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ON SOME HILBERT'S TYPE INEQUALITIES

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Abstract. A generalization of the well-known Hilbert's inequality is given. Several other results of this type in recent years follows as a special case of our result.

1. Introduction

First, let us repeat the well-known Hilbert's integral inequality:

THEOREM A. *If* $f,\ g \in L^2(0,\infty)$*, then the following inequality holds:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy \leqslant \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx \right)^{\frac{1}{2}} \tag{1.1}$$

where π *is the best constant.*

In the recent years a lot of results with generalization of this type of inequality were obtained. Let us mention one of them which take our attention.

Note that in all theorems we suppose that all integrals converge.

THEOREM B. (Yang, Rassias, [1]): *If* f *and* g *are real functions such that:*

$$\int_0^\infty t^{1-\lambda} f^2(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty t^{1-\lambda} g^2(t)\,dt < \infty$$

then:

i) for $0 < b < \infty$*, we have*

A)

$$\int_0^b \int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \left[\int_0^b \left[1 - \frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}} \right] t^{1-\lambda} f^2(t)\,dt \right]^{\frac{1}{2}} \times$$

$$\times \left[\int_0^b \left[1 - \frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}} \right] t^{1-\lambda} g^2(t)\,dt \right]^{\frac{1}{2}}; \tag{1.2}$$

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B)

$$\int_0^b y^{\lambda-1}\left[\int_0^b \frac{f(x)}{(x+y)^\lambda}\,dx\right]^2 dy < \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^2 \int_0^b \left[1-\frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} f^2(t)\,dt; \quad (1.3)$$

ii) for $0<a<\infty$, we have

A)

$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} f^2(t)\,dt\right]^{\frac{1}{2}} \times$$
$$\times\left[\int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} g^2(t)\,dt\right]^{\frac{1}{2}}; \quad (1.4)$$

B)

$$\int_a^\infty y^{\lambda-1}\left[\int_a^\infty \frac{f(x)}{(x+y)^\lambda}\,dx\right]^2 dy < \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^2 \int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} f^2(t)\,dt \quad (1.5)$$

where B is beta-function.

In this paper we generalize inequalities (1.2)–(1.5).

2. Main results

THEOREM 1. *If f and g are real functions and p a real number, $p>1$, such that:*

$$\int_0^\infty t^{p-1-\frac{p\lambda}{2}} f^p(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty t^{p-1-\frac{p\lambda}{2}} g^p(t)\,dt < \infty$$

then:

i) for $0<b<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, we have

A)

$$\int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\int_0^b \left[1-\frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}}\right] t^{p-1-\frac{p\lambda}{2}} f^p(t)\,dt\right]^{\frac{1}{p}} \times$$
$$\times\left[\int_0^b \left[1-\frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}}\right] t^{q-1-\frac{q\lambda}{2}} g^q(t)\,dt\right]^{\frac{1}{q}}; \quad (2.1)$$

B)

$$\int_0^b y^{\frac{\lambda p}{2}-1}\left[\int_0^b \frac{f(x)\,dx}{(x+y)^\lambda}\right]^p dy < \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^p \int_0^b \left[1-\frac{1}{2}\left(\frac{t}{b}\right)^{\frac{\lambda}{2}}\right] t^{p-1-\frac{p\lambda}{2}} f^p(t)\,dt; \tag{2.2}$$

ii) for $0<a<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, *we have*
A)

$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{p-1-\frac{p\lambda}{2}} f^p(t)\,dt\right]^{\frac{1}{p}} \times$$
$$\times \left[\int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{q-1-\frac{q\lambda}{2}} g^q(t)\,dt\right]^{\frac{1}{q}}; \tag{2.3}$$

B)

$$\int_a^\infty y^{\frac{\lambda p}{2}-1}\left[\int_a^\infty \frac{f(x)\,dx}{(x+y)^\lambda}\right]^p dy < \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^p \int_a^\infty \left[1-\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{\lambda}{2}}\right] t^{p-1-\frac{p\lambda}{2}} f^q(t)\,dt. \tag{2.4}$$

Proof: i) A) We start with the following equality:

$$\int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy = \int_0^b \frac{f(x)\frac{x^{\frac{2-\lambda}{2q}}}{y^{\frac{2-\lambda}{2p}}}}{(x+y)^{\frac{\lambda}{p}}}\cdot\frac{g(y)\frac{y^{\frac{2-\lambda}{2p}}}{x^{\frac{2-\lambda}{2q}}}}{(x+y)^{\frac{\lambda}{q}}}\,dxdy. \tag{2.5}$$

