Rad HAZU
Volume 503 (2009), 87–106

# ON SOME INEQUALITIES OF JENSEN–MCSHANE'S TYPE ON RECTANGLE AND APPLICATIONS

V. ČULJAK, B. IVANKOVIĆ AND J. PEČARIĆ

*Abstract.* In this paper we provide the sequence of inequalities which include McShane's generalization of Jensen's inequality for normalized isotonic positive linear functional and convex (concave) function defined on a rectangle. As applications, for different choice of functionals $F$, we present the refinements of the recent results: Diaz-Metcalf's type inequality for bounded random variables in [2], Feyér's and Lupaş type inequality for a function of two variables, and Petrović's type inequality for two nonnegative real n-tuples in [6].

## 1. Introduction

Let $\Omega$ be a nonempty set and $L$ be a linear class of real-valued functions $f:\Omega\to\mathbb{R}$, having the properties:
L1: $f,g\in L\Rightarrow(\alpha f+\beta g)\in L$, for all $\alpha,\beta\in\mathbb{R}$;
L2: $1\in L, i.e.$, if $f(t)=1$ for all $t\in\Omega$, then $f\in L$.

In this paper we consider normalized isotonic positive linear functional $F:L\to\mathbb{R}$, that is, we assume that
A1: $F(\alpha f+\beta g)=\alpha F(f)+\beta F(g)$ for $f,g\in L,\alpha,\beta\in\mathbb{R}$ (linearity);
A2: $f\in L,\ f(t)\geqslant 0$ on $\Omega\Rightarrow F(f)\geqslant 0$ (positive isotonicity);
A3: $F(1)=1$.

For instance, some normalized linear positive functionals are:

$$F(f)=\frac{1}{\nu(\Omega)}\int_\Omega f d\nu,\quad \text{for positive measure } \nu \text{ on } \Omega;$$

$$F(f)=\frac{1}{\sum_{k\in\Omega}p_k}\sum_{k\in\Omega}f_kp_k,\quad \text{for discrete measure on } \Omega=\{1,2,..\},$$

where $p_k\geqslant 0$, $0<\sum_{k\in\Omega}p_k<\infty$.

Jensen's inequality for convex functions is one of the most important inequalities in mathematics and statistics. There are many forms of this famous inequality (discrete form, integral form, etc.). We will use McShane's generalizations of the Jensen's inequality (see [7], [8, p.48-49]).

*Mathematics subject classification* (2000): 26D15.
*Keywords and phrases*: Jensen's inequality, isotonic functional, convex function, Diaz-Metcalf inequality, Fejér inequality, Lupaş inequality, Petrović inequality.
(Accepted January 29, 2008)

THEOREM A 1. *(McShane) Let $\varphi$ be a continuous convex function on a closed convex set $K$ in $\mathbb{R}^n$ and $F$ be a normalized isotonic positive linear functional on $L$. Let $g_i$ be a function in $L, i = 1,\dots,n$, such that $(g_1(t),\dots,g_n(t))$ is in $K$ for all $t \in \Omega$ and the components of $\varphi(g_i)$ are in the class $L$. Then $(F(g_1),\dots,F(g_n))$ is in $K$, $\varphi(F(g_1),\dots,F(g_n))$ is defined and*

$$\varphi(F(g_1),\dots,F(g_n)) \leqslant F(\varphi(g_1),\dots,\varphi(g_n)). \tag{1}$$

*If $\varphi$ is a continuous concave function then the reverse inequality holds.*

Note that Raşa in [10] pointed out that $\varphi$ have to be continuous.

In this paper we provide an extension of this McShane's inequality for $\varphi$ a convex (concave) function defined on rectangle $D = [a,A] \times [b,B]$ and functions $g_1, g_2 \in L$ such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$. This extension gives an upper bound for $F(\varphi(g_1),\dots,\varphi(g_n))$ which includes linear functions $\pi_k : D \to \mathbb{R}$, $k = 1,\dots,4$, determined with vertices $(a,b,\varphi(a,b)),\dots,(A,B,\varphi(A,B))$.

By setting conditions $\Delta\varphi \geqslant 0$ or $\Delta\varphi \leqslant 0$ in the notation

$$\Delta\varphi = \varphi(a,b) - \varphi(a,B) - \varphi(A,b) + \varphi(A,B), \tag{2}$$

we obtain the refinements of the main result.

The methods we use are elementary and are based on the properties of the functionals $F$.

As applications, for different choice of functionals $F$, we present, in the Sections 4.–7., the improvements of the recent results: Diaz-Metcalf's type inequality for mathematical expectation in [2], Feyér's and Lupaş type inequality for a function of two variables and Petrovi'c type inequality for two $n$-tuples of nonnegative real numbers in [6].

## 2. Preliminaries

Notation will be our first issue for clarifications purposes. We are observing rectangle $D = [a,A] \times [b,B]$ separated into triangles in the two different ways.

(i) $D = \Delta_1 \cup \Delta_2$, where $\Delta_1$ is a triangle with vertices $(a,b),(A,b)$ and $(a,B)$, and $\Delta_2 = \Delta((A,B),(a,B),(A,b))$. Note that

$$\Delta_1 \cap \Delta_2 = \{(x,y) : (A-a)y + (B-b)x - AB + ab = 0\}$$
$$(x,y) \in \Delta_1 \Leftrightarrow (A-a)y + (B-b)x - AB + ab \leqslant 0 \tag{3}$$
$$(x,y) \in \Delta_2 \Leftrightarrow (A-a)y + (B-b)x - AB + ab \geqslant 0 \tag{4}$$

(ii) $D = \Delta_3 \cup \Delta_4$ where $\Delta_3$ is a triangle determined with vertices $(a,b),(A,B)$ and $(a,B)$, while the $\Delta_4$ is determined with $(a,b),(A,B)$ and $(A,b)$. Note that:

$$\Delta_3 \cap \Delta_4 = \{(x,y) : (A-a)y - (B-b)x - Ab + aB = 0\}$$
$$(x,y) \in \Delta_3 \Leftrightarrow (A-a)y - (B-b)x - Ab + aB \geqslant 0 \tag{5}$$
$$(x,y) \in \Delta_4 \Leftrightarrow (A-a)y - (B-b)x - Ab + aB \leqslant 0 \tag{6}$$

Let us denote $T_1(a,b,\varphi(a,b))$, $T_2(A,B,\varphi(A,B))$, $T_3(a,B,\varphi(a,b))$ and $T_4(A,b,\varphi(A,B))$ and the planes $\Pi_k$ determined by vertices as follows: $\Pi_1(T_1,T_3,T_4)$, $\Pi_2(T_2,T_3,T_4)$, $\Pi_3(T_3,T_1,T_2)$ and $\Pi_4(T_4,T_1,T_2)$. These planes are the graphs of affine functions $\pi_k : D \to \mathbb{R}$, $k = 1,...,4$:

$$\pi_k(x,y) = \lambda_k x + \mu_k y + \nu_k, \quad k = \{1,2,3,4\} \tag{7}$$

with the coefficients:

$$\begin{aligned} \lambda_1 = \lambda_4 &= \frac{\varphi(A,b)-\varphi(a,b)}{A-a}; \;\; \mu_1 = \mu_3 = \frac{\varphi(a,B)-\varphi(a,b)}{B-b}; \\ \lambda_2 = \lambda_3 &= \frac{\varphi(A,B)-\varphi(a,B)}{A-a}; \;\; \mu_2 = \mu_4 = \frac{\varphi(A,B)-\varphi(A,b)}{B-b}; \\ \nu_1 &= \varphi(a,b) - \lambda_1 a - \mu_1 b; \;\; \nu_2 = \varphi(A,B) - \lambda_2 A - \mu_2 B \\ \nu_3 &= \varphi(a,B) - \lambda_3 a - \mu_3 B; \;\; \nu_4 = \varphi(A,b) - \lambda_4 A - \mu_4 b. \end{aligned} \tag{8}$$

In this geometrical setting, a condition $\Delta\varphi > 0$ means that the edge $T_3T_4$ lies below the edge $T_1T_2$.

