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Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation

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Abstract. In this paper, we study the Ostrovsky, Stepanyams and Tsimring equation. We show that the associated initial value problem is locally well-posed in Sobolev spaces $H^s(\mathbb{R})$ for $s > -3/2$. We also prove that our result is sharp in the sense that the flow map of this equation fails to be C^2 in $H^s(\mathbb{R})$ for $s < -3/2$.
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1. Introduction

This paper is concerned with the well-posedness of the following initial value problem (IVP) for the Ostrovsky, Stepanyams and Tsimring (OST) equation:

$$\begin{cases} u_t + u_{xxx} - \eta(\mathscr{H}u_x + \mathscr{H}u_{xxx}) + uu_x = 0, \; x \in \mathbb{R}, \; t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $u = u(x,t)$ is a real-valued function, $\eta > 0$ and $\mathscr{H}$ denotes the usual Hilbert transformation given by

$$\mathscr{H}\varphi(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\varphi(x-y)}{y}\mathrm{d}y,$$

for $\varphi \in \mathscr{S}(\mathbb{R})$. Equation (1) was derived by Ostrovsky et al. in [18] to describe the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer.

We recall that the IVP for (1) is locally well-posed in Banach space X if the solution uniquely exists in a certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T,T]$ whenever initial data belong to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence), i.e. continuity of application $u_0 \mapsto u(t)$ from X to $C([-T,T];X)$.

Note that the OST equation is a modification of the well-known KdV equation

$$u_t + u_{xxx} + uu_x = 0.$$

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It is known that the KdV equation arises in modeling of one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media [1, 4, 12, 22], as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [21]. Different from the KdV equation which is of purely dispersive type, the OST equation is of the dispersive–dissipative type.

A model similar to (1) is the Korteweg-de Vries-Kuramoto-Sivashinsky (KdV-KS) equation

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + u_x^2 = 0,\ x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2}$$

This equation arises as a model for long waves on a viscous fluid flowing down an inclined plane and describing drift waves in plasma [8, 20]. The IVP for (2) was studied by Biagioni et al. [3]. They proved that (2) is well-posed in $H^s(\mathbb{R})$ for $s \geq 1$, by using the properties of the semi-group associated with the linear problem. They also obtained a global solution in $H^s(\mathbb{R})$ for $s \geq 1$, making use of the conserved quantities for the Korteweg-de Vries equation. Recently, Carvajal and Panthee in [7], considered the derivative equation of (2) and obtained the local well-posedness of (2) in $H^s(\mathbb{R})$ for $s > -3/4$ (see also [6]).

The first work on the well-posedness of the IVP for (1) was carried out by Alvarez in [2]. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > 1/2$ and globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$. In [5], Carvajal improved these results. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$, for $s \geq 0$, and globally well-posed in $L^2(\mathbb{R})$. Zhao and Cui in [23] used the ideas of Molinet and Ribaud in [15, 16, 17], employed the method of bilinear estimate in the Bourgain-type spaces and proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$; which coincides with the sharp local well-posedness result for the KdV equation established by Kenig et al. in [14]. The authors in [24] improved their previous results by showing that the IVP for (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -1$.

In this paper we shall prove that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/2$. Indeed, we use purely dissipative methods as applied by Dix in [9] to study the IVP for the KdV-Burgers equation

$$\begin{cases} u_t + u_{xxx} + uu_x = u_{xx},\ x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = u_0(x). \end{cases} \tag{3}$$

The main ingredient consists of applying a fixed-point theorem to the integral equation associated to (1) in time-weighted spaces.

Regarding the sharpness of our result, we establish that the flow map of the OST equation fails to be C^2 in $H^s(\mathbb{R})$ for $s < -3/2$. This means that a Picard iteration cannot be used to obtain a solution of (1).

Before presenting the precise statement of our main result, let us first introduce some definitions and notations.

Without loss of generality, later on we assume that $\eta = 1$. We shall denote by $\widehat{\varphi}$ the Fourier transform of φ, defined as

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x.$$

For $s \in \mathbb{R}$, by $H^s(\mathbb{R})$ we denote the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \left\{ \varphi \in \mathscr{S}'(\mathbb{R}) \ : \ \|\varphi\|_{H^s(\mathbb{R})} < \infty \right\},$$

where

$$\|\varphi\|_{H^s(\mathbb{R})} = \left\| \left(1+\xi^2\right)^{s/2} \widehat{\varphi}(\xi) \right\|_{L^2(\mathbb{R})},$$

and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions.

