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BROCARD ANGLE OF THE STANDARD TRIANGLE IN AN ISOTROPIC PLANE

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Abstract. The concept of Brocard angle of the standard triangle in an isotropic plane I_2 is introduced. The relationships between Brocard angles of the allowable triangle and circum-Ceva's triangle of its centroid and circum-Ceva's triangle of its Feuerbach point are investigated.

1. Introduction

In Euclidean geometry Brocard angle ω of the triangle ABC with the lengths of the sides a, b, c, the opposite angles A, B, C, the area $\triangle$ and with the radius of circumscribed circle R can be defined by means of equivalent formulas

$$\cot\omega = \cot A + \cot B + \cot C, \qquad \cot\omega = \frac{a^2+b^2+c^2}{4\triangle}, \tag{1}$$

and the following formulas

$$a = 2R\sin A, \qquad b = 2R\sin B, \qquad c = 2R\sin C, \tag{2}$$

$$4\triangle R = abc \tag{3}$$

are also valid.

In an isotropic plane I_2 (more detailed in [4] and [3]) the roles of sine and tangens of the angle are taken over by the very angle, and sides and angles of the triangle are oriented, so the equalities $a+b+c=0$ and $A+B+C=0$ are valid. Therefore for example

$$A^2+B^2+C^2 = -2(B\cdot C + C\cdot A + A\cdot B).$$

The formulas (1) and (2) should be in accordance with the formulas

$$\frac{1}{\omega} = \frac{1}{A}+\frac{1}{B}+\frac{1}{C}, \qquad \frac{1}{\omega} = \frac{a^2+b^2+c^2}{4\triangle}, \tag{4}$$

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$$a = 2RA, \qquad b = 2RB, \qquad c = 2RC, \tag{5}$$

i.e. the formulas

$$\omega = \frac{A \cdot B \cdot C}{B \cdot C + C \cdot A + A \cdot B} = -2\frac{A \cdot B \cdot C}{A^2 + B^2 + C^2},$$

$$\omega = \frac{4\triangle}{a^2 + b^2 + c^2} = \frac{\triangle}{R^2(A^2 + B^2 + C^2)}.$$

These two formulas will be in accordance if the formula $-2R^2 \cdot A \cdot B \cdot C = \triangle$ i.e. because of (5) the formula

$$4\triangle R = -abc \tag{6}$$

is valid.

Because of that we have to use formula (6) instead of formula (3) in an isotropic plane. Let us also add that in an isotropic plane only the so called *allowable triangles*, triangles whose sides are not isotropic, i.e. no two of its vertices are parallel, are studied.

2. Standard triangle

In [2] it is shown that any allowable triangle can be set, by a suitable choice of the coordinate system, in a standard position, in which its circumscribed circle has the equation $y = x^2$, and the vertices of the triangle ABC are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{7}$$

where

$$s = a + b + c = 0. \tag{8}$$

In that case we shall say that ABC is a *standard triangle*. The following numbers

$$p = abc, \qquad q = bc + ca + ab \tag{9}$$

are also important for it.

Let us mention that here and later slightly different labels than in the introduction are used, so for example the lengths of the sides of the triangle ABC will be denoted by BC, CA, AB. From (8) and (9) it immediately follows

$$a^2 + b^2 + c^2 = -2q,$$

and the triangle ABC with the vertices (7) has the centroid

$$G = \left(0, -\frac{2}{3}q\right), \tag{10}$$

and

$$b^2c^2 + c^2a^2 + a^2b^2 = (ab + bc + ca)^2 - 2abc(a + b + c) = q^2.$$

Standard triangle ABC has the area $\triangle$ which is, because of (7), given by the equalities

$$2\triangle = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b) = -BC \cdot CA \cdot AB,$$

and as in accordance with the formula (6) we want that equality

$$4\triangle R = -BC \cdot CA \cdot AB,$$

to be valid, so let $2R = 1$. (In [3] the definition is slightly different; it is chosen that $R = 1$.)

The line AB has the equation $y = -cx - ab$ because for example for the point $A = (a, a^2)$, owing to (8) the equality $a^2 = -ca - ab$ holds. In the same way, the line AC has the equation $y = -bx - ac$. Therefore

$$A = \angle(AB, AC) = -b - (-c) = c - b = BC$$

and similarly $B = CA$, $C = AB$. Thus because of $2R = 1$ the formulas

$$BC = A, \qquad CA = B, \qquad AB = C$$

should be used instead of the formula (5).

3. Brocard angle of the standard triangle

Brocard angle ω of the standard triangle ABC is given by the formula

$$\omega = \frac{4\triangle}{BC^2 + CA^2 + AB^2} = -\frac{2 \cdot BC \cdot CA \cdot AB}{BC^2 + CA^2 + AB^2}$$

and owing to

$$\begin{aligned} BC^2 + CA^2 + AB^2 &= (b-c)^2 + (c-a)^2 + (a-b)^2 \\ &= (b+c)^2 + (c+a)^2 + (a+b)^2 - 4(bc + ca + ab) \\ &= a^2 + b^2 + c^2 - 4q = -2q - 4q = -6q \end{aligned}$$

it follows

$$\omega = -\frac{1}{3q}(b-c)(c-a)(a-b). \tag{11}$$

In the study of the standard triangle the following expressions,

$$p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2), \qquad p_2 = \frac{1}{3}(b^2c + c^2a + a^2b), \tag{12}$$

which are not symmetrical to a, b, c but only cyclical symmetric, will be very useful.

