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HERMITE–HADAMARD TYPE INEQUALITIES FOR STOLARSKY AND RELATED MEANS

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Abstract. Some theorems of the Jensen type for certain classes of means are given in [7]. Some further generalizations and further applications of these results are presented here.

1. Introduction

In [14] H.-J. Seiffert developed an inequality related to the Jensen inequalities for convex and concave functions, which for a certain class of functions, connects the mean of an integral over an interval $[a,b]$ $(a,b>0)$ to the integrand evaluated at, so called, the identric mean $I(a,b)$ of the end points, which is defined by

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & , \quad a\neq b,\\ a & , \quad a=b.\end{cases} \tag{1.1}$$

Seiffert's result provides the following:

THEOREM 1.1. *If f is a strictly increasing continuous function on $[a,b]$, $0<a<b$, having a logarithmically convex inverse function, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant f(I(a,b)), \tag{1.2}$$

while the inequality in (1.2) is reversed if f is strictly decreasing.

A positive function g on $[a,b]$ is logarithmically convex or, simply, logconvex if for every $x,y\in[a,b]$ and $r,s\geqslant 0, r+s=1$ holds that

$$(g(x))^r(g(y))^s \geqslant g(rx+sy), \tag{1.3}$$

while g is logarithmically concave (logconcave) if (1.3) holds with the inequality reversed.

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An analogous result is given by H.Alzer ([1]), that is

$$f(L(a,b)) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \tag{1.4}$$

if $f \in C([a,b])$ is strictly increasing, $1/f^{-1}$ is convex and $L(a,b)$ is the logarithmic mean defined by

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & , \quad a \neq b, \\ a & , \quad a = b, \end{cases} \tag{1.5}$$

while the inequality in (1.4) is reversed if f is strictly decreasing.

The identric and the logarithmic means are rather special cases of the generalized logarithmic mean defined by

$$L_r(a,b) = \begin{cases} \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{\frac{1}{r}} , & r \neq -1, 0 \ , \ a \neq b; \\ \frac{b-a}{\ln b - \ln a} \ , & r = -1, \ a \neq b; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} , & r = 0, \ a \neq b; \\ a \ , & a = b. \end{cases} \tag{1.6}$$

In [7] authors gave the analogous result for this generalized logarithmic mean ([7], Theorem 2.1.):

THEOREM 1.2. *Let a,b be the positive numbers and $f : [a,b] \to \mathbb{R}$ a real-valued function. If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is convex function, or $r = 0$ and $k(x) = f(e^x)$ is convex, then*

$$f(L_r(a,b)) \leqslant \frac{1}{b-a}\int_a^b f(t)dt. \tag{1.7}$$

If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is concave, or $r = 0$ and $k(x) = f(e^x)$ is concave, then (1.7) holds with the inequality reversed.

This result is the generalization of Seiffert's and Alzer's result, what can be easily seen by a short calculation.

In the same paper ([7]) authors gave the generalizations for the Pittenger multidimensional logarithmic mean.

Let E_{n-1} denote the $(n-1)$-dimensional Euclidean simplex given by

$$E_{n-1} = \{(u_1, u_2, \ldots, u_{n-1}) : \ u_i \geqslant 0, \ 1 \leqslant i \leqslant n-1, \ \sum_{i=1}^{n-1} u_i \leqslant 1\},$$

and set $u_n = 1 - \sum_{i=1}^{n-1} u_i$, $\mathbf{u} = (u_1, u_2, ..., u_n)$. For an $n-$tuple $\mathbf{x} = (x_1, ..., x_n)$ of the positive real numbers, A.O.Pittenger in [13] defined the multidimensional logarithmic mean by

$$L(x_1, ..., x_n) = \left[(n-1)! \int_{E_{n-1}} (\mathbf{x} \cdot \mathbf{u})^{-1} du_1 \cdots du_{n-1}\right]^{-1}.$$

In the same paper he also gave its generalization — the multidimensional generalized logarithmic mean:

$$L_r(\mathbf{x}) = \begin{cases} \left(\int_{E_{n-1}} (\mathbf{x}\cdot\mathbf{u})^r \, d\mu(\mathbf{u})\right)^{\frac{1}{r}}, & r \neq 0; \\ \exp(\int_{E_{n-1}} \ln(\mathbf{x}\cdot\mathbf{u}) \, d\mu(\mathbf{u})), & r = 0; \end{cases} \tag{1.8}$$

where μ denotes the probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1 \cdots du_{n-1}$.

In [7] the following result is also given ([7], Theorem 3.1.):

THEOREM 1.3. *Let $x_1,...,x_n$ be the positive numbers belonging to some interval I and let $f : I \to \mathbb{R}$ be a real-valued function. If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is convex, or $r = 0$ and $k(x) = f(e^x)$ is convex, then*

$$f(L_r(\mathbf{x})) \leqslant \int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u}) d\mu(\mathbf{u}). \tag{1.9}$$

If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is concave, or $r = 0$ and $k(x) = f(e^x)$ is concave, then (1.9) holds with the inequality reversed.

In the same paper [7, Theorem 4.1.] the analogous result for the integral power means is also given.

In this paper we give further generalizations of these results.

2. Main results

Let $\Omega \subseteq \mathbb{R}^n$ be a convex set equipped with a probability measure μ. For a strictly monotone continuous function h, the quasi-arithmetic mean $M_h(g;\mu)$ is defined as follows:

$$M_h(g;\mu) = h^{-1}\left(\int_\Omega (h\circ g)(\mathbf{u}) d\mu(\mathbf{u})\right).$$

In this paper we suppose, without further comment, that all involved integrals exist.

THEOREM 2.1. *Let $g : \Omega \to \mathbb{R}$ be a continuous function, h strictly monotone continuous function defined on the image of g and f a real-valued function defined on the image of g. If $k(x) = f\left(h^{-1}(x)\right)$ is convex function, then*

$$f(M_h(g;\mu)) \leqslant \int_\Omega f(g(\mathbf{u})) d\mu(\mathbf{u}) \tag{2.1}$$

is valid.