Let $M_{ij}, m_{ij} : D \to \mathbb{R}$ denote functions defined by

$$M_{ij}(x,y) = \max\{\pi_i(x,y), \pi_j(x,y)\} \text{ and } m_{ij}(x,y) = \min\{\pi_i(x,y), \pi_j(x,y)\}. \tag{9}$$

Note that compositions of functions $M_{ij}(g_1,g_2) : \Omega \to \mathbb{R}$ and $m_{ij}(g_1,g_2) : \Omega \to \mathbb{R}$ are well defined for $g_1, g_2 \in L$ such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$ by

$$\begin{aligned} M_{ij}(g_1,g_2)(t) &= M_{ij}(g_1(t),g_2(t)) = \max\{\pi_i(g_1(t),g_2(t)), \pi_j(g_2(t),g_2(t))\}, \\ m_{ij}(g_1,g_2)(t) &= M_{ij}(g_1(t),g_2(t)) = \min\{\pi_i(g_1(t),g_2(t)), \pi_j(g_2(t),g_2(t))\}. \end{aligned}$$

We introduce functions:

$$\pi_{12}(x,y) = \begin{cases} \pi_1(x,y) & (x,y) \in \Delta_1 \\ \pi_2(x,y) & (x,y) \in \Delta_2 \end{cases} \text{ and } \pi_{34}(x,y) = \begin{cases} \pi_3(x,y) & (x,y) \in \Delta_3 \\ \pi_4(x,y) & (x,y) \in \Delta_4. \end{cases} \tag{10}$$

Note that functions $\pi_{ij}(g_1,g_2) : \Omega \to \mathbb{R}, i = 1,3, j = 2,4$ are also well defined for $g_1, g_2 \in L$ such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$.

The following Lemma is a consequence of previously presented relations.

LEMMA 1. *Let $M_{ij}, m_{ij}$ and $\pi_{ij}$ be functions defined in* (9) *and* (10). *For function $\varphi : D \to \mathbb{R}$ and $\Delta\varphi$ defined by* (2) *we have*

*(i) if $\Delta\varphi \geqslant 0$, then for all $(x,y) \in D$*

$$\pi_{12}(x,y) \leqslant \pi_{34}(x,y), \tag{11}$$

*and*

$$\pi_{12}(x,y) = M_{12}(x,y) \text{ and } \pi_{34}(x,y) = m_{34}(x,y); \tag{12}$$

*(ii) if* $\Delta\varphi \leqslant 0$*, then for all* $(x,y) \in D$

$$\pi_{12}(x,y) \geqslant \pi_{34}(x,y), \tag{13}$$

*and*

$$\pi_{12}(x,y) = m_{12}(x,y) \text{ and } \pi_{34}(x,y) = M_{34}(x,y). \tag{14}$$

*Proof.* Using elementary algebra, we can obtain some convenient formulas. Namely, in the term of $\Delta\varphi$ there exists relations:

$$\pi_2(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{(A-a)y + (B-b)x - AB + ab}{(B-b)(A-a)} \tag{15}$$

$$\pi_4(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{y-b}{B-b} \tag{16}$$

$$\pi_3(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{x-a}{A-a} \tag{17}$$

$$\pi_3(x,y) - \pi_2(x,y) = \Delta\varphi \cdot \frac{B-y}{B-b} \tag{18}$$

$$\pi_4(x,y) - \pi_2(x,y) = \Delta\varphi \cdot \frac{A-x}{A-a} \tag{19}$$

$$\pi_4(x,y) - \pi_3(x,y) = \Delta\varphi \cdot \frac{(A-a)y - (B-b)x - Ab + aB}{(B-b)(A-a)} \tag{20}$$

According (16), (17), (18) and (19) for all $(x,y)$ in $D$ we have

$$\pi_j(x,y) - \pi_i(x,y) \geqslant 0 \text{ for } j \in \{3,4\}, i \in \{1,2\},$$

and (11) holds by (10).

To prove (12), we check that for $(x,y) \in \Delta_1$, (3) and (15) entail $\pi_1 \geqslant \pi_2$ and $M_{12}(x,y) = \pi_1(x,y) = \pi_{12}(x,y)$. If $(x,y) \in \Delta_2$, then (4) and (15) give that $\pi_1 \leqslant \pi_2$, so $M_{12}(x,y) = \pi_2(x,y) = \pi_{12}(x,y)$ . Furthermore, we note that for $(x,y) \in \Delta_3$, (5) and (20) entails $\pi_4 \geqslant \pi_3$ and $m_{34}(x,y) = \pi_3(x,y) = \pi_{34}(x,y)$. Finally, for $(x,y) \in \Delta_4$, (6) and (20) ensure that $\pi_4 \leqslant \pi_3$, so $m_{34}(x,y) = \pi_4(x,y) = \pi_{34}(x,y)$ according definitions (10), as before.

Similarly we can prove *(ii)*.

## 3. Main results

Now, we state the basic result of this paper.

THEOREM 3.1. *Let* $F : L \to \mathbb{R}$ *be a normalized isotonic positive linear functional, where* $L$ *is a linear space of real-valued functions defined on a nonempty set* $\Omega$*. Moreover, let* $g_1, g_2 \in L$ *be functions such that* $g_1(t) \in [a,A], g_2(t) \in [b,B]$ *for all* $t \in \Omega$ *and* $\pi_{12}, \pi_{34}$ *be functions defined by (*10*).*

*If* $\varphi : D \to \mathbb{R}$ *is continuous concave function then*

$$\begin{aligned} &\max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\leqslant F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &\leqslant F(\varphi(g_1,g_2)) \leqslant \varphi(F(g_1),F(g_2)), \end{aligned} \tag{21}$$

*and if* $\varphi : D \to \mathbb{R}$ *is a continuous convex function then*

$$\begin{aligned} &\min\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\geqslant F(\min\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &\geqslant F(\varphi(g_1,g_2)) \geqslant \varphi(F(g_1),F(g_2)). \end{aligned} \tag{22}$$

*Proof.* Note that from the property A1 we can obtain
A1': $F(l(g_1,g_2,...,g_n)) = l(F(g_1),F(g_2),...,F(g_n))$ for every function $l(z_1,...,z_n)$ linear on $\mathbb{R}^n$.

So, for linear functions $\pi_i$ defined by(7) we conclude that

$$F(\pi_i(g_1,g_2)) = \pi_i(F(g_1,g_2)), \quad i = 1,..,4,$$

and $F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\})$ is well defined. A conclusion of McShane's theorem ensure that $(F(g_1),F(g_2)) \in D$.