The equations

$$\begin{aligned}3p_1+3p_2&=bc^2+ca^2+ab^2+b^2c+c^2a+a^2b\\&=bc(b+c)+ca(c+a)+ab(a+b)=-3abc=-3p,\end{aligned}$$

$$3p_1-3p_2=bc^2+ca^2+ab^2-b^2c-c^2a-a^2b=(b-c)(c-a)(a-b),$$

i.e.

$$\begin{aligned}p+p_1+p_2&=0,\\p_1-p_2&=\frac{1}{3}(b-c)(c-a)(a-b)\end{aligned}\tag{13}$$

are valid for them.

Because of (13) the formula (11) can be written in the form

$$\omega=\frac{p_2-p_1}{q}.\tag{14}$$

We have proved:

THEOREM 1. *Brocard angle of the standard triangle is given by the formulas* (11) *and* (14)*, where the numbers* p_1*,* p_2*,* q *are given with* (12) *and* (9)*.*

4. Circum-Ceva's triangle of the centroid of the allowable triangle

For the points $A=(a,a^2)$, $D=(d,d^2)$ on the circumscribed circle of the triangle ABC the line AD has the equation $y=(a+d)x-ad$. That line passes through the centroid G from (10) if $d=\frac{2q}{3a}$. This can similarly be applied on the analogous lines BG and CG, so it is valid:

THEOREM 2. *The medians* AG*,* BG*,* CG *of the standard triangle* ABC *meet its circumscribed circle again in the points* $D=(d,d^2)$*,* $E=(e,e^2)$*,* $F=(f,f^2)$*, where*

$$d=\frac{2q}{3a},\qquad e=\frac{2q}{3b},\qquad f=\frac{2q}{3c}.$$

The triangle DEF is the so called *circum-Ceva's triangle* of the centroid G of the triangle ABC with respect to that triangle.

The triangle DEF has the area $\triangle'$ given by the formula $2\triangle'=(e-f)(f-d)(d-e)$. As for example

$$e-f=-\frac{2q}{3bc}(b-c)=-\frac{2q}{3p}a(b-c),$$

so on one hand this is because of (11)

$$2\triangle'=-\frac{8q^3}{27p^2}(b-c)(c-a)(a-b)=\frac{8q^3}{27p^2}\cdot 3q\omega=\frac{8q^4}{9p^2}\omega,$$

and on the other hand

$$\begin{aligned} EF^2+FD^2+DE^2 &= \frac{4q^2}{9p^2}[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2] \\ &= \frac{4q^2}{9p^2}[2(b^2c^2+c^2a^2+a^2b^2)-2abc(a+b+c)] = \frac{8q^4}{9p^2}, \end{aligned}$$

and it follows

$$\frac{2\triangle'}{EF^2+FD^2+DE^2}=\omega.$$

Therefore the triangle DEF has Brocard angle

$$\omega' = \frac{4\triangle'}{EF^2+FD^2+DE^2}=2\omega,$$

i.e. we have:

THEOREM 3. *Brocard angles of the allowable triangle and circum-Ceva's triangle of its centroid are in the ratio* $1:2$.

In Euclidean geometry a much more complicated statement is analogous to Theorem 3. According to [5] the equality $\omega+\omega'+\delta=\frac{\pi}{2}$ is valid, where δ is angle such that $3\tan\omega\tan\delta=1$.

5. Circum-Ceva's triangle of Feuerbach point of the allowable triangle

According to [1] Feuerbach point of the standard triangle ABC has the form $\Phi=(0,-q)$. The line AD with the equation $y=(a+d)x-ad$ passes through the point Φ if $d=\frac{q}{a}$. Thus it is valid:

THEOREM 4. *Feuerbach point of the standard triangle ABC has circum-Ceva's triangle DEF with the vertices $D=(d,d^2)$, $E=(e,e^2)$, $F=(f,f^2)$, where*

$$d=\frac{q}{a},\qquad e=\frac{q}{b},\qquad f=\frac{q}{c}.$$

Calculating, analogously as in the proof of the Theorem 3 we now get for example $e-f=-\frac{q}{p}a(b-c)$ and then

$$2\triangle'=\frac{3q^4}{p^2}\omega,\qquad EF^2+FD^2+DE^2=\frac{2q^4}{p^2}$$

and therefore

$$\omega'=\frac{4\triangle'}{EF^2+FD^2+DE^2}=3\omega,$$

so we have:

THEOREM 5. *Brocard angles of the allowable triangle and circum-Ceva's triangle of its Feuerbach point are in the ratio* $1:3$.

It would be interesting to find Brocard angle of the circum-Ceva's triangle of Feurbach point in Euclidean geometry.

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