If the function g is bounded and its minimum and maximum value, m and M, are not equal, then we also have that

$$\int_\Omega f(g(\mathbf{u})) d\mu(\mathbf{u}) \leqslant \frac{h(M) - h(M_h(g;\mu))}{h(M) - h(m)} \cdot f(m) + \frac{h(M_h(g;\mu)) - h(m)}{h(M) - h(m)} \cdot f(M). \tag{2.2}$$

If $k(x) = f\left(h^{-1}(x)\right)$ is concave, then (2.1) and (2.2) hold with the inequalities reversed.

Proof. If $k(x) = f\left(h^{-1}(x)\right)$ is convex, then by the Jensen inequality it follows:

$$\begin{aligned} f\left(M_h(g;\mu)\right) &= f\left[h^{-1}\left(\int_\Omega (h\circ g)(\mathbf{u})d\mu(\mathbf{u})\right)\right] = k\left(\int_\Omega (h\circ g)(\mathbf{u})d\mu(\mathbf{u})\right) \\ &\leqslant \int_\Omega (k\circ h)(g(\mathbf{u}))d\mu(\mathbf{u}) = \int_\Omega f(g(\mathbf{u}))d\mu(\mathbf{u}), \end{aligned}$$

and (2.1) is proved.

For the second inequality, we use the following result from Beesack and Pečarić [2] (see also [11], page 98). For a convex function Φ on the interval $I = [m_1, M_1]$ ($m_1 < M_1$) and an isotonic linear functional A, they proved that the following inequality is valid:

$$A(\Phi(g)) \leqslant \frac{M_1 - A(g)}{M_1 - m_1}\Phi(m_1) + \frac{A(g) - m_1}{M_1 - m_1}\Phi(M_1).$$

If we apply this result on our convex function $k(x) = f\left(h^{-1}(x)\right)$, and A is an integral over Ω with the probability measure μ, and then instead of the function g we consider the function $h\circ g$, then we get the following:

$$\begin{aligned} &\int_\Omega f(g(\mathbf{u}))d\mu(\mathbf{u}) \\ &\leqslant \frac{M_1 - \int_\Omega (h\circ g)(\mathbf{u})d\mu(\mathbf{u})}{M_1 - m_1}\cdot f\left(h^{-1}(m_1)\right) + \frac{\int_\Omega (h\circ g)(\mathbf{u})d\mu(\mathbf{u}) - m_1}{M_1 - m_1}\cdot f\left(h^{-1}(M_1)\right) \\ &= \frac{M_1 - h(M_h(g;\mu))}{M_1 - m_1}\cdot f\left(h^{-1}(m_1)\right) + \frac{h(M_h(g;\mu)) - m_1}{M_1 - m_1}\cdot f\left(h^{-1}(M_1)\right), \end{aligned}$$

where m_1 and M_1 ($m_1 < M_1$) are, respectively, the minimum and the maximum value of the function $h\circ g$, i.e. $m_1 \leqslant (h\circ g)(\mathbf{u}) \leqslant M_1$, for all $\mathbf{u}$. If we suppose that h is strictly increasing and denote $m_1 = h(m), M_1 = h(M)$, then $m \leqslant g(\mathbf{u}) \leqslant M$ and $m < M$, and we get (2.2). Analogously, supposing that h is strictly decreasing we get the same.

For k concave function, we get the reverse inequalities in (2.1) and (2.2).

REMARK 2.1 For the functions f, g, h defined as in the Theorem 2.1, the inequalities (2.1) and (2.2) (resp. the reverse inequalities) hold if any of the following cases occurs:

(i) f is strictly increasing, h strictly increasing and $h\circ f^{-1}$ concave (convex)

(ii) f is strictly increasing, h strictly decreasing and $h\circ f^{-1}$ convex (concave)

(iii) f is strictly decreasing, h strictly increasing and $h\circ f^{-1}$ convex (concave)

(iv) f is strictly decreasing, h strictly decreasing and $h\circ f^{-1}$ concave (convex).

3. Applications

From the results in the previous section we can derive the results from [7] and many others.

3.1. Integral power means

Let $\Omega \subseteq \mathbb{R}^n$ be a convex set equipped with a probability measure μ. For $r \in \mathbb{R}$ and $g : \Omega \to \mathbb{R}^+$, the integral For power mean is defined as follows:

$$M_r(g;\mu) = \begin{cases} [\int_\Omega (g(\mathbf{u}))^r d\mu(\mathbf{u})]^{\frac{1}{r}}, & r \neq 0 \\ \exp(\int_\Omega \ln(g(\mathbf{u}))d\mu(\mathbf{u})), & r = 0. \end{cases}$$

Now we have

THEOREM 3.1. *Let the functions f,g be defined as in the Theorem 2.1. If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is convex function, or $r = 0$ and $k(x) = f(e^x)$ is convex, then*

$$f(M_r(g;\mu)) \leqslant \int_\Omega f(g(\mathbf{u}))d\mu(\mathbf{u}) \tag{3.1}$$

is valid.

If the function g is also bounded, and its minimum and maximum value, m and M, are not equal, then we also have

$$\int_\Omega f(g(\mathbf{u}))d\mu(\mathbf{u}) \leqslant \begin{cases} \frac{M^r - M_r^r(g;\mu)}{M^r - m^r} \cdot f(m) + \frac{M_r^r(g;\mu) - m^r}{M^r - m^r} \cdot f(M), \text{ for } r \neq 0 \\ \frac{\ln M - \ln M_0(g;\mu)}{\ln M - \ln m} \cdot f(m) + \frac{\ln M_0(g;\mu) - \ln m}{\ln M - \ln m} \cdot f(M), \text{ for } r = 0. \end{cases} \tag{3.2}$$

If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is a concave function, or $r = 0$ and $k(x) = f(e^x)$ is concave, then (3.1) *and* (3.2) *hold with the inequalities reversed.*

Proof. The statement of the Theorem follows directly from the Theorem 2.1 by setting the function h as

$$h(x) = \begin{cases} x^r, r \neq 0 \\ \ln x, r = 0. \end{cases}$$

REMARK 3.1 For the functions f and g, defined as in the Theorem 2.1, the inequalities (3.1) and (3.2) (resp. the reverse inequalities) hold if any of the following cases occurs:

(i) f is strictly increasing, $r > 0$ and $\left(f^{-1}\right)^r$ concave (convex)

(ii) f is strictly increasing, $r < 0$ and $\left(f^{-1}\right)^r$ convex (concave)

(iii) f is strictly decreasing, $r > 0$ and $\left(f^{-1}\right)^r$ convex (concave)

(iv) f is strictly decreasing, $r < 0$ and $\left(f^{-1}\right)^r$ concave (convex)

(v) f is strictly increasing, $r = 0$ and f^{-1} logconcave (logconvex)

(vi) f is strictly decreasing, $r = 0$ and f^{-1} logconvex (logconcave).

