On the absolute Nörlund summability factors

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Abstract. *In this paper a theorem on the absolute Nörlund summability factors has been proved under more weaker conditions by using an almost increasing sequence.*

Key words: *Nörlund summability, summability factors, almost increasing sequence*

AMS subject classifications: 40D15, 40F05, 40G05

Received July 6, 2000 Accepted October 12, 2000

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively. The series $\sum a_n$ is said to be summable $\mid C, \alpha \mid$, if (see [4], [6])

$$\sum_{n=1}^{\infty} \mid u_n^\alpha - u_{n-1}^\alpha \mid \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid < \infty. \tag{1}$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n \;=\; p_0 + p_1 + p_2 + ... + p_n \neq 0, \quad (n \geq 0) \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_v \tag{3}$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $\mid N, p_n \mid$, if (see [7])

$$\sum_{n=1}^{\infty} \mid \sigma_n - \sigma_{n-1} \mid \;<\; \infty. \tag{4}$$

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In the special case where

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\ ,\ \alpha \geq 0 \tag{5}$$

the Nörlund mean reduces to the (C,α) mean and $\mid N,p_n \mid$ summability becomes $\mid C,\alpha \mid$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C,1)$ mean and then $\mid N,p_n \mid$ summability becomes $\mid C,1 \mid$ summability. For any sequence (λ_n) we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1}$.
In [5] Kishore has proved the following theorem concerning $\mid C,1 \mid$ and $\mid N,p_n \mid$ summability methods.

Theorem 1. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C,1 \mid$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N,p_n \mid$.*

Ahmad [1] proved the following theorem for absolute Nörlund summability factors.

Theorem 2. *Let (p_n) be as in* Theorem 1. *If*

$$\sum_{\nu=1}^{n} \frac{1}{\nu} \mid t_\nu \mid = O(X_n) \quad \text{as} \quad n \to \infty, \tag{6}$$

where (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$X_n \lambda_n = O(1), \tag{7}$$

$$n\Delta X_n = O(X_n), \tag{8}$$

$$\sum nX_n \mid \Delta^2\lambda_n \mid < \infty, \tag{9}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N,p_n \mid$.

Later on Bor [3] has proved *Theorem 2.* under weaker conditions in the following form.

Theorem 3. *Let (p_n) be as in* Theorem 1. *Let (X_n) be a positive non-decreasing sequence. If the conditions (6) and (7) of* Theorem 2. *are satisfied and the sequences (λ_n) and (β_n) are such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{10}$$

$$\beta_n \to 0 \tag{11}$$

$$\sum nX_n \mid \Delta\beta_n \mid < \infty, \tag{12}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N,p_n \mid$.

2. Main results

The aim of this paper is to prove *Theorem 3* under more weaker conditions. For this we need the concept of an almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [2]). Obviously, every increasing sequence is almost increasing but the converse need not be true, as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem replacing the increasing sequence by any almost increasing sequence. Now we shall prove the following theorem.

Theorem 4. *Let* (p_n) *be as in* Theorem 1 *and let* (X_n) *be an almost increasing sequence. If the conditions (6), (7), (10) and (12) of* Theorem 2. *and* Theorem 3. *are satisfied, then the series* $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $\mid N, p_n \mid$.

We need the following Lemma for the proof of our theorem.

Lemma 1. *Under the conditions on* (X_n)*,* (λ_n) *and* (β_n)*, as taken in the statement of the theorem, the following conditions hold, when (12) is satisfied:*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \tag{14}$$

Proof. Let $Ac_n \leq b_n \leq Bc_n$, where (c_n) is an increasing sequence. In this case

$$\begin{aligned} n\beta_n X_n &\leq nBc_n \mid \sum_{\nu=n}^{\infty} \Delta\beta_\nu \mid \leq nBc_n \sum_{\nu=n}^{\infty} \mid \Delta\beta_\nu \mid \leq B \sum_{\nu=n}^{\infty} \nu c_\nu \mid \Delta\beta_\nu \mid \\ &\leq (A/B) \sum_{\nu=n}^{\infty} \nu X_\nu \mid \Delta\beta_\nu \mid < \infty. \end{aligned}$$

Hence, $n\beta_n X_n = O(1)$ as $n \to \infty$.

