$

from which follows

$$x_1=\frac{R^2+d^2-r^2-t_1^2}{2d},\quad x_2=\frac{R^2+d^2-r^2-t_2^2}{2d}. \tag{12}$$

Now, we have

$$(t_1-t_2)^2=|A_1A_2|^2=(x_1-x_2)^2+(y_1-y_2)^2=2R^2-2x_1x_2-2y_1y_2$$

or $2R^2-2x_1x_2-(t_1-t_2)^2=2y_1y_2$. The relation

$$\left[2R^2-2x_1x_2-(t_1-t_2)^2\right]^2=(2y_1y_2)^2$$

using $x_1^2+y_1^2=x_2^2+y_2^2=R^2$ and (12), can be written as

$$(t_1-t_2)^2\left[(r^2+t_1^2)t_2^2-2t_1t_2(d^2-R^2)+r^2t_1^2-4R^2d^2+(R^2+d^2-r^2)^2\right]=0.$$

We shall show that for every t_1 which can be drawn from C_2 to C_1 we can find t_2 as a solution of the equation

$$(r^2+t_1^2)t_2^2-2t_1t_2(d^2-R^2)+r^2t_1^2-4R^2d^2+(R^2+d^2-r^2)^2=0. \tag{13}$$

For this purpose we shall in (13) replace t_2^2 by t_1^2. The obtained equation can be written as

$$\left[t_1-d^2+(R-r)^2\right]\left[t_1-d^2+(R+r)^2\right]=0. \tag{14}$$

Thus, if t_1 is a solution of the above equation, then $t_2=t_1$. In this case triangle $A_1A_2A_3$ is degenerate. (See Figure 4.)

Also, it can be found that solutions of the equation (13) for t_2 are given by (11) Thus Lemma 1 is proved.

In this connection let us remark that, if t_2 is given such that $t_2=t_1$, then t_3 is one of the solutions of the equation

$$(r^2+t_2^2)t_3^2-2t_2t_3(d^2-R^2)+r^2t_2^2-4R^2d^2+(R^2+d^2-r^2)^2=0. \tag{15}$$

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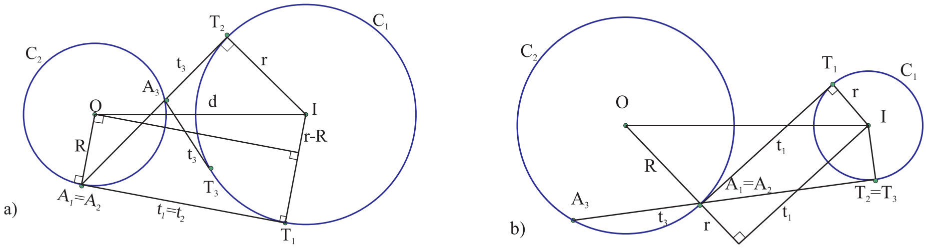


Figure 4

The other solution is $t_1 = t_2$. See, for example, Figure 4, where $t_3 = |A_3T_3|$. As will be shown, for degenerate triangle $A_1A_2A_3$ also holds

$$|A_1A_3| = (t_1 + t_3)\frac{2rR}{d^2 - R^2}.$$

In the case when $t_3 = 0$, then $2Rr = d^2 - R^2$. Namely, in this case the relation is Euler's relation for triangle where excircle instead of incircle is under consideration. (See, for example, [2, Corollary 2.1.3].)

Now we prove Theorem 1. First from Figure 3 we see that

$$|A_1A_3|^2 = (t_1 - t_2)^2 + (t_2 - t_3)^2 - 2(t_1 - t_2)(t_2 - t_3)\frac{t_2^2 - r^2}{t_2^2 + r^2}, \tag{16}$$

since

$$\cos 2\beta_2 = \frac{1 - \tan^2\beta_2}{1 + \tan^2\beta 2} = \frac{1 - \left(\frac{r}{t_2}\right)^2}{1 + \left(\frac{r}{t_2}\right)^2} = \frac{t_2^2 - r^2}{t_2^2 + r^2}. \tag{17}$$

