Grüss-Lupas type inequality and its applications for the estimation of p-moments of guessing mappings

S. S. DRAGOMIR* AND G. L. BOOTH†

Abstract. *An inequality of Grüss-Lupas type in normed spaces is proved. Some applications in estimating the p-moments of guessing mapping which complement the recent results of Massey [1], Arikan [2], Boztas [3] and Dragomir-van der Hoek [5]-[7] are also given.*

Key words: *Grüss-Lupas type inequalities, guessing mapping, p−moments*

AMS subject classifications: 26D15, 94Axx

Received November 29, 1999 Accepted April 15, 2000

1. Introduction

In 1935, G. Grüss proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \qquad (1)$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfying the assumption

$$\varphi \le f(x) \le \Phi, \gamma \le g(x) \le \Gamma \qquad (2)$$

for each $x \in [a, b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1) as well as for some other integral inequalities of Grüss' type see Chapter X of the recent book [4] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski established the following discrete version of Grüss' inequality [4, Chap. X]:

*School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, MC 8001, Australia, e-mail: `sever@matilda.vu.edu.au`

†Department of Mathematics, University of Port Elisabeth, South Africa

Theorem 1. *Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n)$ be two n-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, ..., n$. Then one has*

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right| \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s) \tag{3}$$

where $[x]$ is the integer part of $x, x \in \mathbb{R}$.

A weighted version of Grüss' discrete inequality was proved by J.E. Pečarić in 1979, [4, Chap. X]:

Theorem 2. *Let a, b be two monotonic n-tuples and p a positive one. Then*

$$\begin{aligned} &\left|\tfrac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \tfrac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \tfrac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right| \\ \leq\ & |a_n - a_1|\,|b_n - b_1| \max_{1\leq k\leq n-1}\left(\tfrac{P_k \bar{P}_{k+1}}{P_n^2}\right) \end{aligned} \tag{4}$$

where $P_n := \sum_{i=1}^{n} p_i$, $\bar{P}_{k+1} = P_n - P_{k+1}$.

In 1981 , A. Lupas [4, Chap. X] proved some similar results for the first difference of a as follows :

Theorem 3. *Let a, b two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned} &\min_{1\leq i\leq n-1}|a_{i+1} - a_i| \min_{1\leq i\leq n-1}|b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} i p_i\right)^2\right] \\ \leq\ & \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i \\ \leq\ & \max_{1\leq i\leq n-1}|a_{i+1} - a_i| \max_{1\leq i\leq n-1}|b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} i p_i\right)^2\right] \end{aligned} \tag{5}$$

If there exist numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (5) the equality holds.

For some generalizations of Grüss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [4] where further references are given.

2. Some Grüss-Lupas type inequalities

The following inequality of Grüss-Lupas type in normed linear spaces holds:

Theorem 4. *Let* $(X, \|\cdot\|)$ *be a normed linear space over* $K = (\mathbb{R}, \mathbb{C})$, $x_i \in X$, $\alpha_i \in K$ *and* $p_i \geq 0$ $(i = 1, ..., n)$ *such that* $\sum_{i=1}^{n} p_i = 1$. *Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\|$$
$$\leq \max_{1\leq j\leq n-1} |\alpha_{j+1} - \alpha_j| \max_{1\leq j\leq n-1} \|x_{j+1} - x_j\| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right]. \quad (6)$$

Inequality (6) *is sharp in the sense that the constant* $C = 1$ *in the right membership cannot be replaced by a smaller one.*

Proof. Let us start with the following identity which can be proved by direct computation:

$$\begin{aligned} \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i &= \tfrac{1}{2} \sum_{i,j=1}^{n} p_i p_j (\alpha_j - \alpha_i)(x_j - x_i) \\ &= \sum_{1\leq i<j\leq n}^{n} p_i p_j (\alpha_j - \alpha_i)(x_j - x_i). \end{aligned}$$