The first inequality follows by properties of functional $F$ as follows:

$$\begin{aligned} \pi_{12}(g_1,g_2) &\leqslant \max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}, \\ \pi_{34}(g_1,g_2) &\leqslant \max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}, \\ F(\pi_{12}(g_1,g_2)) &\leqslant F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}), \\ F(\pi_{34}(g_1,g_2)) &\leqslant F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}). \end{aligned}$$

Concavity of $\varphi : D \to \mathbb{R}$ provides that for all $t \in \Omega$ :

$$(g_1(t),g_2(t)) \in \Delta_i \Rightarrow \pi_i(g_1(t),g_2(t)) \leqslant \varphi(g_1(t),g_2(t)), \text{ for } i = 1,..,4. \tag{23}$$

In the case that $\varphi$ is a convex function, the inequalities are opposite.

Using inequalities (23) and (10) we achieve that

$$\pi_{12}(g_1,g_2) \leqslant \varphi(g_1,g_2) \text{ and } \pi_{34}(g_1,g_2) \leqslant \varphi(g_1,g_2)$$

hold for concave functions $\varphi$, for all $t \in \Omega$. Hence we have

$$\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\} \leqslant \varphi(g_1,g_2).$$

Applying isotonic normalized positive linear functional $F$ we obtain the second inequality in (21).

The third inequality in (21) is the well-known Jensen's inequality which was modified by Jessen and generalized by McShane (1). To prove (22), note that if $\varphi$ is convex, then $-\varphi$ is the concave function.

By setting condition $\Delta\varphi \geqslant (\leqslant) 0$, Theorem 3.1 can be generalized as follows:

THEOREM 3.2. *Let $F : L \to \mathbb{R}$ be a normalized isotonic positive linear functional, where $L$ is a linear space of real-valued functions defined on a nonempty set $\Omega$. Moreover, let $g_1, g_2 \in L$ be functions such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$ and $\pi_{12}, \pi_{34}$ be functions defined by (10).*

*(i) Suppose that $\varphi : D \to \mathbb{R}$ is a continuous and concave function.*

*($i_1$) If $\Delta\varphi \geqslant 0$, then*

$$\begin{aligned} &M_{12}(F(g_1),F(g_2)) \leqslant F(M_{12}(g_1,g_2)) \\ &\leqslant \max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\leqslant F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(m_{34}(g_1,g_2)) \leqslant F(\varphi(g_1,g_2)) \leqslant \varphi(F(g_1),F(g_2)). \end{aligned} \tag{24}$$

*($i_2$) If $\Delta\varphi \leqslant 0$, then*

$$\begin{aligned} &M_{34}(F(g_1),F(g_2)) \leqslant F(M_{34}(g_1,g_2)) \\ &\leqslant \max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\leqslant F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(m_{12}(g_1,g_2)) \leqslant F(\varphi(g_1,g_2)) \leqslant \varphi(F(g_1),F(g_2)). \end{aligned} \tag{25}$$

*(ii) Suppose that $\varphi : D \to \mathbb{R}$ is a continuous and convex function.*

*($ii_1$) If $\Delta\varphi \leqslant 0$, then*

$$\begin{aligned} &m_{12}(F(g_1),F(g_2)) \geqslant F(m_{12}(g_1,g_2)) \\ &\geqslant \min\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\geqslant F(\min\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(M_{34}(g_1,g_2)) \geqslant F(\varphi(g_1,g_2)) \geqslant \varphi(F(g_1),F(g_2)). \end{aligned}$$

*($ii_2$) If $\Delta\varphi \geqslant 0$, then*

$$\begin{aligned} &m_{34}(F(g_1),F(g_2)) \geqslant F(m_{34}(g_1,g_2)) \\ &\geqslant \min\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\geqslant F(\min\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(M_{12}(g_1,g_2)) \geqslant F(\varphi(g_1,g_2)) \geqslant \varphi(F(g_1),F(g_2)). \end{aligned}$$

*Proof.* $(i)$ First, we consider a concave function $\varphi : D \to \mathbb{R}$.

$(i_1)$ Since $\pi_1(g_1,g_2) \leqslant M_{12}(g_1,g_2)$ and $\pi_2(g_1,g_2) \leqslant M_{12}(g_1,g_2)$, properties of functional $F$ ensure that $F(\pi_1(g_1,g_2)) = \pi_1(F(g_1),F(g_2)) \leqslant F(M_{12}(g_1,g_2))$ and $F(\pi_2(g_1,g_2)) = \pi_2(F(g_1),F(g_2)) \leqslant F(M_{12}(g_1,g_2))$, so the first inequality in (24) states. Since $\varphi$ is a concave function with $\Delta\varphi \geqslant 0$, the second, fourth and fifth inequalities in (24) are consequence of (11) and (12) in Lemma 1. The third inequality and the last one are rewritten from (21).

$(i_2)$ If we assume that $\varphi$ is a concave function with $\Delta\varphi \leqslant 0$, first inequality in (25) is consequence of isotonicity. The second, fourth and fifth inequalities in (25) are consequence of (13) and (14) in Lemma1. The third inequality and the last one are rewritten from (21).

(ii) Similarly we can prove *( $ii_1$ )* and *( $ii_2$ ).*

## 4. Diaz-Metcalf type inequality

The Diaz-Metcalf inequality in the probabilistic setting (see [3], [9]) inspired authors in [2] to prove the following generalization (in our notation):

THEOREM A 2. *Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a concave function. Let $(X,Y)$ be a random vector, $P[(X,Y) \in D] = 1$ and $E[X], E[Y]$ be the expectations of random variables $X$ and $Y$ with respect to probability $P$.*

*If $\Delta\varphi \geqslant 0$, then*

$$M_{12}(E[X],E[Y]) \leqslant E[\varphi(X,Y)] \leqslant \varphi(E[X],E[Y])$$

*holds and if $\Delta\varphi \leqslant 0$, then*

$$M_{34}(E[X],E[Y]) \leqslant E[\varphi(X,Y)] \leqslant \varphi(E[X],E[Y]),$$

*holds, where $M_{12}$ and $M_{34}$ are defined by (9).*

As an application of Theorem 3.2 for mathematical expectations and bounded random variables $X,Y : \Omega \to \mathbb{R}$ we obtain the following refinement of the Theorem A2.

THEOREM 4.1. *Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a concave function. Let $(X,Y)$ be a random vector with $P[(X,Y) \in D] = 1$ and $E[X], E[Y]$ be the expectations of random variables $X$ and $Y$ with respect to probability $P$.*

*If $\Delta\varphi \geqslant 0$, then*

$$\begin{aligned} &M_{12}(E[X],E[Y]) \leqslant E(M_{12}(X,Y)) \\ &\leqslant \max\{E[\pi_{12}(X,Y)], E[\pi_{34}(X,Y)]\} \leqslant E[\max\{\pi_{12}(X,Y), \pi_{34}(X,Y)\}] \\ &= E[m_{34}(X,Y)] \leqslant E[\varphi(X,Y)] \leqslant \varphi(E[X],E[Y]), \end{aligned}$$

*and if $\Delta\varphi \leqslant 0$, then*

$$\begin{aligned} &M_{34}(E[X],E[Y]) \leqslant E[M_{34}(X,Y)] \\ &\leqslant \max\{E[\pi_{12}(X,Y)], E[\pi_{34}(f,g)]\} \leqslant E[\max\{\pi_{12}(X,Y), \pi_{34}(X,Y)\}] \\ &= E[m_{12}(X,Y)] \leqslant E[\varphi(X,Y)] \leqslant \varphi(E[X],E[Y]). \end{aligned}$$

## 5. Hadamard's and Fejér's inequalities

In the [6] authors obtained the following Corollary considering the extension of the weighted version of Hadamard's inequality by Fejér for the functions of two-variables defined on a rectangle (see [5], [8, p.138]).