REMARK 3.2 We can, naturally, apply these results on different means which can be obtained from previously mentioned means (the integral power means).

3.1.1. Tobey mean

Let E_{n-1} represent the $(n-1)$-dimensional Euclidean simplex given by

$$E_{n-1} = \{(u_1, u_2, \dots, u_{n-1}) : \ u_i \geqslant 0, \ 1 \leqslant i \leqslant n-1, \ \sum_{i=1}^{n-1} u_i \leqslant 1\}$$

and set $u_n = 1 - \sum_{i=1}^{n-1} u_i$. With $\mathbf{u} = (u_1, ..., u_n)$, let $\mu(\mathbf{u})$ be a probability measure on E_{n-1}.

The power mean of order p ($p \in \mathbb{R}$) of the positive n-tuple $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$ with the weights $\mathbf{u} = (u_1, ..., u_n)$, is defined by

$$\bar{M}_p(\mathbf{x}, \mathbf{u}) = \begin{cases} (\sum_{i=1}^{n} u_i x_i^p)^{\frac{1}{p}}, & p \neq 0 \\ \prod_{i=1}^{n} x_i^{u_i}, & p = 0. \end{cases}$$

Then, the Tobey mean $L_{p,r}(\mathbf{x}, \mu)$ is defined by

$$L_{p,r}(\mathbf{x}, \mu) = M_r(\bar{M}_p(\mathbf{x}, \mathbf{u}); \mu),$$

where $M_r(g; \mu)$ denotes the integral power mean in which Ω is the $(n-1)$-dimensional Euclidean simplex E_{n-1}.

The following result is valid:

THEOREM 3.2. *Let I be an interval containing all x_i $(i = 1, ..., n)$ and let $f: I \to \mathbb{R}$ be a real-valued function. If $r \neq 0$ and $k(x) = f\left(x^{\frac{1}{r}}\right)$ is a convex function, or $r = 0$ and $k(x) = f(e^x)$ is convex, then*

$$f(L_{p,r}(\mathbf{x}, \mu)) \leqslant \int_{E_{n-1}} f(\bar{M}_p(\mathbf{x}, \mathbf{u})) d\mu(\mathbf{u}). \tag{3.3}$$

If not all x_i $(i = 1, ..., n)$ are equal then we also have

$$\int_{E_{n-1}} f(\bar{M}_p(\mathbf{x}, \mathbf{u})) d\mu(\mathbf{u}) \leqslant \begin{cases} \frac{M^r - L^r_{p,r}(\mathbf{x},\mu)}{M^r - m^r} \cdot f(m) + \frac{L^r_{p,r}(\mathbf{x},\mu) - m^r}{M^r - m^r} \cdot f(M), \text{ for } r \neq 0 \\ \frac{\ln M - \ln L_{p,0}(\mathbf{x},\mu)}{\ln M - \ln m} \cdot f(m) + \frac{\ln L_{p,0}(\mathbf{x},\mu) - \ln m}{\ln M - \ln m} \cdot f(M), \text{ for } r = 0 \end{cases} \tag{3.4}$$

where m and M are, respectively, the minimum and the maximum value of x_i $(i = 1, ..., n)$.

If $r \neq 0$ *and* $k(x) = f\left(x^{\frac{1}{r}}\right)$ *is a concave function, or* $r = 0$ *and* $k(x) = f(e^x)$ *is concave, then* (3.3) *and* (3.4) *hold with the inequalities reversed.*

Proof. Note that as $\bar{M}_p(\mathbf{x},\mathbf{u})$ is a mean, we have that

$$m \leqslant \bar{M}_p(\mathbf{x},\mathbf{u}) \leqslant M$$

where m and M are, respectively, the minimum and the maximum value of x_i $(i = 1,...,n)$. If not all x_i $(i = 1,...,n)$ are equal, then $m < M$. Now setting the function g as $g(\mathbf{u}) = \bar{M}_p(\mathbf{x},\mathbf{u})$, from (3.1) we get (3.3) and from (3.2) we get (3.4).

REMARK 3.3 For strictly monotone function $f : I \to \mathbb{R}$ on the interval I, the inequalities (3.3) and (3.4) (resp. the reverse inequalities) hold if any of the cases $(i) - (vi)$ from the Remark 3.1 occurs.

3.1.2. Stolarsky-Tobey mean

The Stolarsky-Tobey mean $\mathscr{E}_{p,q}(\mathbf{x},\mu)$ is defined (in [12]) as follows:

$$\mathscr{E}_{p,q}(\mathbf{x},\mu) = \begin{cases} \left[\int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i x_i^p\right)^{\frac{q-p}{p}} d\mu(\mathbf{u})\right]^{\frac{1}{q-p}} & , \quad p(q-p) \neq 0, \\ \exp\left(\int_{E_{n-1}} \ln(\sum_{i=1}^{n} u_i x_i^p)^{\frac{1}{p}} d\mu(\mathbf{u})\right) & , \quad p = q \neq 0, \\ \left[\int_{E_{n-1}} \left(\prod_{i=1}^{n} x_i^{u_i}\right)^{q} d\mu(\mathbf{u})\right]^{\frac{1}{q}} & , \quad p = 0, q \neq 0, \\ \exp\left(\int_{E_{n-1}} \ln(\prod_{i=1}^{n} x_i^{u_i}) d\mu(\mathbf{u})\right) & , \quad p = q = 0, \end{cases} \tag{3.5}$$

or, in other words,

$$\mathscr{E}_{p,q}(\mathbf{x},\mu) = L_{p,q-p}(\mathbf{x},\mu) = M_{q-p}(\bar{M}_p(\mathbf{x},\mathbf{u});\mu),$$

where $L_{p,r}(\mathbf{x},\mu)$ is the Tobey mean.