As $i < j$, we can write that

$$\alpha_j - \alpha_i = \sum_{k=i}^{j-1} (\alpha_{k+1} - \alpha_k)$$

and

$$x_j - x_i = \sum_{k=i}^{j-1} (x_{k+1} - x_k).$$

Using the generalized triangle inequality we have successively:

$$\begin{aligned} \left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| &= \left\| \sum_{1\leq i<j\leq n}^{n} p_i p_j \sum_{k=i}^{j-1} (\alpha_{k+1} - \alpha_k) \sum_{k=i}^{j-1} (x_{k+1} - x_k) \right\| \\ &\leq \sum_{1\leq i<j\leq n}^{n} p_i p_j \left| \sum_{k=i}^{j-1} (\alpha_{k+1} - \alpha_k) \right| \left\| \sum_{k=i}^{j-1} (x_{k+1} - x_k) \right\| \\ &\leq \sum_{1\leq i<j\leq n}^{n} p_i p_j \sum_{k=i}^{j-1} |\alpha_{k+1} - \alpha_k| \sum_{k=i}^{j-1} \|x_{k+1} - x_k\| =: A. \end{aligned}$$

Note that

$$|\alpha_{k+1} - \alpha_k| \leq \max_{1\leq s\leq n-1} |\alpha_{s+1} - \alpha_s|$$

and

$$\|x_{k+1} - x_k\| \leq \max_{1\leq s\leq n-1} \|x_{s+1} - x_s\|$$

for all $k = i, ..., j-1$ and then by summation,

$$\sum_{k=i}^{j-1} |\alpha_{k+1} - \alpha_k| \leq (j - i) \max_{1\leq s\leq n-1} |\alpha_{s+1} - \alpha_s|$$

and

$$\sum_{k=i}^{j-1}\|x_{k+1}-x_k\| \le (j-i)\max_{1\le s\le n-1}\|x_{s+1}-x_s\|.$$

Taking into account the above estimations, we can write

$$A \le \left[\sum_{1\le i<j\le n}^{n} p_i p_j (j-i)^2\right]\max_{1\le s\le n-1}|\alpha_{s+1}-\alpha_s|\max_{1\le s\le n-1}\|x_{s+1}-x_s\|.$$

As a simple calculation shows that

$$\sum_{1\le i<j\le n}^{n} p_i p_j (j-i)^2 = \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2,$$

inequality (6) is proved.
Assume that inequality (6) holds with a constant $c>0$, i.e.,

$$\begin{aligned}&\left\|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i\sum_{i=1}^{n} p_i x_i\right\| \\ \le\ & c\max_{1\le j\le n-1}|\alpha_{j+1}-\alpha_j|\max_{1\le j\le n-1}\|x_{j+1}-x_j\|\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]. \qquad (7)\end{aligned}$$

Now, choose the sequences $\alpha_k=\alpha+k\beta\,(\beta\ne 0)$, $x_k=x+ky\,(y\ne 0)$ $(k=1,...,n)$. We get

$$\begin{aligned}&\left\|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i\sum_{i=1}^{n} p_i x_i\right\| \\ =\ & \frac{1}{2}\left\|\sum_{i,j=1}^{n} p_i p_j (i-j)^2\beta y\right\| = |\beta|\,\|y\|\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}&\max_{1\le j\le n-1}|\alpha_{j+1}-\alpha_j|\max_{1\le j\le n-1}\|x_{j+1}-x_j\|\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] \\ =\ & |\beta|\,\|y\|\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]\end{aligned}$$

and then by (7) we get $c\ge 1$, which proves the sharpness of the constant $c=1$. □

The following corollary holds:

Corollary 1. *Under the above assumptions for α_i, x_i $(i = 1, ..., n)$ we have the inequality:*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq \frac{n^2 - 1}{12} \max_{1 \leq j \leq n-1} |\alpha_{j+1} - \alpha_j| \max_{1 \leq j \leq n-1} \|x_{j+1} - x_j\| . \tag{8}$$

The constant $\frac{1}{12}$ is sharp in the sense that it cannot be replaced by a smaller one.

The proof follows by the above theorem, putting $p_i = \frac{1}{n}$ and taking into account that:

$$\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 = \frac{n^2 - 1}{12}.$$

3. Applications for the moments of guessing mappings

J. L. Massey in [1] considered the problem of guessing the value of realization of random variable X by asking questions of the form: "Is X equal to x ? " until the answer is "Yes" .

Let $G(X)$ denote the number of guesses required by a particular guessing strategy when $X = x$.