The tangent length $t_2 = (t_2)_1$ is given by (11) and tangent length t_3 can be written as

$$t_3 = \frac{t_2(d^2 - R^2) + \sqrt{D_2}}{r^2 + t_2^2}, \tag{18a}$$

where

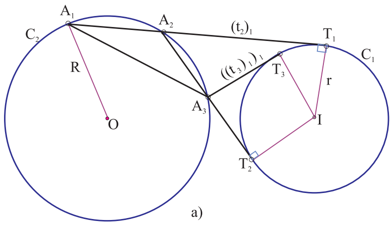
$$D_2 = t_2^2(d^2 - R^2)^2 + (r^2 + t_2^2)\left[4R^2d^2 - r^2t_2^2 - (R^2 + d^2 - r^2)^2\right]. \tag{18b}$$

Using computer, it is not difficult to find that

$$k(t_1 + t_3) - |A_1A_3| = 0 \iff d^4k^2 - 2d^2R^2k^2 - 4r^2R^2 + k^2R^4 = 0. \tag{19}$$

But $d^4k^2 - 2d^2R^2k^2 - 4r^2R^2 + k^2R^4 = 0$ if $k = \frac{2rR}{d^2 - R^2}$. This proves Theorem 1

Here are some illustrations.



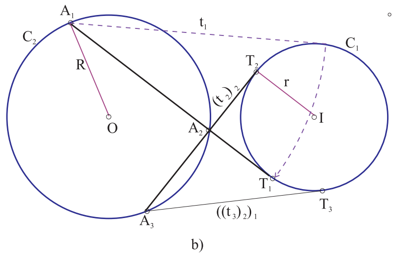


Figure 5

EXAMPLE 2.1. Let $R = 3.8$, $r = 2.75$, $d = 7.7$, $t_1 = 9.1$. (See Figure 5.) Then

$$D_1 = t_1^2(d^2-R^2)^2 + (r^2+t_1^2)\left[4d^2R^2 - r^2t_1^2 - (d^2+R^2-r^2)^2\right] = 154.28498977,$$

$$(t_2)_{1,2} = \frac{t_1(d^2-R^2)\pm\sqrt{D_1}}{r^2+t_1^2} \to (t_2)_1 = 6.223353231,\ (t_2)_2 = 2.808929822, \quad (20)$$

$$(D_2)_1 = (t_2)_1^2(d^2-R^2)^2 + (r^2+(t_2)_1^2)\left[4R^2d^2 - r^2(t_2)_1^2 - (R^2+d^2-r^2)^2\right],$$
$$(D_2)_2 = (t_2)_2^2(d^2-R^2)^2 + (r^2+(t_2)_2^2)\left[4R^2d^2 - r^2(t_2)_2^2 - (R^2+d^2-r^2)^2\right],$$

$$\sqrt{(D_2)_1} = 142.145499, \quad \sqrt{(D_2)_2} = 14.63807559,$$
$$((t_3)_1)_{1,2} = \frac{(t_2)_1(d^2-R^2)\pm\sqrt{(D_2)_1}}{r^2+(t_2)_1^2}$$
$$((t_3)_2)_{1,2} = \frac{(t_2)_2(d^2-R^2)\pm\sqrt{(D_2)_2}}{r^2+(t_2)_2^2}$$

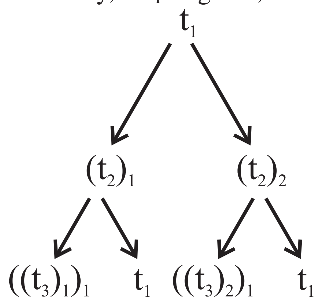
$$((t_3)_1)_1 = 2.958827504, \qquad ((t_3)_1)_2 = 9.1 = t_1,$$
$$((t_3)_2)_1 = 7.20542319, \qquad ((t_3)_2)_2 = 9.1 = t_1.$$

It holds

$$\left(t_1 + ((t_3)_1)_1\right)\cdot\frac{2Rr}{d^2-R^2} = 5.61938673, \qquad |A_1A_3| = 5.61938673,$$
$$\left(t_1 + ((t_3)_2)_1\right)\cdot\frac{2Rr}{d^2-R^2} = 7.598290851, \qquad |A_1A_3| = 7.598290851,$$

where the first relation referred to Figure 5a and the second to Figure 5b.