For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$; and we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $s \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{R})$, consider the following linear problem associated to (1):

$$\begin{cases} u_t + u_{xxx} - \mathscr{H}u_x - \mathscr{H}u_{xxx} = 0, & x \in \mathbb{R}, \ \ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{4}$$

The unique solution of (4) is given by the semigroup $\{U(t)\}_{t \geq 0}$ defined as follows:

$$u(t) = U(t)u_0 = \int_{\mathbb{R}} \mathrm{e}^{t\left(\mathrm{i}\xi^3 - |\xi|^3 + |\xi|\right)} \mathrm{e}^{\mathrm{i}x\xi} \, \hat{u}_0(\xi) \, \mathrm{d}\xi.$$

The main results of this paper read as follows:

Theorem 1. *Let $s > -3/2$. Then for all $u_0 \in H^s(\mathbb{R})$, there exist*

$$T = T\left(\|u_0\|_{H^s(\mathbb{R})}\right) > 0,$$

a space

$$\mathscr{X}_T^s \hookrightarrow C\left([0,T]; H^s(\mathbb{R})\right)$$

and a unique solution $u(t)$ of (1) such that $u(0) = u_0$. Moreover, $u \in C((0,T); H^\infty(\mathbb{R}))$ and the map solution

$$\mathscr{F} : H^s(\mathbb{R}) \longrightarrow \mathscr{X}_T^s \cap C\left([0,T]; H^s(\mathbb{R})\right), \quad u_0 \mapsto u,$$

is smooth.

Theorem 2. *Let $s < -3/2$, if there exists some $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$, then the flow-map data solution*

$$\mathscr{F} : H^s(\mathbb{R}) \longrightarrow C([0,T]; H^s(\mathbb{R})), \quad u_0 \longmapsto u(t)$$

is not C^2 at zero.

The rest of this paper is as follows. In Section 2 we present the time-weighted space $\mathscr{X}_T^s$ and obtain some basic linear and bilinear estimates in this space. Section 3 is devoted to proving the local well-posedness in this space. We also establish that the flow map of the OST equation fails to be C^2 in $H^s(\mathbb{R})$ for $s < -3/2$.

2. Linear and bilinear estimates

In this section, we introduce a suitable Banach space in order to derive appropriates linear and bilinear estimates.

To prove Theorem 1, we will make the assumption $-3/2 < s < 0$, since the case $0 \leq s$ follows by similar arguments. Our strategy is to use a contraction argument on the integral equation associated to (1):

$$u(t) = \Phi(u(t)) := U(t)u_0 + \frac{1}{2}\int_0^t U(t-t')\partial_x(u^2(t'))\,\mathrm{d}t'. \tag{5}$$

For $0 < T \leq T^* = \min\{1, 9|s|/2\}$, we define the Banach space

$$\mathscr{X}_T^s = \left\{u \in C\left([0,T]; H^s(\mathbb{R})\right) \,:\, \|u\|_{\mathscr{X}_T^s} < \infty\right\},$$

where

$$\|u\|_{\mathscr{X}_T^s} = \sup_{t\in[0,T]}\left(\|u(t)\|_{H^s(\mathbb{R})} + t^{|s|/3}\|u(t)\|_{L^2(\mathbb{R})}\right).$$

We note that $T^* = 1$, if $s \leq -2/9$.

First we state the following lemma which is useful in establishing smoothness properties for the semigroup of (1). The proof is straightforward.

Lemma 1. *For any $a > 0$ and $0 < t \leq 9a$, we have for all $\xi \in \mathbb{R}$,*

$$\xi^{2a}\,\mathrm{e}^{-t(|\xi|^3-|\xi|)} \leq \rho^{2a}\mathrm{e}^{-t(\rho^3-\rho)} =: \psi(a,t), \tag{6}$$

where

$$\rho = \frac{\left(9a+\sqrt{81a^2-t^2}\right)^{1/3}}{3}t^{-1/3} + \frac{t^{1/3}}{3\left(9a+\sqrt{81a^2-t^2}\right)^{1/3}}$$

Moreover, if $a = 0$, then (6) holds for $\psi(0,t) = \exp(\frac{2t}{3\sqrt{3}})$.

Now, we will turn our attention to estimate the linear part in $\mathscr{X}_T^s$.

Proposition 1. *Let $0 < T \leq T^*$, $s < 0$ and $u_0 \in H^s(\mathbb{R})$, then*

$$\sup_{t\in[0,T]}\|U(t)u_0\|_{H^s(\mathbb{R})} \leq \mathrm{e}^{\frac{2T}{3\sqrt{3}}}\|u_0\|_{H^s(\mathbb{R})}, \tag{7}$$

and

$$\sup_{t\in[0,T]} t^{|s|/3}\|U(t)u_0\|_{L^2(\mathbb{R})} \lesssim \Upsilon_s(T)\|u_0\|_{H^s(\mathbb{R})}, \tag{8}$$

where

$$\Upsilon_s(t) = \mathrm{e}^{\frac{2t}{3\sqrt{3}}} + t^{|s|/3}\psi(|s|/2, t)$$

is a continuous nondecreasing function on $[0, T^*]$ and ψ is defined as in Lemma 1.