Massey observed that $E(G(x))$, the average number of guesses, is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [2].

Let (X, Y) be a pair of random variables with X taking values in a finite set χ of size n, Y taking values in a countable set $\mathcal{Y}$. Call a function $G(X)$ of the random variable X a *guessing function* for X if $G : \chi \to \{1, ..., n\}$ is one-to-one. Call a function $G(X \mid Y)$ a *guessing function for X given Y* if for any fixed value $Y = y, G(X \mid y)$ is a guessing function for X . $G(X \mid y)$ will be thought of as the number of guessing required to determine X when the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X|Y)$ were proved by E. Arikan in the recent paper [2].

Theorem 5. *For an arbitrary guessing function $G(X)$ and $G(X \mid Y)$ and any $p > 0$, we have:*

$$E(G(X)^p) \geq (1 + \ln n)^{-p} \left[\sum_{x \in \chi} P_X(x)^{\frac{1}{1+p}} \right]^{1+p} \tag{9}$$

and

$$E(G(X \mid Y)^p) \geq (1 + \ln n)^{-p} \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \chi} P_{X,Y}(x, y)^{\frac{1}{1+p}} \right]^{1+p} \tag{10}$$

where $P_{X,Y}$ and P_X are probability distributions of (X, Y) and X, respectively.

Note that, for $p=1$, we get the following estimations on the average number of guesses:

$$E(G(X)) \geq \frac{\left[\sum_{x\in\chi} P_X(x)^{\frac{1}{2}}\right]^2}{1+\ln n}$$

and

$$E(G(X)) \geq \frac{\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\chi} P_{X,Y}(x,y)^{\frac{1}{2}}\right]^2}{1+\ln n}.$$

In paper [3], Boztas proved the following analytic inequality and applied it for the moments of guessing mappings:

Theorem 6. *The relation*

$$\left[\sum_{k=1}^{n} p_k{}^{\frac{1}{r}}\right]^r \geq \sum_{k=1}^{n}\left(k^r-(k-1)^r\right)p_k \tag{11}$$

where $r \geq 1$ *holds for any positive integer* n*, provided that the weights* $p_1, ..., p_n$ *are nonnegative real numbers satisfying the condition:*

$$p_{k+1}^{\frac{1}{r}} \leq \frac{1}{k}\left(p_1^{\frac{1}{r}}+...+p_k^{\frac{1}{r}}\right), k=1,2,...n-1 \tag{12}$$

To simplify the notation further, we assume that the x_i are numbered such that x_k is always the k^{th} guess. This yields:

$$E(G^p)=\sum_{k=1}^{n} k^p p_k, p\geq 0.$$

If we now consider the guessing problem, we note that (9) can be written (see for example [3]) as:

$$\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+p}}\right]^{1+p} \geq E\left(G^{1+p}\right)-E\left((G-1)^{1+p}\right)$$

for guessing sequences obeying (12).

In particular, using the binomial expansion of $(G-1)^{1+p}$ we have the following corollary [3] :

Corollary 2. *For guessing sequences obeying* (12) *with* $r=1+m$ *, the* m^{th} *guessing moment, when* $m\geq 1$ *is an integer satisfies:*

$$\begin{aligned} E(G^m) &\leq \tfrac{1}{1+m}\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+m}}\right]^{1+m} \\ &+\tfrac{1}{1+m}\left\{\tbinom{m+1}{2}E\left(G^{m-1}\right)-\tbinom{m+1}{3}E\left(G^{m-2}\right)+...+(-1)^{m+1}\right\}. \end{aligned} \tag{13}$$

The following inequalities immediately follow from *Corollary 2.*:

$$E(G) \le \frac{1}{2}\left[\sum_{k=1}^{n} p_k^{\frac{1}{2}}\right]^2 + \frac{1}{2}$$

and

$$E\left(G^2\right) \le \frac{1}{3}\left[\sum_{k=1}^{n} p_k^{\frac{1}{3}}\right]^3 + E(G) - \frac{1}{3}.$$

We are able now to point out some new results for the p-moment of guessing mapping as follows.