Generally, if t_1 is given, then situation can be described by the following scheme:



$$\left(t_1+((t_3)_1)_1\right)\cdot\frac{2Rr}{d^2-R^2}=|A_1A_3|_1\,,$$
$$\left(t_1+((t_3)_2)_1\right)\cdot\frac{2Rr}{d^2-R^2}=|A_1A_3|_2\,.$$

EXAMPLE 2.2. This example will be in some way connected with Example 2.1. Namely, let r, R, d be as in Example 2.1, but let $t_1=(t_2)_1$, where $(t_2)_1$ is given by (20). Thus $t_1=6.223353231$. In this case we have

$$t_2=(t_2)_1=9.1,\quad t_3=((t_3)_1)_1=2.808929822$$
$$(t_1+t_3)\frac{2rR}{d^2-R^2}=4.209023879.$$

Using expression

$$|A_1A_3|^2=(t_1-t_2)^2+(t_2-t_3)^2+2(t_1-t_2)(t_2-t_3)\frac{t_2^2-r^2}{t_2^2+r^2},$$

we get $|A_1A_3|=4.209023879$. (See Figure 6.)

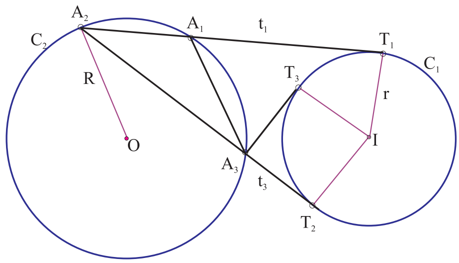


Figure 6

Also can be found that analogously holds for t_1 and $((t_3)_2)_1$. (For brevity we have not drawn Figure 6b.)

Now, something else about the case when triangle $A_1A_2A_3$ is degenerate one. It may be interesting.

Let t_1 be one solution of the equation (14) given by

$$t_1=\sqrt{d^2-(R-r)^2}. \tag{21}$$

Since in this case $t_2 = t_1$, we shall get t_3 as one of the solutions of the equation (15). It can be found that one solution of this equation is $t_2 = t_1$, where t_1 is given by (21), and the other solution t_3 is given by

$$t_3 = \frac{d^2 - 2Rr - R^2}{d^2 + 2Rr - R^2} \cdot t_1. \tag{22}$$

As can be seen from Figure 4a), it holds $|A_1A_3|^2 = (t_1 - t_3)^2$. Thus, we can write

$$\begin{aligned} |A_1A_3|^2 &= (t_1 - t_3)^2 = t_1^2 \left(1 - \frac{d^2 - 2Rr - R^2}{d^2 + 2Rr - R^2}\right)^2 = \frac{16R^2r^2t_1^2}{(d^2 + 2Rr - R^2)^2}, \\ |A_1A_3| &= \frac{4Rrt_1}{d^2 + 2Rr - R^2}, \end{aligned}$$

$$\begin{aligned} (t_1 + t_3)^2 \left(\frac{2Rr}{d^2 - R^2}\right)^2 &= \frac{4t_1^2(d^2 - R^2)^2}{(d^2 + 2Rr - R^2)^2} \left(\frac{2Rr}{d^2 - R^2}\right)^2 = \frac{16R^2r^2t_1^2}{(d^2 + 2Rr - R^2)^2} \\ (t_1 + t_3) \cdot \frac{2Rr}{d^2 - R^2} &= \frac{4Rrt_1}{d^2 + 2Rr - R^2}. \end{aligned}$$

Hence $|A_1A_3| = (t_1 + t_3)\frac{2Rr}{d^2 - R^2}$.

In the same way can be proved that for degenerate triangle $A_1A_2A_3$ shown in Figure 4b) it holds

$$t_1 = \sqrt{d^2 - (R + r)^2}, \quad t_3 = \frac{d^2 - 2Rr - R^2}{d^2 + 2Rr - R^2} \cdot t_1, \quad |A_1A_3| = (t_1 + t_3)\frac{2Rr}{d^2 - R^2}.$$

3. The case when circles are intersecting

The situation can be expressed by the following theorem.