COROLLARY F 1. *Let* $w : D = [a,A] \times [b,B] \to \mathbb{R}$ *be a nonnegative integrable function such that* $w(s,t) = u(s)v(t)$, *where* $u : [a,A] \to \mathbb{R}$ *is an integrable function such that* $\int_a^A u(s)ds = 1$, $u(s) = u(a+A-s)$, *for all* $s \in [a,A]$, *and* $v : [b,B] \to \mathbb{R}$ *is an integrable function such that* $\int_b^B v(t)dt = 1$, $v(t) = v(b+B-t)$,*for all* $s \in [a,A]$. *If* $\varphi : D \to \mathbb{R}$ *is a concave function, then*

$$\frac{\varphi(A,b)+\varphi(a,B)}{2} \leqslant \int_D w(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} \leqslant \varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right).$$

As application of Theorem 3.1 for a functional defined as weighted integral over the rectangle $D$, we obtain a refinement of Feyér's inequalities calculated by $|\Delta\varphi|$.

THEOREM 5.1. *Let* $w : D = [a,A] \times [b,B] \to \mathbb{R}$ *be a nonnegative integrable function such that* $w(s,t) = u(s)v(t)$, *where* $u : [a,A] \to \mathbb{R}$ *is an integrable function,* $\int_a^A u(s)ds = 1$, $u(s) = u(a+A-s)$, *for all* $s \in [a,A]$, *and* $v : [b,B] \to \mathbb{R}$ *is an integrable function,* $\int_b^B v(t)dt = 1$, $v(t) = v(b+B-t)$, *for all* $t \in [b,B]$. *If* $\varphi : D \to \mathbb{R}$ *is a continuous concave function, then*

$$\begin{aligned}&\max\{\frac{\varphi(a,b)+\varphi(A,B)}{2},\frac{\varphi(A,b)+\varphi(a,B)}{2}\} - O(|\Delta\varphi|)\\ &\leqslant \int_D w(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \leqslant \varphi(\frac{a+A}{2},\frac{b+B}{2})\end{aligned} \tag{26}$$

*If* $\varphi : D \to \mathbb{R}$ *is a continuous convex function, then*

$$\begin{aligned}&\varphi(\frac{a+A}{2},\frac{b+B}{2}) \leqslant \int_D w(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\\ &\leqslant \min\{\frac{\varphi(a,b)+\varphi(A,B)}{2},\frac{\varphi(A,b)+\varphi(a,B)}{2}\} + O(|\Delta\varphi|),\end{aligned} \tag{27}$$

*where*

$$\begin{aligned}O(|\Delta\varphi|) = \frac{|\Delta\varphi|}{2}\Bigg[&\frac{1}{A-a}\int_a^A su(s)(\int_{b+\frac{B-b}{A-a}(s-a)}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt)ds\\ &+\frac{1}{B-b}\int_b^B tv(t)(\int_{a+\frac{A-a}{B-b}(t-b)}^{A-\frac{A-a}{B-b}(t-b)} u(s)ds)dt\Bigg].\end{aligned}$$

*Proof.* Suppose that $\varphi : D \to \mathbb{R}$ is a concave function. We apply Theorem 3.1 and the Lemma 1 for a functional $F$ defined on $L$, the class of integrable real functions on $\Omega = D$ by $F(f) = \int_D w(\mathbf{x})f(\mathbf{x})d\mathbf{x}$, for $f : \Omega = D \to \mathbb{R}$. Let $g_1, g_1 \in L$ be such that

$g_1(s,t)=s, g_2(s,t)=t,$ for all $(s,t)$ in $\Omega=D$. Using the properties of functions $w,u$ and $v$, we may check that $F(g_1)=\frac{a+A}{2}$ as follows:

$$\begin{aligned}
F(g_1) &= \int_D w(\mathbf{x})g_1(\mathbf{x})d\mathbf{x} \\
&= \int_a^A \int_b^B w(s,t)sdsdt \\
&= \int_a^A su(s)(\int_b^B v(t)dt)ds \\
&= \int_a^{\frac{a+A}{2}} su(s)ds + \int_{\frac{a+A}{2}}^A su(a+A-s)ds \\
&= \text{use the substitution } a+A-s=x \\
&= \int_a^{\frac{a+A}{2}} su(s)ds + \int_a^{\frac{a+A}{2}} (a+A-x)u(x)dx \\
&= (a+A)\int_a^{\frac{a+A}{2}} u(x)dx \\
&= \frac{a+A}{2}.
\end{aligned}$$

Similarly, we can show that $F(g_2)=\frac{b+B}{2}$. If $\Delta\varphi \geqslant 0,$ then we calculate $F(m_{34}(g_1,g_2))$ using a fact $\min\{a,b\}=\dfrac{a+b-|a-b|}{2}$:

$$\begin{aligned}
&F(m_{34}(g_1,g_2)) = \\
&= \frac{1}{2}\int_D [(\lambda_3+\lambda_4)s+(\mu_3+\mu_4)t+\nu_3+\nu_4]u(s)v(t)dsdt \\
&\quad -\frac{1}{2}\int_D |(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4|u(s)v(t)dsdt \\
&= \frac{\varphi(a,b)+\varphi(A,B)}{2} - O(\Delta\varphi) \\
&= \frac{\varphi(a,b)+\varphi(A,B)}{2} - O(|\Delta\varphi|)
\end{aligned}$$

where the final expression for $O(\Delta\varphi)$ we obtain by elementary calculus as fellows:

$$O(\Delta\varphi)=\frac{1}{2}\int_D |(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4|u(s)v(t)dsdt.$$

Since $\Delta\varphi \geqslant 0$, $(s,t) \in \Delta_3$ implies $(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4 \geqslant 0$, and $(s,t) \in \Delta_4$ implies $(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4 \leqslant 0$

$$\begin{aligned}
O(\Delta\varphi) &= \frac{1}{2}\int_{\Delta_3}[(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4]u(s)v(t)dsdt \\
&\quad -\frac{1}{2}\int_{\Delta_4}[(\lambda_3-\lambda_4)s+(\mu_3-\mu_4)t+\nu_3-\nu_4]u(s)v(t)dsdt \\
&= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_{\Delta_3} su(s)v(t)dsdt-\int_{\Delta_4} su(s)v(t)dsdt] \\
&\quad -\frac{1}{(B-b)}[\int_{\Delta_3} tu(s)v(t)dsdt-\int_{\Delta_4} tu(s)v(t)dsdt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_{\Delta_3} u(s)v(t)dsdt-\int_{\Delta_4} u(s)v(t)dsdt)]\} \\
&= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_a^A su(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt)ds-\int_a^A su(s)(\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt)ds] \\
&\quad -\frac{1}{(B-b)}[\int_b^B tv(t)(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds)dt-\int_b^B tv(t)(\int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(s)ds)dt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_a^A u(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt)ds \\
&\qquad\qquad -\int_a^A u(s)(\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt)ds]\}. \\
&= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_a^A su(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt-\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt)ds] \\
&\quad -\frac{1}{(B-b)}[\int_b^B tv(t)(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds-\int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(s)ds)dt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_a^A u(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt-\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt)ds]\}.
\end{aligned}$$