Therefore, from Theorem 3.2 we have:

THEOREM 3.3. *Let* I *be an interval containing* x_i $(i = 1,...,n)$ *and let* $f : I \to \mathbb{R}$ *be a real-valued function. If* $q - p \neq 0$ *and* $k(x) = f\left(x^{\frac{1}{q-p}}\right)$ *is convex, or* $q - p = 0$ *and* $k(x) = f(e^x)$ *is convex, then*

$$f(\mathscr{E}_{p,q}(\mathbf{x},\mu)) \leqslant \int_{E_{n-1}} f(\bar{M}_p(\mathbf{x},\mathbf{u}))d\mu(\mathbf{u}). \tag{3.6}$$

If not all x_i $(i=1,...,n)$ *are equal then we also have*

$$\int_{E_{n-1}} f(\bar{M}_p(\mathbf{x},\mathbf{u}))d\mu(\mathbf{u}) \leqslant \begin{cases} \frac{M^{q-p}-\mathscr{E}_{p,q}^{q-p}(\mathbf{x},\mu)}{M^{q-p}-m^{q-p}} \cdot f(m) + \frac{\mathscr{E}_{p,q}^{q-p}(\mathbf{x},\mu)-m^{q-p}}{M^{q-p}-m^{q-p}} \cdot f(M), \text{ for } q-p \neq 0 \\ \frac{\ln M - \ln \mathscr{E}_{p,p}(\mathbf{x},\mu)}{\ln M - \ln m} \cdot f(m) + \frac{\ln \mathscr{E}_{p,p}(\mathbf{x},\mu) - \ln m}{\ln M - \ln m} \cdot f(M), \text{ for } q-p=0 \end{cases} \tag{3.7}$$

where m *and* M *are, respectively, the minimum and the maximum value of* x_i $(i=1,...,n)$.

If $q-p \neq 0$ *and* $k(x) = f\left(x^{\frac{1}{q-p}}\right)$ *is concave, or* $q-p=0$ *and* $k(x)=f(e^x)$ *is concave, then* (3.6) *and* (3.7) *hold with the inequalities reversed.*

REMARK 3.4 For strictly monotone function $f: I \to \mathbb{R}$, the inequalities (3.6) and (3.7) (resp. the reverse inequalities) hold if any of the following cases occurs:

(i) f is strictly increasing, $q-p>0$ and $\left(f^{-1}\right)^{q-p}$ concave (convex)

(ii) f is strictly increasing, $q-p<0$ and $\left(f^{-1}\right)^{q-p}$ convex (concave)

(iii) f is strictly decreasing, $q-p>0$ and $\left(f^{-1}\right)^{q-p}$ convex (concave)

(iv) f is strictly decreasing, $q-p<0$ and $\left(f^{-1}\right)^{q-p}$ concave (convex)

(v) f is strictly increasing, $q-p=0$ and f^{-1} logconcave (logconvex)

(vi) f is strictly decreasing, $q-p=0$ and f^{-1} logconvex (logconcave).

From this, as a special case, follows Theorem 1.3 ([7, Theorem 3.1.]) for the Pittenger multidimensional generalized logarithmic mean.

As $L_r(\mathbf{x},\mu) = \mathscr{E}_{\mathbf{1},\mathbf{r+1}}(\mathbf{x},\mu)$, it follows:

THEOREM 3.4. *Let* I *be an interval containing* x_i $(i=1,...,n)$ *and let* $f: I \to \mathbb{R}$ *be a real-valued function. If* $r \neq 0$ *and* $k(x) = f\left(x^{\frac{1}{r}}\right)$ *is convex, or* $r=0$ *and* $k(x) = f(e^x)$ *is convex, then*

$$f(L_r(\mathbf{x},\mu)) \leqslant \int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})d\mu(\mathbf{u}). \tag{3.8}$$

If not all x_i $(i=1,...,n)$ *are equal then we also have*

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})d\mu(\mathbf{u}) \leqslant \begin{cases} \frac{M^r - L_r^r(\mathbf{x},\mu)}{M^r-m^r} \cdot f(m) + \frac{L_r^r(\mathbf{x},\mu)-m^r}{M^r-m^r} \cdot f(M), \text{ for } r \neq 0 \\ \frac{\ln M - \ln L_0(\mathbf{x},\mu)}{\ln M - \ln m} \cdot f(m) + \frac{\ln L_0(\mathbf{x},\mu) - \ln m}{\ln M - \ln m} \cdot f(M), \text{ for } r=0 \end{cases} \tag{3.9}$$

where m *and* M *are, respectively, the minimum and the maximum value of* x_i $(i=1,...,n)$.

If $r \neq 0$ *and* $k(x) = f\left(x^{\frac{1}{r}}\right)$ *is concave, or* $r=0$ *and* $k(x)=f(e^x)$ *is concave, then* (3.8) *and* (3.9) *hold with the inequalities reversed.*

REMARK 3.5 For strictly monotone function $f : I \to \mathbb{R}$ on the interval I, the inequalities (3.8) and (3.9) (resp. the reverse inequalities) hold if any of the cases $(i)-(vi)$ from the Remark 3.1 occurs.

REMARK 3.6 If μ is the probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1\cdots du_{n-1}$, $L_r(\mathbf{x},\mu)$ is equal to the Pittenger multidimensional generalized logarithmic mean $L_r(\mathbf{x})$ defined in the Introduction. So, in this case we have the Theorem 1.3 ([7, Theorem 3.1.]).

REMARK 3.7 In [12] an explicit form of the Stolarsky-Tobey mean in $n-$variables is given (for distinct positive x_i, $i = 1,...n$), when μ is the probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1\cdots du_{n-1}$.