THEOREM 2. *Let C_1 and C_2 be any given circles in the same plane such that holds one of the inequalities*

$$R < d < R + r, \tag{23}$$

$$d < R, \tag{24}$$

where $r =$ radius of C_1, $R =$ radius of C_2, $d = |IO|$, I is the center of C_1 and O is the center of C_2. Let $A_1A_2A_3$ be any given triangle whose circumcircle is C_2 and lines A_1A_2 and A_2A_3 are tangents to C_1 at points T_1 and T_2 respectively. Then

$$(t_1 + t_3)^2 \left(\frac{2Rr}{d^2 - R^2}\right)^2 = |A_1A_3|^2 \tag{25}$$

in the following two cases:

i) Both of A_1 and A_3 are on the same side of the line T_1T_2.

ii) A_3 *is on the line* T_1T_2.

The proof that holds above theorem is completely analogous to the proof of Theorem 1. In this connection let us remark that it is easy to see that in the case when one circle is outside of the other always holds assertion i) or ii).

Also, let us remark that the relation (25) is used instead of two relations

$$(t_1+t_3)\frac{2Rr}{d^2-R^2}=|A_1A_3|, \quad (t_1+t_3)\frac{2Rr}{R^2-d^2}=|A_1A_3|.$$

The first relation corresponds to the inequality (23) and the second to the inequality (24).

Here are some examples. (For brevity we shall restrict ourselves to the relation $(t_1+(t_3)_1)\frac{2Rr}{d^2-R^2}=|A_1A_3|$.)

EXAMPLE 3.1. Let $R=3.6$, $r=5.9$, $d=8.9$, $t_1=5.75$. Then

$$\sqrt{D_1}=349.8395482, \qquad t_2=10.76690925,$$
$$\sqrt{D_2}=153.426188, \qquad t_3=3.714310492,$$

$$|A_1A_3|=6.068587322, \quad \frac{2rR}{d^2-R^2}=0.641207547,$$

$$(t_1+t_3)\frac{2rR}{d^2-R^2}=6.068587322.$$

See Figure 7.

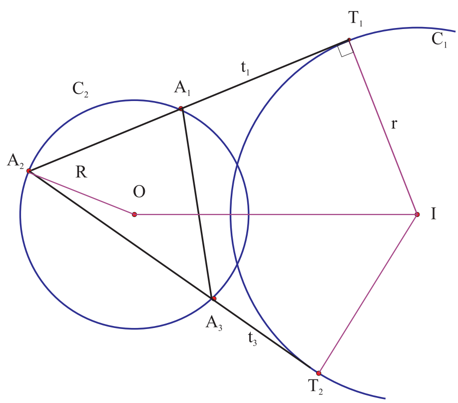


Figure 7

EXAMPLE 3.2. Let $R = 4.1$, $r = 2.5$, $d = 4.4$, $t_1 = 2.5$. Then

$$t_2 = 5.956090318,\ t_3 = 1.771993978,\ |A_1A_3| = 5.851505091.$$

But

$$(t_1 + t_3)\frac{2rR}{d^2 - R^2} = 34.343481 > |A_1A_3|.$$

In this case A_1 and A_3 are on different sides of the line T_1T_2. See Figure 8.

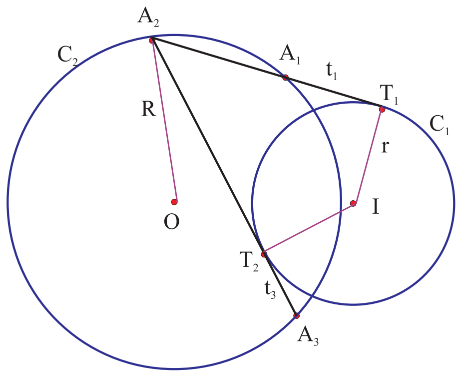


Figure 8

Theorem 2 also holds for some cases when triangle $A_1A_2A_3$ is degenerate one. See, for example, Figure 9. Since in this case also holds relation (22), the proof that holds

$$|A_1A_3| = \frac{4Rrt_1}{d^2 + 2Rr - R^2}, \quad (t_1 + t_3)\frac{2Rr}{d^2 - R^2} = \frac{4Rrt_1}{d^2 + 2rR - R^2},$$

is in the same way as the proof for triangle $A_1A_2A_3$ shown in Figure 4a).