Since $u(s)=u(a+A-s)$, for all $s\in[a,A]$, and $v(t)=v(b+B-t)$, for all $t\in[b,B]$, we have

$$\begin{aligned}
O(\Delta\varphi) &= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_a^A su(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt-\int_b^{\frac{B-b}{A-a}(s-a)+b} v(b+B-t)dt)ds] \\
&\quad -\frac{1}{(B-b)}[\int_b^B tv(t)(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds+\int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(a+A-s)ds)dt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_a^A u(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt-\int_b^{\frac{B-b}{A-a}(s-a)+b} v(b+B-t)dt)ds]\};
\end{aligned}$$

Use the substitution $a+A-s=x,\ b+B-t=y$;

$$\begin{aligned}
O(\Delta\varphi) &= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_a^A su(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt+\int_B^{B-\frac{B-b}{A-a}(s-a)} v(y)dy)ds] \\
&\quad -\frac{1}{(B-b)}[\int_b^B tv(t)(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds+\int_{A-\frac{A-a}{B-b}(t-b)}^{a} u(x)dx)dt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_a^A u(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt+\int_B^{B-\frac{B-b}{A-a}(s-a)} v(y)dy)ds]. \\
&= \frac{\Delta\varphi}{2}\{\frac{1}{(A-a)}[\int_a^A su(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt)ds] \\
&\quad +\frac{1}{(B-b)}[\int_b^B tv(t)(\int_{\frac{A-a}{B-b}(t-b)+a}^{A-\frac{A-a}{B-b}(t-b)} u(s)ds)dt] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}[\int_a^A u(s)(\int_{\frac{B-b}{A-a}(s-a)+b}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt)ds]\}.
\end{aligned}$$

It ease to see that third integral equals zero.

For $\Delta\varphi \leqslant 0$ we have to calculate $F(m_{12}(g_1,g_2))$:

$$\begin{aligned}
&F(m_{12}(g_1,g_2)) = \\
&= \frac{1}{2}\int_D [(\lambda_1+\lambda_2)s+(\mu_1+\mu_2)t+\nu_1+\nu_2]u(s)v(t)dsdt- \\
&-\frac{1}{2}\int_D |(\lambda_1-\lambda_2)s+(\mu_1-\mu_2)t+\nu_1-\nu_2|u(s)v(t)dsdt \\
&= \frac{\varphi(a,B)+\varphi(A,b)}{2}+O(\Delta\varphi) \\
&= \frac{\varphi(a,B)+\varphi(A,b)}{2}-O(|\Delta\varphi|)
\end{aligned}$$

Now, the inequalities in (21)

$$\begin{aligned}
&\max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\
&\leqslant F(\varphi(g_1,g_2)) \leqslant \varphi(F(g_1),F(g_2)),
\end{aligned}$$

Lemma 1, relations (12) and (14) imply (26).

In the case $\varphi$ is a convex one, note that $-\varphi$ is the concave and use the previous proof. The article $O(|\Delta\varphi|)$ is a consequence of (22) together with (14) and the fact $\max\{a,b\}=\dfrac{a+b+|a-b|}{2}$.

Special choice of $u,v$ in Theorem 5.1 gives refinement of Hadamard's inequalities for a concave and convex function of two variables obtained in the [6].

COROLLARY 5.1. *Suppose $\varphi : D=[a,A]\times[b,B]\to\mathbb{R}$ is a continuous concave function.*

*(i) If* $\Delta\varphi \geqslant 0$*, then*

$$\begin{aligned} &\frac{2\varphi(a,b)+2\varphi(A,B)+\varphi(a,B)+\varphi(A,b)}{6} \\ \leqslant &\frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \\ \leqslant &\varphi(\frac{a+A}{2},\frac{b+B}{2}) \end{aligned} \tag{28}$$

*(ii) If* $\Delta\varphi \leqslant 0$*, then*

$$\begin{aligned} &\frac{2\varphi(a,B)+2\varphi(A,b)+\varphi(a,b)+\varphi(A,B)}{6} \\ \leqslant &\frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \\ \leqslant &\varphi(\frac{a+A}{2},\frac{b+B}{2}) \end{aligned} \tag{29}$$

*Proof.* Substituting $u(s) = \frac{1}{A-a}$ and $v(t) = \frac{1}{B-b}$ in (27) and (26) one can get $O(|\Delta\varphi|) = \frac{|\Delta\varphi|}{6}$.

COROLLARY 5.2. *Suppose that* $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ *is a continuous convex function.*

*(i) If* $\Delta\varphi \geqslant 0$, *then*

$$\begin{aligned} &\varphi(\frac{a+A}{2},\frac{b+B}{2}) \\ \leqslant &\frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \\ \leqslant &\frac{2\varphi(a,B)+2\varphi(A,b)+\varphi(a,b)+\varphi(A,B)}{6}. \end{aligned} \tag{30}$$

*(ii) If* $\Delta\varphi \leqslant 0$, *then*

$$\begin{aligned} &\varphi(\frac{a+A}{2},\frac{b+B}{2}) \\ \leqslant &\frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \\ \leqslant &\frac{2\varphi(a,b)+2\varphi(A,B)+\varphi(a,B)+\varphi(A,b)}{6}. \end{aligned} \tag{31}$$

REMARK 5.1. Allasia in [1, Theorem 1] gives Hermite-Hadamard's inequality for a triangle which imply our result in Corollary 5.2. The right side of the inequality in

Theorem 1, for a convex function $\varphi$ and for the special choice of triangles $\Delta_1, \Delta_2$ and $\Delta_3, \Delta_4$ gives

$$\begin{aligned}
&\frac{\int_a^A \int_b^B \varphi(s,t)dsdt}{(A-a)(B-b)} \\
&\leqslant \min\left\{ \frac{\varphi(a,b)+2\varphi(A,b)+2\varphi(a,B)+\varphi(A,B)}{6}, \frac{2\varphi(a,b)+\varphi(A,b)+\varphi(a,B)+2\varphi(A,B)}{6} \right. \\
&= \begin{cases} \frac{1}{6}[\varphi(a,b)+2\varphi(A,b)+2\varphi(a,B)+\varphi(A,B)], \ \Delta\varphi \geqslant 0 \\ \frac{1}{6}[2\varphi(a,b)+\varphi(A,b)+\varphi(a,B)+2\varphi(A,B)], \Delta\varphi \leqslant 0. \end{cases}
\end{aligned}$$

## 6. Lupaş inequalities

A local property of the concave functions inspired Feyér's inequalities has been given by Vasić, Lacković (1974,1976) and Lupaş (1976), (see [4, p.5]).