For $p,q \in \mathbb{R}$, $\mathbf{x} = (x_1,...,x_n) \in \mathbb{R}^n_+$ and $x_i \neq x_j$ (for $i \neq j$) we have:

$$(i) \qquad \mathscr{E}_{p,q}(\mathbf{x}) = \left[\frac{(n-1)!p^{n-1}}{q(q+p)...(q+(n-2)p)}\sum_{i=1}^{n}\frac{x_i^{q+(n-2)p}}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(x_i^p-x_j^p)}\right]^{\frac{1}{q-p}},$$

for $p \neq 0,\ q \neq -kp,\ -1 \leqslant k \leqslant n-2$;

$$(ii) \qquad \mathscr{E}_{p,-kp}(\mathbf{x}) = \left[(-1)^k(k+1)\binom{n-1}{k+1}\sum_{i=1}^{n}\frac{x_i^{(n-k-2)p}\ln(x_i)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(x_i^p-x_j^p)}\right]^{-\frac{1}{(k+1)p}},$$

for $p \neq 0,\ 0 \leqslant k \leqslant n-2$;

$$(iii) \qquad \mathscr{E}_{0,q}(\mathbf{x}) = \left[\frac{(n-1)!}{q^{n-1}}\sum_{i=1}^{n}\frac{x_i^q}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\ln(\frac{x_i}{x_j})}\right]^{\frac{1}{q}}, \qquad \text{for } q \neq 0\,;$$

$$(iv) \qquad \mathscr{E}_{p,p}(\mathbf{x}) = \exp\left(\frac{1}{p}\sum_{i=1}^{n}\frac{x_i^{p(n-1)}\left(\ln x_i^p - \sum\limits_{k=1}^{n-1}\frac{1}{k}\right)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}(x_i^p-x_j^p)}\right), \qquad \text{for } p \neq 0;$$

$$(v) \qquad \mathscr{E}_{0,0}(\mathbf{x}) = \left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}.$$

Pittenger in [13] gave the explicit form for $L_r(\mathbf{x})$ only when the index r is a negative integer, $-n \leqslant r \leqslant 0$. Using the fact that $L_r(\mathbf{x}) = \mathscr{E}_{1,r+1}(\mathbf{x})$ we get the explicit form for $L_r(\mathbf{x})$ for all possible r:

For $r \in \mathbb{R}$ and x_i $(i = 1,\dots,n)$ distinct positive real numbers, we have the following:

$$L_r(x_1,\dots,x_n) = \begin{cases} \left((-1)^{-r-1} \cdot (-r) \cdot \binom{n-1}{-r} \cdot \sum\limits_{i=1}^{n} \frac{x_i^{n+r-1} \ln x_i}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j)} \right)^{\frac{1}{r}}, & \text{if } -r \in N,\ -n < r < 0; \\ \exp\left(\sum\limits_{i=1}^{n} \frac{x_i^{n-1} \left(\ln x_i - \sum\limits_{k=1}^{n-1} \frac{1}{k} \right)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j)} \right), & \text{if } r = 0; \\ \left(\frac{(n-1)!}{(r+1)(r+2)\dots(r+n-1)} \sum\limits_{i=1}^{n} \frac{x_i^{r+n-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j)} \right)^{\frac{1}{r}}, & \text{in all other cases.} \end{cases}$$

REMARK 3.8 An extension of the result (1.4) given by Alzer in [1] is the following inequality:

$$f(L(a,b)) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{bf(b) - af(a)}{b-a} - ab\frac{f(b) - f(a)}{b-a} \frac{1}{L(a,b)} \tag{3.10}$$

where $0 < a < b$.

REMARK 3.9 An extension of the result (1.2) given by Seiffert in [14] is the following inequality:

$$f(I(a,b)) \geqslant \frac{1}{b-a} \int_a^b f(x)dx \geqslant \frac{\ln b - \ln I(a,b)}{\ln b - \ln a} \cdot f(a) + \frac{\ln I(a,b) - \ln a}{\ln b - \ln a} \cdot f(b) \tag{3.11}$$

where $0 < a < b$.

3.2. Functional Stolarsky means

For strictly monotone continuous functions h and g, the functional Stolarsky means are defined by ([9]):

$$m_{h,g}(\mathbf{x};\mu) = h^{-1} \left(\int_{E_{n-1}} (h \circ g^{-1})(\mathbf{u} \cdot \mathbf{g}) d\mu(\mathbf{u}) \right)$$

where $\mathbf{g} = (g(x_1),\dots,g(x_n))$ and μ is a probability measure on E_{n-1}.

In the same way as we developed results for the quasi-arithmetic means, we can get analogous results for the functional Stolarsky means using $\Omega = E_{n-1}$.

THEOREM 3.5. *Let I be an interval containing x_i $(i=1,...,n)$, $g,h: I \to \mathbb{R}$ strictly monotone continuous functions and $f: I \to \mathbb{R}$ a real-valued function. If $k(x) = f\left(h^{-1}(x)\right)$ is convex, then*

$$f\left(m_{h,g}(\mathbf{x};\mu)\right) \leqslant \int_{E_{n-1}} f(g^{-1}(\mathbf{u}\cdot\mathbf{g}))d\mu(\mathbf{u}). \tag{3.12}$$

If not all x_i $(i=1,...,n)$ are equal then we also have

$$\int_{E_{n-1}} f(g^{-1}(\mathbf{u}\cdot\mathbf{g}))d\mu(\mathbf{u}) \leqslant \frac{h(M)-h(m_{h,g}(\mathbf{x};\mu))}{h(M)-h(m)}\cdot f(m) + \frac{h(m_{h,g}(\mathbf{x};\mu))-h(m)}{h(M)-h(m)}\cdot f(M), \tag{3.13}$$

where m and M are, respectively, the minimum and the maximum value of x_i, $(i = 1,...,n)$.
If $k(x) = f\left(h^{-1}(x)\right)$ is concave, then (3.12) and (3.12) hold with the inequalities reversed.

Proof. The proof is analogous to that of the Theorem 2.1; we just consider the function $g^{-1}(\mathbf{u}\cdot\mathbf{g})$ instead of the function $g(\mathbf{u})$.

REMARK 3.10 For strictly monotone function $f: I \to \mathbb{R}$ on the interval I, the inequalities (3.12) and (3.13) (resp. the reverse inequalities) hold if any of the cases $(i)-(iv)$ from the Remark 2.1 occurs.

3.3. Symmetric means

3.3.1. Complete symmetric polynomial mean

The $r-$th complete symmetric polynomial mean (or, simply, the complete symmetric mean) of the positive real $n-$tuple $\mathbf{x}$, is defined by ([3])

$$Q_n^{[r]}(\mathbf{x}) = \left(q_n^{[r]}(\mathbf{x})\right)^{\frac{1}{r}} = \left(\frac{c_n^{[r]}(\mathbf{x})}{\binom{n+r-1}{r}}\right)^{\frac{1}{r}},$$

where

$$c_n^{[0]} = 1 \text{ and } c_n^{[r]} = \sum\left(\prod x_i^{i_j}\right)$$

and the sum is taken over all $\binom{n+r-1}{r}$ non-negative integer n-tuples $(i_1,...,i_n)$ with $\sum_{j=1}^n i_j = r,\ r \neq 0$.