For degenerate triangle $A_1A_2A_3$ shown in Figure 10 it holds $t_1 + t_3 = |A_1A_3|$, but it is not $(t_1 + t_3)\frac{2Rr}{d^2 - R^2} = |A_1A_3|$. In this case A_1 and A_3 are on different sides of the line T_1T_2.

Now about some special cases of Theorem 2.

COROLLARY 2.1. *Let $r > d - R$. (See Figure 11.) If t_1 is given by*

$$t_1 = \frac{2(d^2 - R^2)t_2}{r^2 + t_2^2}, \tag{26a}$$

where

$$t_2 = \frac{\sqrt{4d^2R^2 - (d^2 + R^2 - r^2)^2}}{r}, \tag{26b}$$

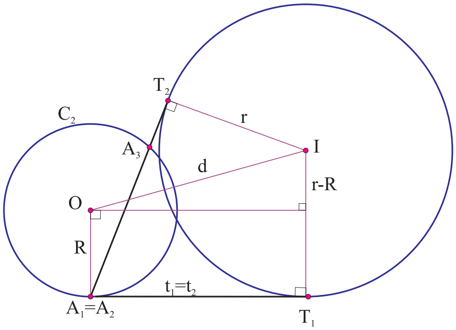


Figure 9

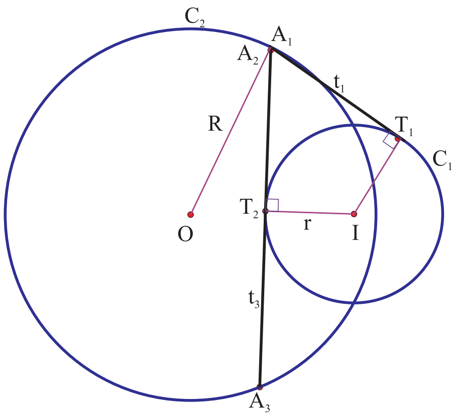


Figure 10

then $t_3 = 0$ *and*

$$|A_1A_3|^2 = t_1^2 \left(\frac{2Rr}{d^2 - R^2}\right)^2 \tag{27a}$$

or

$$|A_1A_3|^2 = \frac{4Rr^3 t_2}{2\,(d^2 + R^2)\,r^2 - (d^2 - R^2)^2}. \tag{27b}$$

Proof. First let us remark that from

$$D_2 = t_2^2(d^2 - R^2)^2 + (r^2 + t_2^2)\left[4R^2d^2 - r^2t_2^2 - (R^2 + d^2 - r^2)^2\right],$$

putting $4d^2R^2 - r^2t_2^2 - (d^2 + R^2 - r^2)^2 = 0$, follows t_2 given by (26b) and

$$\sqrt{D_2} = t_2(d^2 - R^2).$$

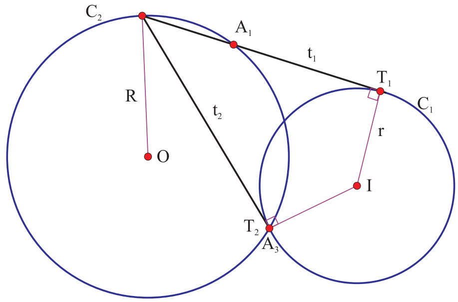


Figure 11

Now from

$$t_3 = \frac{t_2(d^2-R^2) \pm t_2(d^2-R^2)}{r^2+t_2^2}$$

follows

$$(t_3)_1 = \frac{2t_2(d^2-R^2)}{r^2+t_2^2}, \quad (t_3)_2 = 0.$$

That $t_1 = (t_3)_1$ can also be seen from the equation (13), that is from

$$(r^2+t_1^2)t_2^2 - 2t_1t_2(d^2-R^2) + r^2t_1^2 - 4R^2d^2 + (R^2+d^2-r^2)^2 = 0,$$

putting t_2 given by (26b) and then to be solved obtained equation for t_1. The one solution is t_1 given by (26a) and the other is $t_1 = 0$.