THEOREM A 3. *Let $p, q$ be given positive real numbers and $a_1 \leqslant a \leqslant b \leqslant b_1$. Moreover, let $w : [a_1, b_1] \to \mathbb{R}$ be a positive symmetric function with respect to $x_0 = \dfrac{pa+qb}{p+q}$, i.e. $w(x_0+s) = w(x_0-s)$, for $0 \leqslant s \leqslant h$. Then the inequalities*

$$\frac{p\varphi(a)+q\varphi(b)}{p+q} \leqslant \frac{\displaystyle\int_{x_0-h}^{x_0+h} w(x)\varphi(x)dx}{\displaystyle\int_{x_0-h}^{x_0+h} w(x)dx} \leqslant \varphi(x_0). \tag{32}$$

*hold for all continuous concave functions $\varphi : [a_1, b_1] \to \mathbb{R}$ iff $h \leqslant \frac{b-a}{p+q} \cdot min\{p,q\}$.*

Theorem A3 inspired us to give the following result related with Theorem 3.1

THEOREM 6.1. *Let $L$ be a linear space of real-valued functions defined on a nonempty set $\Omega$ and $g_1, g_2 \in L$ be functions such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$. Moreover, let $F : L \to \mathbb{R}$ be a normalized isotonic positive linear functional such that*

$$F(g_1) = \frac{pa+qA}{p+q} \quad \text{and} \quad F(g_2) = \frac{qb+pB}{p+q} \text{ or } F(g_2) = \frac{pb+qB}{p+q}. \tag{33}$$

*where $p, q \geqslant 0$ and $p^2+q^2 > 0$.*

*$(i)$ Suppose $\varphi : D \to \mathbb{R}$ is a concave continuous function. Then*

$$\begin{aligned}
&\max\left\{ \frac{p\varphi(a,B)+q\varphi(A,b)}{p+q}, \frac{p\varphi(a,b)+q\varphi(A,B)}{p+q} - \frac{|\Delta\varphi|}{2} \right\} \\
&\leqslant F(\varphi(g_1,g_2)) \\
&\leqslant \min\left\{ \varphi\left(\frac{pa+qA}{p+q}, \frac{pb+qB}{p+q}\right), \varphi\left(\frac{pa+qA}{p+q}, \frac{qb+pB}{p+q}\right) \right\}.
\end{aligned} \tag{34}$$

*(ii) Suppose $\varphi : D \to \mathbb{R}$ is a continuous convex function. Then*

$$\begin{aligned}&\max\left\{\varphi\left(\frac{pa+qA}{p+q},\frac{pb+qB}{p+q}\right),\varphi\left(\frac{pa+qA}{p+q},\frac{qb+pB}{p+q}\right)\right\}\\ &\leqslant F(\varphi(g_1,g_2)) \\ &\leqslant \min\left\{\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q},\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q}-\frac{|\Delta\varphi|}{2}\right\}\end{aligned} \tag{35}$$

*Proof. (i)* First we suppose $\varphi : D \to \mathbb{R}$ is a concave continuous function, $\Delta\varphi \geqslant 0$ and $F(g_2) = \dfrac{qb+pB}{p+q}$. According Theorem 3.1, Theorem 3.2 and Lemma 1, we have to calculate

$$F(m_{34}(g_1,g_2)) = F\left(\frac{\pi_3(g_1,g_2)+\pi_4(g_1,g_2)-|\pi_3(g_1,g_2)-\pi_4(g_1,g_2)|}{2}\right).$$

The properties of $F$ and (8) enable us to continue with

$$\begin{aligned}F(m_{34}(g_1,g_2)) &= \frac{1}{2}\left(\lambda_3\frac{pa+qA}{p+q}+\mu_3\frac{qb+pB}{p+q}+\varphi(a,B)-\lambda_3 a-\mu_3 B\right)\\ &+\frac{1}{2}\left(\lambda_4\frac{pa+qA}{p+q}+\mu_4\frac{qb+pB}{p+q}+\varphi(A,b)-\lambda_3 A-\mu_3 b\right)\\ &-\frac{1}{2}F(|\pi_3(g_1,g_2)-\pi_4(g_1,g_2)|).\end{aligned}$$

Some algebra together with (20) achieve

$$\begin{aligned}F(m_{34}(g_1,g_2)) &= \frac{(p+q)(\varphi(a,b)+\varphi(A,B))+(p-q)(\varphi(a,B)-\varphi(A,b))}{2(p+q)}\\ &-\frac{1}{2}\frac{|\Delta\varphi|}{(B-b)(A-a)}F(|(A-a)g_2-(B-b)g_1-Ab+aB|)\end{aligned}$$

Note that the maximum value of $|(A-a)g_2-(B-b)g_1-Ab+aB|$ is $(B-b)(A-a)$, so we can state that

$$F(m_{34}(g_1,g_2)) \geqslant \frac{(p+q)(\varphi(a,b)+\varphi(A,B))+(p-q)(\varphi(a,B)-\varphi(A,b))}{2(p+q)}-\frac{\Delta\varphi}{2}$$

Using Theorem 3.2 and (33) we have

$$\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q} \leqslant F(\varphi(g_1,g_2)) \leqslant \varphi\left(\frac{pa+qA}{p+q},\frac{qb+pB}{p+q}\right). \tag{36}$$

The same analysis can be used with the assumption that $F(g_2) = \dfrac{pb+qB}{p+q}$, to prove:

$$\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q}-\frac{|\Delta\varphi|}{2} \leqslant F(\varphi(g_1,g_2)) \leqslant \varphi\left(\frac{pa+qA}{p+q},\frac{pb+qB}{p+q}\right). \tag{37}$$

Taking the maximum of (36) and (37), we obtain the desired inequality (34).

Very similar efforts for $\Delta\varphi \leqslant 0$ give the same result (34).

For the convex case we use Theorem 3.1, Theorem 3.2, Lemma 1, and calculate

$$F(M_{12}(g_1,g_2)) = F(\frac{\pi_1(g_1,g_2)+\pi_2(g_1,g_2)+|\pi_1(g_1,g_2)-\pi_2(g_1,g_2)|}{2}).$$

Generalization of Theorem A3 for concave functions of two variable is obtained in the following Corollary.

COROLLARY 6.1. *Let* $w : D = [a,A]\times[b,B] \to \mathbb{R}$ *be a nonnegative integrable function such that* $w(s,t) = u(s)v(t)$*, where* $u : [a,A] \to \mathbb{R}$ *and* $v : [b,B] \to \mathbb{R}$ *are integrable functions with properties:*

$u(x_1+s) = u(x_1-s)$*, for all* $s \in [0,h]$*,* $v(y_i+t) = v(y_i-t)$*,* $i = 1,2.$*, for all* $t \in [0,k]$ *where* $x_1 = \dfrac{pa+qA}{p+q}$*,* $y_1 \dfrac{pb+qB}{p+q}$ *and* $y_2 = \dfrac{qb+pB}{p+q}$ *are fixed and determined with given numbers* $p,q \geqslant 0$*,* $p^2+q^2 > 0$*. For all* $h,k > 0$ *such that*

$$0 \leqslant h \leqslant \frac{A-a}{p+q}min\{p,q\}, \quad 0 \leqslant k \leqslant \frac{B-b}{p+q}min\{p,q\}, \tag{38}$$