The complete symmetric polynomial mean can also be written in an integral form as follows:

$$Q_n^{[r]}(\mathbf{x}) = \left(\int_{E_{n-1}} (\sum_{i=1}^n x_i u_i)^r d\mu(\mathbf{u})\right)^{\frac{1}{r}}$$

where μ represents the probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1\cdots du_{n-1}$.

As we can see, this is a special case of the integral power mean $M_r(g;\mu)$ where $g(\mathbf{u}) = \sum_{i=1}^{n} x_i u_i$, μ is the probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1 \cdots du_{n-1}$ and Ω is the $(n-1)$-dimensional simplex E_{n-1}.

We have the following theorem:

THEOREM 3.6. *Let I be an interval containing all x_i $(i=1,\ldots,n)$ and let $f:I\to\mathbb{R}$ be a real-valued function. If $k(x) = f\left(x^{\frac{1}{r}}\right)$ ($r \neq 0$) is a convex function, then*

$$f\left(Q_n^{[r]}(\mathbf{x})\right) \leqslant \int_{E_{n-1}} f(\sum_{i=1}^{n} x_i u_i) d\mu(\mathbf{u}). \tag{3.14}$$

If not all x_i $(i=1,\ldots,n)$ are equal then we also have

$$\int_{E_{n-1}} f(\sum_{i=1}^{n} x_i u_i) d\mu(\mathbf{u}) \leqslant \frac{M^r - \left(Q_n^{[r]}(\mathbf{x})\right)^r}{M^r - m^r} \cdot f(m) + \frac{\left(Q_n^{[r]}(\mathbf{x})\right)^r - m^r}{M^r - m^r} \cdot f(M) \tag{3.15}$$

where m and M are, respectively, the minimum and the maximum value of x_i $(i = 1,\ldots,n)$.

If $k(x) = f\left(x^{\frac{1}{r}}\right)$, ($r \neq 0$) is a concave function, then (3.14) *and* (3.15) *hold with the inequalities reversed.*

REMARK 3.11 For strictly monotone function $f: I \to \mathbb{R}$ on the interval I, the inequalities (3.14) and (3.15) (resp. the reverse inequalities) hold if any of the cases $(i)-(iv)$ from the Remark 3.1 occurs.

REMARK 3.12 The generalization of the complete symmetric polynomial means are so called the Whiteley means (see [3]).

3.3.2. Whiteley means and the generalization

Let $\mathbf{x}$ be a positive real $n-$tuple, $s \in \mathbb{R}, s \neq 0$ and $r \in \mathbb{N}$. Then the $s-$th function of degree r, the Whiteley symmetric function $t_n^{[r,s]}(\mathbf{x})$, is defined by the following generating function (see [3]):

$$\sum_{r=0}^{\infty} t_n^{[r,s]}(\mathbf{x}) t^r = \begin{cases} \prod_{i=1}^{n}(1+x_i t)^s, & \text{if } s > 0, \\ \prod_{i=1}^{n}(1-x_i t)^s, & \text{if } s < 0. \end{cases}$$

The Whiteley mean is now defined by

$$\mathscr{W}_n^{[r,s]}(\mathbf{x}) = \left(w_n^{[r,s]}(\mathbf{x})\right)^{\frac{1}{r}} = \begin{cases} \left(\frac{t_n^{[r,s]}(\mathbf{x})}{\binom{ns}{r}}\right)^{\frac{1}{r}}, & s > 0, \\ \left(\frac{t_n^{[r,s]}(\mathbf{x})}{(-1)^r\binom{ns}{r}}\right)^{\frac{1}{r}}, & s < 0. \end{cases}$$

REMARK 3.13 If $s<0$ then $(-1)^r\binom{ns}{r}=\binom{-ns+r-1}{r}$.

REMARK 3.14 An alternative definition of $t_n^{[r,s]}(\mathbf{x})$ is given by:

$$t_n^{[r,s]}(\mathbf{x})=\sum\left(\prod_{j=1}^{n}\lambda_{i_j}x_j^{i_j}\right)$$

where

$$\lambda_i=\begin{cases}\binom{s}{i}, & s>0,\\ (-1)^i\binom{s}{i}, & s<0,\end{cases}$$

and the summation is over all non-negative integer n-tuples $(i_1,\dots,i_n)$ with $\sum_{j=1}^{n} i_j=r$.

REMARK 3.15 For $s=-1$ the Whiteley mean becomes the complete symmetric polynomial mean.

For $s<0$ the Whiteley symmetric function can be further generalized if we slightly change its definition and define $h_n^{[r,\sigma]}(\mathbf{x})$ as follows:

$$\sum_{r=0}^{\infty}h_n^{[r,\sigma]}(\mathbf{x})t^r=\prod_{i=1}^{n}\frac{1}{(1-x_it)^{\sigma_i}}$$

where $\sigma=(\sigma_1,\dots,\sigma_n)$, $\sigma_i\in\mathbb{R}_+$ $(i=1,\dots,n)$.

Now the following generalization of the Whiteley mean for $s<0$ is defined by (see [10])

$$\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})=\left(H_n^{[r,\sigma]}(\mathbf{x})\right)^{\frac{1}{r}}=\left(\frac{h_n^{[r,\sigma]}(\mathbf{x})}{\binom{\sum_{i=1}^{n}\sigma_i+r-1}{r}}\right)^{\frac{1}{r}}.$$

REMARK 3.16 The previous definition can be written as

$$\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})=\left(H_n^{[r,\sigma]}(\mathbf{x})\right)^{\frac{1}{r}}=\left(\frac{h_n^{[r,\sigma]}(\mathbf{x})}{(-1)^r\binom{-\sum_{i=1}^{n}\sigma_i}{r}}\right)^{\frac{1}{r}}.$$

REMARK 3.17 If we put

$$\sigma_1=\dots=\sigma_n=-s,\ (s<0)$$

we get the $s-$th function of degree r, (that is, the Whiteley symmetric function $t_n^{[r,s]}(\mathbf{x})$), and the Whiteley mean $\mathscr{W}_n^{[r,s]}(\mathbf{x})$.