Using Pečarić's result (4), we can state the following inequality for the moments of a guessing mapping $G(X)$:

Theorem 7. *Let $p, q > 0$. Then we have the inequality:*

$$\begin{aligned} 0 &\le E\left(G^{p+q}\right) - E\left(G^p\right) E\left(G^q\right) \\ &\le \left(n^p - 1\right)\left(n^q - 1\right) \max_{1 \le k \le n-1} \left\{P_k\left(1 - P_k\right)\right\} \end{aligned} \tag{14}$$

where $P_k = \sum\limits_{i=1}^{k} p_i$.

Proof. Define the sequences $a_i = i^p$, $b_i = i^q$ which are monotonous nondecreasing. Using both Čebyšev's and Pečarić's results we can state

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} i^{p+q} p_i - \sum_{i=1}^{n} i^p p_i \sum_{i=1}^{n} i^q p_i \\ &\le \left(n^p - 1\right)\left(n^q - 1\right) \max_{1 \le k \le n-1} \left\{P_k\left(1 - P_k\right)\right\} \end{aligned}$$

which is exactly (14). □

Now, let us define the mappings m_n, $M_n : (0, \infty) \longrightarrow (0, \infty)$ given by

$$m_n(t) := \begin{cases} n^t - (n-1)^t, & \text{if } t \in (0,1) \\ 2^t - 1, & \text{if } t \in [1, \infty) \end{cases}$$

and

$$M_n(t) := \begin{cases} 2^t - 1, & \text{if } t \in (0,1) \\ n^t - (n-1)^t, & \text{if } t \in [1, \infty) \end{cases} .$$

Now, using Lupas' result (see *Theorem 3.*) we can state the following result:

Theorem 8. *Let $p, q > 0$. Then we have the inequality*

$$\begin{aligned} m_n(p)\, m_n(q) \left[E\left(G^2\right) - E^2(G)\right] &\le E\left(G^{p+q}\right) - E\left(G^p\right) E\left(G^q\right) \\ &\le M_n(p)\, M_n(q) \left[E\left(G^2\right) - E^2(G)\right]. \end{aligned} \tag{15}$$

Proof. Consider the sequences $a_i = i^p$, $b_i = i^q$ in Lupas' theorem (note that a_i, b_i are monotonous nondecreasing) to get:

$$\begin{aligned} & \min_{1\leq i\leq n-1}\left[(i+1)^p - i^p\right]\min_{1\leq i\leq n-1}\left[(i+1)^q - i^q\right]\left[E\left(G^2\right) - E^2\left(G\right)\right] \\ \leq\ & E\left(G^{p+q}\right) - E\left(G^p\right)E\left(G^q\right) \\ \leq\ & \max_{1\leq i\leq n-1}\left[(i+1)^p - i^p\right]\min_{1\leq i\leq n-1}\left[(i+1)^q - i^q\right]\left[E\left(G^2\right) - E^2\left(G\right)\right]. \quad (16) \end{aligned}$$

Now, let us observe that if $p \in (0,1)$, then the sequence $\alpha_i = i^p$ is concave, i.e.,

$$\alpha_{i+1} - \alpha_i \leq \alpha_i - \alpha_{i-1} \text{ for all } i = 2, ..., n-1$$

and if $p \in [1,\infty)$ then $\alpha_i = i^p$ is convex, i.e.,

$$\alpha_{i+1} - \alpha_i \geq \alpha_i - \alpha_{i-1} \text{ for all } i = 2, ..., n-1.$$

Consequently

$$\min_{1\leq j\leq n-1}\left[(j+1)^p - j^p\right] = m_n\left(p\right)$$

and

$$\max_{1\leq j\leq n-1}\left[(j+1)^p - j^p\right] = M_n\left(p\right).$$

Using (16) we get the desired inequality (15). □

Now, for a given $p > 0$, consider the sum

$$S_p\left(n\right) := \sum_{i=1}^{n} i^p.$$

We know that

$$S_1\left(n\right) = \frac{n\left(n+1\right)}{2},$$

$$S_2\left(n\right) = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

and

$$S_3\left(n\right) = \left[\frac{n\left(n+1\right)}{2}\right]^2.$$

Using Biernaki-Pidek-Nardzewski's result (see *Theorem 1.*) we can state and prove the following approximation result concerning the p -moment of guessing mapping $G\left(X\right)$.