$(i)$ *if* $\varphi : D \to \mathbb{R}$ *is a continuous concave function, then*

$$\begin{aligned}&\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q}+\frac{\Delta\varphi-|\Delta\varphi|}{4}\\ &\leqslant \frac{\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)\varphi(s,t)dsdt}{\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt}\\ &\leqslant \varphi(x_1,y_2);\end{aligned} \tag{39}$$

*and*

$$\begin{aligned}&\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q}-\frac{\Delta\varphi+|\Delta\varphi|}{4}\\ &\leqslant \frac{\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)\varphi(s,t)dsdt}{\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)dtds}\\ &\leqslant \varphi(x_1,y_1);\end{aligned} \tag{40}$$

*(ii) if $\varphi : D \to \mathbb{R}$ is a continuous convex function then*

$$\begin{aligned}&\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q}-\frac{\Delta\varphi-|\Delta\varphi|}{4}\\ &\leqslant \frac{\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)\varphi(s,t)dsdt}{\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)dsdt}\\ &\leqslant \varphi\left(x_1,y_1\right).\end{aligned}$$

*and*

$$\begin{aligned}&\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q}+\frac{\Delta\varphi+|\Delta\varphi|}{4}\\ &\leqslant \frac{\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)\varphi(s,t)dsdt}{\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt}\\ &\leqslant \varphi\left(x_1,y_2\right).\end{aligned}$$

*Proof for (*39*).* Conditions $[x_1-h,x_2+h]\subset[a,A]$ and $[y_i-k,y_i+k]\subset[b,B]$ are satisfied by (38). We use Lemma 1 and Theorem 3.1 for $\Omega=D$, functions $g_1(s,t)=s$ and $g_2(s,t)=t$. A functional is defined by

$$F(f)=\frac{\int_{x_1-h}^{x_1+h}\int_{y_i-k}^{y_i+k} f(s,t)\cdot w(s,t)dsdt}{\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt}, i=1,2, \text{ for } f:\Omega=D\to\mathbb{R}. \tag{41}$$

The functional (41) is positive, linear and $F(1)=1$, so the proof is similar to the proof of Theorem 6.1.

REMARK 6.1. In the cases $p=q$, $h=\dfrac{A-a}{2}$ and $k=\dfrac{B-b}{2}$, Theorem 6.1 is expressed as the Corollary 3.1 in [6].

Integral version of Hadammard's inequalities for concave function of two variables is consequence of Theorem 6.1.

REMARK 6.2. Using the Corollary 6.1 for $w(s,t)=u(s)=v(t)=1$ the functional acquires the shape

$$F(g_1)=\frac{1}{4hk}\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} sdsdt = x_1.$$

The inequalities obtained from the Corollary 6.1 with the mentioned shape of the functional, show the local feature from the the Hadamard's inequalities enlarged on the functions of two variables.

REMARK 6.3. For $p = q$, $h = \dfrac{A-a}{2}$, $k = \dfrac{B-b}{2}$, $u, v$ such that $\int_a^A u(s)ds = \int_b^B v(t)dt = 1$ one can get the Corollary 3.2 in [6].

## 7. Petrović's inequality

In the [6] authors achieved the following generalization of the famous Petrović's inequality.

THEOREM A 4. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be* $n$*-tuples of nonnegative real numbers and put* $P_n := \sum_{i=1}^n p_i\ (> 0)$ *and* $Q_n := \sum_{j=1}^n q_j\ (> 0)$. *Suppose that* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *are* $n$*-tuples of nonnegative real numbers such that* $0 \leqslant x_k \leqslant \sum_{i=1}^n p_i x_i \leqslant c$ *and* $0 \leqslant y_k \leqslant \sum_{j=1}^n q_j y_j \leqslant d$, *for* $k = 1, 2, \ldots, n$. *Let* $\varphi : [0,c] \times [0,d] \to R$ *be a concave function.*

*(i) Suppose* $\varphi(0,0) + \varphi(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j) \geqslant \varphi(\sum_{i=1}^n p_i x_i, 0) + \varphi(0, \sum_{j=1}^n q_j y_j)$.

*If* $\dfrac{1}{P_n} + \dfrac{1}{Q_n} \leqslant 1$, *then*

$$\frac{1}{P_n}\varphi\left(\sum_{i=1}^n p_i x_i, 0\right) + \frac{1}{Q_n}\varphi\left(0, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{P_n} - \frac{1}{Q_n}\right)\varphi(0,0)$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^n \sum_{j=1}^n p_i q_j \varphi(x_i, y_j).$$

*If* $\dfrac{1}{P_n} + \dfrac{1}{Q_n} \geqslant 1$, *then*

$$\left(\frac{1}{P_n} + \frac{1}{Q_n} - 1\right)\varphi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{Q_n}\right)\varphi\left(\sum_{i=1}^n p_i x_i, 0\right)$$
$$+ \left(1 - \frac{1}{P_n}\right)\varphi\left(0, \sum_{j=1}^n q_j y_j\right)$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^n \sum_{j=1}^n p_i q_j \varphi(x_i, y_j).$$

*(ii) Suppose* $\varphi(0,0) + \varphi(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j) \leqslant \varphi(\sum_{i=1}^n p_i x_i, 0) + \varphi(0, \sum_{j=1}^n q_j y_j)$.

*If* $P_n \geqslant Q_n$, *then*

$$\frac{1}{P_n}\varphi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) + \left(\frac{1}{Q_n} - \frac{1}{P_n}\right)\varphi\left(0, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{Q_n}\right)\varphi(0,0)$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^n \sum_{j=1}^n p_i q_j \varphi(x_i, y_j).$$

*If* $Q_n \geqslant P_n$, *then*

$$\frac{1}{Q_n}\varphi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right) - \left(\frac{1}{Q_n} - \frac{1}{P_n}\right)\varphi\left(\sum_{i=1}^{n} p_i x_i, 0\right) + \left(1 - \frac{1}{P_n}\right)\varphi(0,0)$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \varphi(x_i, y_j).$$

We obtain a refinement of Theorem A4 as application of Theorem 3.2 in a discrete case including the special choice of $\Omega$ and $F$.

THEOREM 7.1. *Let* $\mathbf{p} = (p_1, \dots, p_n)$ *and* $\mathbf{q} = (q_1, \dots, q_n)$ *be* $n$*-tuples of nonnegative real numbers and put* $P_n := \sum_{i=1}^{n} p_i\ (>0)$ *and* $Q_n := \sum_{j=1}^{n} q_j\ (>0)$. *Suppose that* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\mathbf{y} = (y_1, \dots, y_n)$ *are* $n$*-tuples of nonnegative real numbers such that* $0 \leqslant x_k \leqslant \sum_{i=1}^{n} p_i x_i = A \leqslant c$ *and* $0 \leqslant y_k \leqslant \sum_{j=1}^{n} q_j y_j = B \leqslant d$, *for* $k = 1, 2, \dots, n$. *Let* $\varphi : [0,c) \times [0,d) \to \mathbb{R}$ *be a concave function.*