REMARK 3.18 In [10] the mean $H_n^{[r,\sigma]}(\mathbf{x})$ is considered, some useful results there were given, including its integral representation.

$\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})$ is normalized as it is $\mathscr{H}_n^{[r,\sigma]}(\mathbf{x}) = x$, when $\mathbf{x} = (x, x, \ldots, x)$.

Further, if we denote with μ the measure on the simplex E_{n-1} such that

$$d\mu(\mathbf{u}) = \frac{\Gamma(\sum_{i=1}^n \sigma_i)}{\prod_{i=1}^n \Gamma(\sigma_i)} \prod_{i=1}^n u_i^{\sigma_i - 1} du_1 \ldots du_{n-1},$$

then we have that μ is a probability measure and we can also write the mean $\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})$ in an integral form as follows

$$\mathscr{H}_n^{[r,\sigma]}(\mathbf{x}) = \left(\int_{E_{n-1}} \left(\sum_{i=1}^n x_i u_i \right)^r d\mu(\mathbf{u}) \right)^{\frac{1}{r}}.$$

Now we can develop the analogous result as we did in the previous section.

THEOREM 3.7. *Let I be an interval containing x_i $(i = 1, \ldots, n)$ and let $f : I \to \mathbb{R}$ be a real-valued function.*

If $k(x) = f\left(x^{\frac{1}{r}}\right)$ ($r \neq 0$) is convex, then

$$f\left(\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})\right) \leqslant \int_{E_{n-1}} f\left(\sum_{i=1}^n x_i u_i \right) d\mu(\mathbf{u}). \tag{3.16}$$

If not all x_i $(i = 1, \ldots, n)$ are equal then we also have

$$\int_{E_{n-1}} f\left(\sum_{i=1}^n x_i u_i \right) d\mu(\mathbf{u}) \leqslant \frac{M^r - \left(\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r}{M^r - m^r} \cdot f(m) + \frac{\left(\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r - m^r}{M^r - m^r} \cdot f(M) \tag{3.17}$$

where m and M are, respectively, the minimum and the maximum value of x_i $(i = 1, \ldots, n)$.

If $k(x) = f\left(x^{\frac{1}{r}}\right)$ ($r \neq 0$) is concave, then (3.16) and (3.17) hold with the inequalities reversed.

Proof. As μ is a probability measure, for the convex function k by the Jensen inequality we get the following:

$$\begin{aligned} f\left[\left(\int_{E_{n-1}} \left(\sum_{i=1}^n x_i u_i \right)^r d\mu(\mathbf{u}) \right)^{\frac{1}{r}} \right] &= k\left(\int_{E_{n-1}} \left(\sum_{i=1}^n x_i u_i \right)^r d\mu(\mathbf{u}) \right) \\ &\leqslant \int_{E_{n-1}} k\left(\left(\sum_{i=1}^n x_i u_i \right)^r \right) d\mu(\mathbf{u}) = \int_{E_{n-1}} f\left(\sum_{i=1}^n x_i u_i \right) d\mu(\mathbf{u}) \end{aligned}$$

and this proves the inequality in (3.16).

For the second inequality we use the same reasoning as we did in the proof of the Theorem 2.1. For the convex function $k(x) = f(x^{\frac{1}{r}})$ (as $\Phi(x)$) and the function $\left(\sum_{i=1}^{n} x_i u_i\right)^r$ (as $g(\mathbf{u})$) we get the following:

$$\int_{E_{n-1}} f\left(\sum_{i=1}^{n} x_i u_i\right) d\mu(\mathbf{u})$$
$$\leqslant \frac{M_1 - \int_{E_{n-1}} \left(\sum_{i=1}^{n} x_i u_i\right)^r d\mu(\mathbf{u})}{M_1 - m_1} \cdot f(m_1^{\frac{1}{r}}) + \frac{\int_{E_{n-1}} \left(\sum_{i=1}^{n} x_i u_i\right)^r d\mu(\mathbf{u}) - m_1}{M_1 - m_1} \cdot f(M_1^{\frac{1}{r}})$$
$$= \frac{M_1 - \left(\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r}{M_1 - m_1} \cdot f(m_1^{\frac{1}{r}}) + \frac{\left(\mathscr{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r - m_1}{M_1 - m_1} \cdot f(M_1^{\frac{1}{r}})$$

where m_1 and M_1 are, respectively, the minimum and the maximum value of the function $\left(\sum_{i=1}^{n} x_i u_i\right)^r$, i.e. $m_1 \leqslant \left(\sum_{i=1}^{n} x_i u_i\right)^r \leqslant M_1$, for all $\mathbf{u}$, and $m_1 < M_1$. If we denote $m_1 = m^r, M_1 = M^r$, then $m \leqslant \sum_{i=1}^{n} x_i u_i \leqslant M$ and $m < M$. Now we get the inequality (3.17) where m and M are, respectively, the minimum and the maximum value of x_i $(i = 1, \ldots, n)$.
For k concave, we get the reverse inequalities.

REMARK 3.19 For strictly monotone function $f : I \to \mathbb{R}$ on the interval I, the inequalities (3.16) and (3.17) (resp. the reverse inequalities) hold if any one of the cases $(i) - (iv)$ from the Remark 3.1 occurs.

3.4. Inequalities for divided differences

In the next theorem we connect our main results, applied on the Pittenger multidimensional logarithmic mean, with the divided differences.

Let now $\mathbf{x} = (x_1, \ldots, x_n)$ be a $n-$tuple of distinct positive real numbers.