By Hölder's inequality and (2.5) we have:

$$\int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy$$
$$\leqslant \left[\int_0^b\int_0^b \frac{f^p(x)}{(x+y)^\lambda}\cdot\frac{x^{\frac{p(2-\lambda)}{2q}}}{y^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{p}}\left[\int_0^b\int_0^b \frac{g^q(y)}{(x+y)^\lambda}\cdot\frac{y^{\frac{q(2-\lambda)}{2p}}}{x^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{q}}$$
$$= \left[\int_0^b\int_0^b \frac{f^p(x)}{(x+y)^\lambda}\cdot\frac{x^{\frac{p(2-\lambda)}{2q}}x^{\frac{2-\lambda}{2}}}{x^{\frac{2-\lambda}{2}}y^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{p}}\left[\int_0^b\int_0^b \frac{g^q(y)}{(x+y)^\lambda}\cdot\frac{y^{\frac{q(2-\lambda)}{2p}}y^{\frac{2-\lambda}{2}}}{y^{\frac{2-\lambda}{2}}x^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{q}}$$
$$= \left[\int_0^b f^p(x)x^{\frac{(2-\lambda)(p-q)}{2q}}\left(\int_0^b \frac{\left(\frac{y}{x}\right)^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dy\right)dx\right]^{\frac{1}{p}}\times$$

$$\times\left[\int_0^b g^q(y)y^{\frac{(2-\lambda)(q-p)}{2p}}\left(\int_0^b \frac{\left(\frac{x}{y}\right)^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dx\right)dy\right]^{\frac{1}{q}}$$

$$=\left[\int_0^b f^p(x)x^{\frac{(2-\lambda)(p-q)}{2q}} I_x\,dx\right]^{\frac{1}{p}}\left[\int_0^b g^q(y)y^{\frac{(2-\lambda)(q-p)}{2p}} I_y\,dy\right]^{\frac{1}{q}}, \tag{2.6}$$

where we denote:

$$I_x=\int_0^b \left(\frac{y}{x}\right)^{\frac{\lambda-2}{2}}(x+y)^{-\lambda}\,dy, \qquad I_y=\int_0^b \left(\frac{x}{y}\right)^{\frac{\lambda-2}{2}}(x+y)^{-\lambda}\,dx.$$

Using the change of variables $y=xt$, $dy=x\,dt$, we have for I_x:

$$I_x=\int_0^{\frac{b}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(x+xt)^\lambda}x\,dt=x^{1-\lambda}\int_0^{\frac{b}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt=x^{1-\lambda}W_\lambda\left(\frac{b}{x}\right)$$

and similarly:

$$I_y=y^{1-\lambda}W_\lambda\left(\frac{b}{y}\right)$$

where we denote:

$$W_\lambda(z)=\int_0^z \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt, \qquad z>1 \tag{2.7}$$

Using expressions for I_x and I_y, (2.6) can be rewritten as:

$$\int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy$$

$$\leqslant\left[\int_0^b f^p(x)x^{\frac{(2-\lambda)(p-q)}{2q}+1-\lambda}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\left[\int_0^b g^q(y)y^{\frac{(2-\lambda)(q-p)}{2p}+1-\lambda}W_\lambda\left(\frac{b}{y}\right)dy\right]^{\frac{1}{q}}$$

$$=\left[\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\left[\int_0^b g^q(y)y^{q-1-\frac{q\lambda}{2}}W_\lambda\left(\frac{b}{y}\right)dy\right]^{\frac{1}{q}}. \tag{2.8}$$

In [2] Gavrea proved the inequality:

$$\int_\alpha^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt>\frac{1}{2}\alpha^{-\frac{\lambda}{2}}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right), \qquad \alpha>1. \tag{2.9}$$

Using (2.9) we have for $W_\lambda\left(\frac{b}{x}\right)$:

$$\begin{aligned}
W_\lambda\left(\frac{b}{x}\right) &= \int_0^{\frac{b}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt = \int_0^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt - \int_{\frac{b}{x}}^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt \\
&= B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) - \int_{\frac{b}{x}}^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt \\
&< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) - \frac{1}{2}\left(\frac{b}{x}\right)^{-\frac{\lambda}{2}} B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \\
&= \left[1-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{\lambda}{2}}\right] B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \qquad (2.10)
\end{aligned}$$

and similarly

$$W_\lambda\left(\frac{b}{y}\right) < \left[1-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{\lambda}{2}}\right] B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right). \qquad (2.11)$$

Using inequalities (2.10) and (2.11) in (2.8), and taking into account $\frac{1}{p}+\frac{1}{q}=1$, we obtain (2.1).