(*i*) *If* $\varphi(0,0) + \varphi(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j) \geqslant \varphi(\sum_{i=1}^{n} p_i x_i, 0) + \varphi(0, \sum_{j=1}^{n} q_j y_j)$, *then*

$$\varphi(0,0) + \frac{\varphi(A,B) - \varphi(0,0)}{2}\left(\frac{1}{P_n} + \frac{1}{Q_n}\right) + \frac{\varphi(A,0) - \varphi(0,B)}{2}\left(\frac{1}{P_n} - \frac{1}{Q_n}\right)$$
$$-\frac{\Delta\varphi}{2P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} - \frac{y_j}{B}\right| \tag{42}$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \varphi(x_i, y_j).$$

(*ii*) *If* $\varphi(0,0) + \varphi(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j) \geqslant \varphi(\sum_{i=1}^{n} p_i x_i, 0) + \varphi(0, \sum_{j=1}^{n} q_j y_j)$, *then*

$$\varphi(0,0) + \frac{\varphi(A,B) - \varphi(0,0)}{2}\left(\frac{1}{P_n} + \frac{1}{Q_n}\right) + \frac{\varphi(A,0) - \varphi(0,B)}{2}\left(\frac{1}{P_n} - \frac{1}{Q_n}\right)$$
$$-\frac{\Delta\varphi}{2}\left[1 - \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} + \frac{y_j}{B} - 1\right|\right] \tag{43}$$
$$\leqslant \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \varphi(x_i, y_j).$$

*Proof.* We will use Theorem 3.2. Let $L$ be a linear class of real-valued functions defined on $\Omega^2 = \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ having the conditions L1 and L2. We consider a functional $F$ on $L$ defined by

$$F(h) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j h(i,j)}{\sum_{i=1}^{n} p_i \sum_{j=1}^{n} q_j}, \quad \text{for} \quad h : \Omega^2 \to \mathbb{R} \tag{44}$$

where $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ are given nonnegative $n$-tuples.

Put $a = b = 0$, $A = \sum_{i=1}^{n} p_i x_i$, $B = \sum_{j=1}^{n} q_j y_j$ and define $g_1, g_2 : \Omega^2 \to \mathbb{R}$ by $g_1(i,j) = x_i$ and $g_2(i,j) = y_j$. Using definition (44) we have $F(g_1) = \frac{\sum_{i=1}^{n} p_i x_i}{P_n} = \frac{A}{P_n}$ and $F(g_2) = \frac{B}{Q_n}$. If $\Delta\varphi \geqslant 0$ then we obtain the result in (42) using the inequality (24) in Theorem 3.2 $F(m_{34}(g_1,g_2)) \leqslant F(\varphi(g_1,g_2))$. Similarly, if $\Delta\varphi \leqslant 0$, the inequality in (43) we obtain using the inequality (25) for $F(m_{12}(g_1,g_2))$.

REMARK 7.1. The left side of (42) can be rewritten as:

$$\varphi(0,0)\left(1 - \frac{1}{P_n} - \frac{1}{Q_n}\right) + \frac{1}{P_n}\varphi(A,0) + \frac{1}{Q_n}\varphi(0,B) + \frac{\Delta\varphi}{2}\left[\frac{1}{P_n} + \frac{1}{Q_n} - \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} - \frac{y_j}{B}\right|\right].$$

Note that for $\Delta\varphi \geqslant 0$ and $\frac{1}{P_n} + \frac{1}{Q_n} \leqslant 1$ in Theorem A4 there is the similar right side. According to Theorem 3.2 we obtain:

$$\frac{1}{P_n} + \frac{1}{Q_n} - \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} - \frac{y_i}{B}\right| \geqslant 0.$$

Similarly, for all cases in Theorem A4 we can obtain the following results as the consequences of Theorem 3.2

$$1 - \left|\frac{1}{P_n} + \frac{1}{Q_n} - 1\right| - \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} - \frac{y_j}{B}\right| \geqslant 0;$$

$$1 - \left|\frac{1}{Q_n} - \frac{1}{P_n}\right| - \frac{1}{P_n Q_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A} + \frac{y_j}{B} - 1\right| \geqslant 0.$$

We obtain the following estimations as a consequence of the refinement made in the Remark 7.1.

COROLLARY 7.1. *Let* $\mathbf{p} = (p_1, \dots, p_n)$ *and* $\mathbf{q} = (q_1, \dots, q_n)$ *be* $n$*-tuples of nonnegative real numbers. Suppose that* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\mathbf{y} = (y_1, \dots, y_n)$ *are* $n$*-tuples of nonnegative real numbers such that* $0 \leqslant x_k \leqslant \sum_{i=1}^{n} p_i x_i$ *and* $0 \leqslant y_k \leqslant \sum_{j=1}^{n} q_j y_j$*, for* $k = 1, 2, \dots, n$*. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{\sum_{k=1}^{n} x_k p_k} - \frac{y_j}{\sum_{k=1}^{n} y_k q_k}\right| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j - \left|\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j - \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i\right|;$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{\sum_{k=1}^{n} x_k p_k} + \frac{y_j}{\sum_{k=1}^{n} y_k q_k} - 1\right| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j - \left|\sum_{i=1}^{n} p_i - \sum_{j=1}^{n} q_j\right|.$$

## REFERENCES

[1] G. ALLASIA, *Approximate Integration Formulas and Related Inequalities for Bivariate Convex Functions*, Varahmihir Journal of Mathematical Sciences, 6 (1) (2006), 139–152.

[2] V. CSISZÁR AND T. F. MÓRI, *The convexity method of proving moment type inequalities*, Stat. Probab. Lett. **66** (2004), no. 3, 303–313.

[3] J. B. DIAZ AND F. T. METCALF, *Stronger forms of a class of inequalities of G. Pólya-G. Szegö and V. Kantorovich*, Bull. Amer. Math. Soc., **69** (1963), 415–418.

[4] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (online: http://rgmia.vu.edu.au/monographs/)

[5] L. FEJÉR, *Über die Fourierreihen*, II. Math. Naturwiss, Ant. Ungar. Acad. Wiss, **24** (1906), 369–390.

[6] B. IVANKOVIĆ, S. IZUMINO, J. PEČARIĆ AND M. TOMINAGA, *On an inequality of V. Csiszár and T. F. Móri for convex functions of two variables*, JIPAM, J. Inequal. Pure Appl. Math., **8** (2007), no. 3, article 88, (electronic).

[7] E. J. MCSHANE, *Jensen's inequality*, Bull. Amer. Math. Soc., **43**, (1937), 521–527.

[8] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc, 1992.

[9] T. K. POGANY, *A new (probabilistic) proof on the Diaz-Matcalf and Pólya-Szegö inequalities and some applications*, Theor. Probability and Math. Statist., (2005), no. 70, 113–122 (electronic).

[10] I. RAŞA, *A note on Jensen's inequality*, It. Sem. Funct. Eq. Approx. Conv., Cluj-Napoca, (1988), 275–280.

*V. Čuljak, Department of Mathematics, Faculty of Civil Engineering, University of Zagreb, Kaciceva 26, 10 000 Zagreb, Croatia*
*e-mail:* vera@master.grad.hr

*B. Ivanković, Faculty of transport and traffic engineering, University of Zagreb, Vukelićeva 4, 10 000 Zagreb, Croatia*
*e-mail:* ivankovb@fpz.hr

*J. Pečarić, Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10 000 Zagreb, Croatia*
*e-mail:* pecaric@hazu.hr