THEOREM 3.8. *Let f be a real function such that it has continuous $(n-1)-th$ derivative. If $r \neq 0$ and $k(x) = f^{(n-1)}\left(x^{\frac{1}{r}}\right)$ is a convex function, or $r = 0$ and $k(x) = f^{(n-1)}(e^x)$ is convex, then*

$$f^{(n-1)}(L_r(\mathbf{x})) \leqslant (n-1)! \cdot [x_1, \ldots, x_n] f$$
$$\leqslant \begin{cases} \frac{M^r - L_r^r(\mathbf{x})}{M^r - m^r} \cdot f^{(n-1)}(m) + \frac{L_r^r(\mathbf{x}) - m^r}{M^r - m^r} \cdot f^{(n-1)}(M), \text{ for } r \neq 0 \\ \frac{\ln M - \ln L_0(\mathbf{x})}{\ln M - \ln m} \cdot f^{(n-1)}(m) + \frac{\ln L_0(\mathbf{x}) - \ln m}{\ln M - \ln m} \cdot f^{(n-1)}(M), \text{ for } r = 0 \end{cases} \tag{3.18}$$

where m *and* M *are, respectively, the minimum and the maximum value of* x_i $(i = 1,\ldots,n)$, *and* $[x_1,\ldots,x_n]f$ *represents the* $(n-1)-$*th divided difference of the function* f.

If $r \neq 0$ *and* $k(x) = f^{(n-1)}\left(x^{\frac{1}{r}}\right)$ *is concave, or* $r = 0$ *and* $k(x) = f^{(n-1)}(e^x)$ *is concave, then (*3.18*) holds with the inequalities reversed.*

Proof. The divided difference of the function f in the distinct points $x_1,\ldots,x_n$ can be written in the integral form as:

$$[x_1,\ldots,x_n]f = \int_{E_{n-1}} f^{(n-1)}(\sum_{i=1}^{n} x_i u_i)du_1\ldots du_{n-1}.$$

Now the statement of our Theorem follows immediately from the Theorem 3.4, for the probability measure μ such that $d\mu(\mathbf{u}) = (n-1)!du_1\ldots du_{n-1}$.

REMARK 3.20 For real function f such that its continuous $(n-1)-$th derivative is strictly monotone function, the inequalities (3.18) (resp. the reverse inequalities) hold if any of the following occurs:

(i) $f^{(n-1)}$ is strictly increasing, $r > 0$ and $\left(\left(f^{(n-1)}\right)^{-1}\right)^r$ concave (convex)

(ii) $f^{(n-1)}$ is strictly increasing, $r < 0$ and $\left(\left(f^{(n-1)}\right)^{-1}\right)^r$ convex (concave)

(iii) $f^{(n-1)}$ is strictly decreasing, $r > 0$ and $\left(\left(f^{(n-1)}\right)^{-1}\right)^r$ convex (concave)

(iv) $f^{(n-1)}$ is strictly decreasing, $r < 0$ and $\left(\left(f^{(n-1)}\right)^{-1}\right)^r$ concave (convex)

(v) $f^{(n-1)}$ is strictly increasing, $r = 0$ and $\left(f^{(n-1)}\right)^{-1}$ logconcave (logconvex)

(vi) $f^{(n-1)}$ is strictly decreasing, $r = 0$ and $\left(f^{(n-1)}\right)^{-1}$ logconvex (logconcave).

In [5] the following result is given (with the slightly changed notation):

THEOREM 3.9. *For the function* $f \in C^{n-1}(I)$, *(*I *open interval,* $n \in N$*) with the* $(n-1)-$*th derivative strictly positive, the next inequality is valid*

$$[x_1,\ldots,x_n]f \geqslant \prod_{i=1}^{n}\left(\left[\underbrace{x_i,\ldots,x_i}_{n-times}\right]f\right)^{\frac{1}{n}} \tag{3.19}$$

if the function $c(x) = \left(\frac{1}{f^{(n-1)}(x)}\right)^{\frac{1}{n}}$ *is convex.*

If the function c *is concave, the inequality is reversed.*

Here we give a generalization of this result.

THEOREM 3.10. *Let I and J be open intervals in $\mathbb{R}$, $f \in C^{n-1}(I)$, $H \in C^{n-1}(J)$ such that $\mathscr{R}(f^{(n-1)}) \subseteq \mathscr{R}(H^{(n-1)})$ and let $H^{(n-1)}$ be monotonous function. Define $c : I \to \mathbb{R}$ by $c(x) = \left(H^{(n-1)}\right)^{-1}\left(f^{(n-1)}(x)\right)$. If*

(i) c is convex and $H^{(n-1)}$ increasing, or

(ii) c is concave and $H^{(n-1)}$ decreasing,

then

$$[x_1,\dots,x_n]f \leqslant \left[\left(H^{(n-1)}\right)^{-1}\left(f^{(n-1)}(x_1)\right),\dots,\left(H^{(n-1)}\right)^{-1}\left(f^{(n-1)}(x_n)\right)\right]H.$$

If:

(i) c is concave and $H^{(n-1)}$ increasing, or

(ii) c is convex and $H^{(n-1)}$ decreasing,

then the reverse inequality is valid.

Proof. Suppose c is convex and $H^{(n-1)}$ increasing (case (i)). Then by the Jensen inequality we have:

$$\begin{aligned}
[x_1,\dots,x_n]f &= \int_{E_{n-1}} f^{(n-1)}(\mathbf{x}\cdot\mathbf{u})du_1\dots du_{n-1} \\
&= \int_{E_{n-1}} H^{(n-1)}\left(c\left(\sum_{i=1}^{n} u_i x_i\right)\right)du_1\dots du_{n-1} \\
&\leqslant \int_{E_{n-1}} H^{(n-1)}\left(\sum_{i=1}^{n} u_i c(x_i)\right)du_1\dots du_{n-1} \\
&= [c(x_1),\dots,c(x_n)]H \\
&= \left[\left(H^{(n-1)}\right)^{-1}\left(f^{(n-1)}(x_1)\right),\dots,\left(H^{(n-1)}\right)^{-1}\left(f^{(n-1)}(x_n)\right)\right]H.
\end{aligned}$$

The proof is similar for all other cases.

REMARK 3.21 Consider the function $H(t) = \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{t}$, $t > 0$, $n \in \mathbb{N}$. Then $H^{(n-1)}(t) = \frac{1}{t^n}$ is decreasing function. Now applying the previous theorem and the simple fact that in this case $[x_1,\dots,x_n]H = \frac{1}{(n-1)!}\frac{1}{x_1\cdots x_n}$, we get the quoted result (3.19) from [5]. Moreover, setting $f(x) = \exp x$ we get the result from [4] (Appendix 1):

$$[x_1,\dots,x_n]\exp \geqslant \frac{1}{(n-1)!}\exp\left(\frac{x_1+\dots+x_n}{n}\right).$$

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