Again

$$\begin{aligned} \sum_{n=1}^{\infty} \beta_n X_n &\leq B \sum_{n=1}^{\infty} c_n \beta_n = B \sum_{n=1}^{\infty} c_n \mid \sum_{\nu=n}^{\infty} \Delta\beta_\nu \mid \\ &\leq B \sum_{n=1}^{\infty} c_n \sum_{\nu=n}^{\infty} \mid \Delta\beta_\nu \mid = B \sum_{\nu=1}^{\infty} \mid \Delta\beta_\nu \mid \sum_{n=1}^{\nu} c_n \\ &\leq B \sum_{\nu=1}^{\infty} \nu c_\nu \mid \Delta\beta_\nu \mid \leq (B/A) \sum_{\nu=1}^{\infty} \nu X_\nu \mid \Delta\beta_\nu \mid < \infty. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \beta_n X_n < \infty$. □

Proof of Theorem 4. In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is summable $\mid C,1 \mid$. Our theorem will then follow by means of *Theorem 1.* Let T_n be the n-th $(C,1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu \tag{15}$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu = \frac{1}{n+1}\sum_{\nu=1}^{n} \Delta\lambda_\nu(\nu+1)t_\nu + \lambda_\nu t_\nu \\ &= T_{n,1} + T_{n,2} \quad , \ say. \end{aligned}$$

By (1), to completes the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}(1/n) \mid T_{n,r} \mid < \infty \qquad \text{for } r = 1, 2. \tag{16}$$

Now, we have

$$\begin{aligned} \sum_{n=2}^{m+1}(1/n) \mid T_{n,1} \mid &\leq \sum_{n=2}^{m+1}(1/n(n+1)) \left\{ \sum_{\nu=1}^{n-1}((\nu+1)/\nu)\nu \mid \Delta\lambda_\nu \mid\mid t_\nu \mid \right\} \\ &= O(1) \sum_{n=2}^{m+1}(1/n^2) \left\{ \sum_{\nu=1}^{n-1} \nu\beta_\nu \mid t_\nu \mid \right\} \\ &= O(1) \sum_{\nu=1}^{m} \nu\beta_\nu \mid t_\nu \mid \sum_{n=\nu+1}^{m+1} 1/n^2 = O(1) \sum_{\nu=1}^{m} \nu\beta_\nu \mid t_\nu \mid /\nu \\ &= O(1) \sum_{\nu=1}^{m-1} \Delta(\nu\beta_\nu) \sum_{r=1}^{\nu} \mid t_r \mid /r + O(1)m\beta_m \sum_{\nu=1}^{m} \mid t_\nu \mid /\nu \\ &= O(1) \sum_{\nu=1}^{m-1} \mid \Delta(\nu\beta_\nu) \mid X_\nu + O(1)m\beta_m X_m \\ &= O(1)\sum_{\nu=1}^{m-1} \mid \Delta\beta_\nu \mid \nu X_\nu + O(1)\sum_{\nu=1}^{m-1} \mid \beta_{\nu+1} \mid X_{\nu+1} + O(1)m\beta_m X_m \\ &= O(1) \quad as \quad m \to \infty, \end{aligned}$$

by (6), (10), (12), (13) and (14).
Again

$$\begin{aligned} \sum_{n=1}^{m}(1/n) \mid T_{n,2} \mid &= \sum_{n=1}^{m} \mid \lambda_n \mid (\mid t_n \mid n) \\ &= \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{\nu=1}^{n} \mid t_\nu \mid \nu + \mid \lambda_m \mid \sum_{n=1}^{m} \mid t_n \mid n \\ &= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\ &= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty \ , \end{aligned}$$

by (6), (7), (10) and (14). This completes the proof of the theorem. □

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