In the similar way we prove ii) A), using the equation:

$$\int_{\frac{a}{x}}^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt = \int_0^{\frac{x}{a}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt = W_\lambda\left(\frac{x}{a}\right). \qquad (2.12)$$

REMARK 1. Inequalities (2.1) and (2.3) are generalizations of (1.2) and (1.4), respectively. We obtain (1.2) and (1.4) by putting $p=q=2$ in (2.1) and (2.3).

We begin our proof of i) B) by using the equation:

$$\begin{aligned}
J &= \int_0^b y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}\left(\int_0^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^p dy \\
&= \int_0^b y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}\left(\int_0^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p-1}\left(\int_0^b \frac{f(x)\,dx}{(x+y)^\lambda}\right) dy \\
&= \int_0^b g(y)\left(\int_0^b \frac{f(x)\,dx}{(x+y)^\lambda}\right) dy = \int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy \qquad (2.13)
\end{aligned}$$

where we denote

$$g(y)=y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}\left(\int_0^b\frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p-1}. \tag{2.14}$$

Using (2.8) and (2.14) we have:

$$\begin{aligned}
J &\leqslant \left[\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\left[\int_0^b g^q(y)y^{q-1-\frac{q\lambda}{2}}W_\lambda\left(\frac{b}{y}\right)dy\right]^{\frac{1}{q}}\\
&= \left[\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\times\\
&\quad\times\left[\int_0^b y^{q\left(\frac{\lambda p}{2}-1\right)}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{q(1-p)}\left(\int_0^b\frac{f(x)\,dx}{(x+y)^\lambda}\right)^{q(p-1)}y^{q-1-\frac{q\lambda}{2}}W_\lambda\left(\frac{b}{y}\right)dy\right]^{\frac{1}{q}}\\
&= \left[\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\left[\int_0^b y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}\left(\int_0^b\frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p}dy\right]^{\frac{1}{q}}.
\end{aligned}$$

Thus we obtain:

$$J\leqslant\left[\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx\right]^{\frac{1}{p}}\cdot J^{\frac{1}{q}}$$

wherefrom it follows:

$$J=\int_0^b y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}\left(\int_0^b\frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p}dy\leqslant\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}W_\lambda\left(\frac{b}{x}\right)dx. \tag{2.15}$$

As $1-p<0$ and $\left[1-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{\lambda}{2}}\right]^{1-p}>1$ for $y\in(0,b)$, we obtain from (2.11):

$$\left[W_\lambda\left(\frac{b}{y}\right)\right]^{1-p}>\left\{\left[1-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{\lambda}{2}}\right]B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right\}^{1-p}>\left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{1-p}.$$

Using the last inequation in (2.15) we finally obtain:

$$\begin{aligned}
&\int_0^b y^{\frac{\lambda p}{2}-1}\left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{1-p}\left(\int_0^b\frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p}dy\\
&\qquad<\int_0^b f^p(x)x^{p-1-\frac{p\lambda}{2}}\left[1-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{\lambda}{2}}\right]B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)dx
\end{aligned}$$

wherefrom it follows (2.2).

The proof of ii) B) is similar to the proof of i) B). Indeed, instead of the function $g(y)$ used in (2.14), here we define:

$$g(y) = y^{\frac{\lambda p}{2}-1}\left[W_\lambda\left(\frac{y}{a}\right)\right]^{1-p}\left(\int_a^\infty \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p-1}, \qquad y > a,$$

and later we use the inequality

$$\left[W_\lambda\left(\frac{y}{a}\right)\right]^{1-p} > \left\{\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{\lambda}{2}}\right]B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right\}^{1-p} > \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{1-p} \qquad \text{for } y > a.$$

REMARK 2. Inequalities (2.2) and (2.4) are generalization of (1.3) and (1.5), respectively. We obtain (1.3) and (1.5) by putting $p = q = 2$ in (2.2) and (2.4).

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