Theorem 9. *Let $p > 0$. Then we have the estimation*

$$\left|E\left(G^p\left(X\right)\right) - \frac{1}{n}S_p\left(n\right)\right| \leq \left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\left(n^p - 1\right)\left(p_M - p_m\right) \quad (17)$$

where $p_M := \max\left\{p_i \mid i = 1, ..., n\right\}$ and $p_m := \min\left\{p_i \mid i = 1, ..., n\right\}$.

Proof. Let us choose in *Theorem 1.*, $a_i = p_i$, $b_i = i^p$. Then $p_m \le a_i \le p_M$, $1 \le b_i \le n^p$ for all $i = 1, ..., n$ and by (3) we get

$$\left| \sum_{i=1}^{n} i^p p_i - \frac{1}{n} \sum_{i=1}^{n} i^p \sum_{i=1}^{n} p_i \right| \le \left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) (n^p - 1)(p_M - p_m),$$

which proves the theorem. □

Remark 1. *1. If in* (17) *we put* $p = 1$, *we get*

$$\left| E(G(X)) - \frac{n+1}{2} \right| \le (n-1)\left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) (p_M - p_m) \qquad (18)$$

which is an estimation of the average number of guesses in term of the size n *of* X *and* $p_M - p_m$.

2. Note that if $p = (p_1, ..., p_n)$ *is close to the uniform distribution* $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$, *i.e.,*

$$0 \le p_M - p_m \le \frac{\varepsilon}{(n-1)\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)}, \varepsilon > 0 \qquad (19)$$

then the error of approximating $E(G(X))$ *by* $\frac{n+1}{2}$ *is less than* $\varepsilon > 0$.

Now, using our new inequality in *Corollary 1.* we shall be able to prove another type of estimation for the p-moment of guessing mapping $G(X)$ as follows:

Theorem 10. *Let* $p > 0$. *Then we have the estimation:*

$$\left| E(G^p(X)) - \frac{1}{n} S_p(n) \right| \le \frac{(n^2 - 1)\, n}{12} M_n(p) \max_{1 \le j \le n-1} |p_{j+1} - p_j|. \qquad (20)$$

Proof. Follows by *Corollary 1.*, choosing $\alpha_i = i^p, x_i = p_i$ and $\|\cdot\|$ is the usual modulus $|\cdot|$ from the real number field $\mathbb{R}$. □

Remark 2. *1. If in* (20) *we put* $p = 1$, *we get*

$$\left| E(G(X)) - \frac{n+1}{2} \right| \le \frac{n(n^2 - 1)}{12} \max_{1 \le j \le n-1} |p_{j+1} - p_j|, \qquad (21)$$

which is another type of estimation for the average number of guesses in terms of the size of X *and of the "step size" of probabilities* p_i.

2. Note that if we choose

$$\max_{1 \le j \le n-1} |p_{j+1} - p_j| < \frac{12\varepsilon}{n(n^2 - 1)}, \quad \varepsilon > 0$$

then

$$\left| E(G^p(X)) - \frac{n+1}{2} \right| < \varepsilon.$$

Acknowledgement. The authors would like to thank the anonymous referee for some valuable suggestions in improving this paper.

References

[1] J. L. Massey, *Guessing and entropy*, in: Proc. 1994, IEEE Int. Symp. on Information Theory, Trodheim, Norway, 1994, p. 204.

[2] E. Arikan, *An inequality on guessing and its applications to sequential decoding*, IEEE Transactions on Information Theory **42**(1996), 99–105.

[3] S. Boztas, *Comments on " An inequality on guessing and its applications to sequential decoding"*, IEEE Transactions on Information Theory, to appear

[4] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[5] S. S. Dragomir, J. van der Hoek, *Some new inequalities for the average number of guesses*, Kyungpook Math. Journal, to appear.

[6] S. S. Dragomir, J. van der Hoek, *New inequalities for the moments of guessing mapping*, East Asian Math. Journal **14**(1998), 1–14.

[7] S. S. Dragomir, J. van der Hoek, *Some new analytic inequalities and their applications in guessing theory*, J. Math. Anal. Appl. **225**(1998), 542–556.