That holds (27a), that is

$$t_1 \cdot \frac{2Rr}{d^2-R^2} = |A_1A_3| \tag{28a}$$

or

$$\frac{4Rrt_2}{r^2+t_2^2} = |A_1A_3|, \tag{28b}$$

can be proved in the following elementary way. See Figure 12. The equation (28b) can be written as

$$2\frac{t_2}{\sqrt{r^2+t_2^2}} \cdot \frac{r}{\sqrt{r^2+t_2^2}} = \frac{|A_1A_3|}{2R}$$

or $2\cos\alpha\sin\alpha = \sin 2\alpha$. Let us remark that $t_2 = |A_2A_3|$.

COROLLARY 2.2. *Let $t_1 = t_3$. See Figure 13. Since $t_2 = |A_2T_2|$ and*

$$\frac{t_1}{d-R} = \frac{t_2-t_1}{2R} \text{ or } t_1 = \frac{d-R}{d+R}t_2,$$

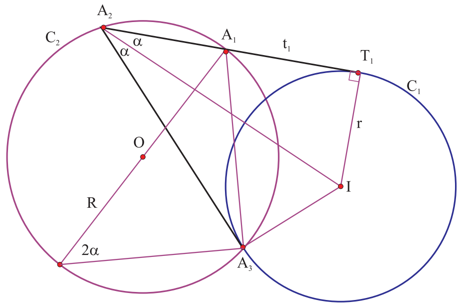


Figure 12

we can write

$$2t_1 \cdot \frac{2Rr}{d^2 - R^2} = 4R \cdot \frac{r}{d+R} \cdot \frac{t_2}{d+R} = 4R\,\sin\alpha\cos\alpha = 2 \cdot R \cdot \sin 2\alpha$$
$$= 2\frac{|A_1A_3|}{2} = |A_1A_3|\,.$$

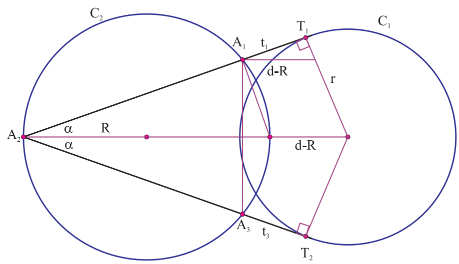


Figure 13

REMARK 1. It is quite possible that the Theorem 1 and Theorem 2 can be proved using only planimetry and trigonometry (as in the above corollaries). But we have not succeed (at least for the time being).

4. Connection with bicentric polygons

As it is known, a polygon which is both chordal and tangential is called bicentric polygon. In connection with Theorem 1 and Theorem 2 we have the following theorem concerning bicentric polygons.

THEOREM 3. *Let $A_1 \ldots A_n$ be a tangential polygon, where instead of incircle there is excircle. Then this polygon is also a chordal one if and only if it holds*

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{|A_2A_4|}{t_2+t_4} = \cdots = \frac{|A_nA_2|}{t_4+t_2} = \frac{2Rr}{d^2-R^2}. \tag{29}$$

Proof. First it is easy to see that , if $A_1 \ldots A_n$ is a bicentric polygon, then holds (29). See, for example, Figure 14, where $A_1 \ldots A_n$ is a bicentric quadrilateral.

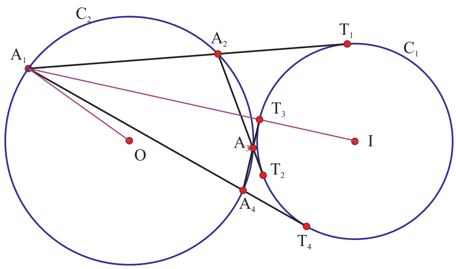


Figure 14

That conversely also holds, can be proved in the same way as it is proved Corollary 2 in [1]. (There is complete analogy between this proof and the proof of Corollary 2 in [1].)

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