Proof. Inequality (7) follows immediately from Lemma 1. To prove inequality (8), we first observe from $0 < T \leq 1$ that

$$t^{|s|/3} \leq \frac{\left(1+t^{2/3}\xi^2\right)^{|s|/2}}{(1+\xi^2)^{|s|/2}},$$

for all $t \in [0,T]$. Hence, by using the Plancherel theorem and the definition of $U(t)$, we deduce that

$$\begin{aligned} t^{|s|/3}&\|U(t)u_0\|_{L^2(\mathbb{R})} \\ &\leq \left\| \left(1+t^{2/3}\xi^2\right)^{|s|/2} \mathrm{e}^{-t(|\xi^3|-|\xi|)} \left(1+\xi^2\right)^{s/2} \widehat{u_0}(\xi) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \left(\left\|\mathrm{e}^{-t(|\xi^3|-|\xi|)}\right\|_{L^\infty(\mathbb{R})} + \left\|(t^{2/3}\xi^2)^{|s|/2}\mathrm{e}^{-t(|\xi^3|-|\xi|)}\right\|_{L^\infty(\mathbb{R})} \right) \|u_0\|_{H^s(\mathbb{R})}. \end{aligned}$$

Lemma 1 implies the desired inequality in (8). □

The next step is to derive the bilinear estimate.

Proposition 2. *Let* $0 \leq t \leq T \leq T^*$ *and* $s \in (-3/2, 0)$*; then*

$$\left\| \int_0^t U(t-t')\partial_x(uv)(t')\,\mathrm{d}t' \right\|_{\mathscr{X}_T^s} \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} T^{(2s+3)/6} \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s}, \tag{9}$$

for all $u, v \in \mathscr{X}_T^s$*, where the constant of the above inequality depends only on* s*.*

Proof. Let $0 \leq t \leq T$. We have $(1+\xi^2)^{s/2} \leq |\xi|^s$, since $s < 0$. So by using the Minkowski inequality and the definition of $U(t)$, we obtain that

$$\begin{aligned} &\left\| \int_0^t U(t-t')\partial_x(uv)(t')\mathrm{d}t' \right\|_{H^s(\mathbb{R})} \\ &\quad \leq \int_0^t \left\| \xi(1+\xi^2)^{s/2}\, \mathrm{e}^{(t-t')\left(|\xi|-|\xi|^3\right)} (u(t')v(t'))^\wedge(\xi) \right\|_{L^2(\mathbb{R})} \mathrm{d}t' \qquad (10) \\ &\quad \leq \int_0^t \left\| |\xi|^{1+s}\, \mathrm{e}^{(t-t')\left(|\xi|-|\xi|^3\right)} \right\|_{L^2(\mathbb{R})} \left\| \widehat{u(t')} * \widehat{v(t')}(\xi) \right\|_{L^\infty(\mathbb{R})} \mathrm{d}t'. \end{aligned}$$

The Young inequality implies that

$$\left\| \widehat{u(t')} * \widehat{v(t')}(\xi) \right\|_{L^\infty(\mathbb{R})} \leq \frac{\|u\|_{\mathscr{X}_T^s}\|v\|_{\mathscr{X}_T^s}}{|t'|^{2|s|/3}}. \tag{11}$$

Therefore, by changing the variable, we obtain

$$\begin{aligned} &\left\| \int_0^t U(t-t')\partial_x(uv)(t')\,\mathrm{d}t' \right\|_{H^s(\mathbb{R})} \\ &\quad \leq \left(\int_0^t \left\| |\xi|^{1+s}\, \mathrm{e}^{-t'\left(|\xi|^3-|\xi|\right)} \right\|_{L^2(\mathbb{R})} \frac{1}{|t-t'|^{2|s|/3}} \mathrm{d}t' \right) \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s}. \end{aligned} \tag{12}$$

To estimate the integral on the right-hand side of (12), we use a change of the variable to deduce that

$$\begin{aligned} &\left\| |\xi|^{1+s}\, \mathrm{e}^{t'\left(|\xi|-|\xi|^3\right)} \right\|_{L^2(\mathbb{R})} \\ &\quad \le |t'|^{-(2s+3)/6} \left\| \mathrm{e}^{\left(|\xi| t'^{2/3} - |\xi|^3/2\right)} \right\|_{L^\infty(\mathbb{R})} \left\| |\xi|^{1+s} \mathrm{e}^{-|\xi|^3/2} \right\|_{L^2(\mathbb{R})} \\ &\quad \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} |t'|^{-(2s+3)/6}, \end{aligned} \tag{13}$$

where in the last inequality we used the following inequality

$$\mathrm{e}^{(|\xi| t'^{2/3} - |\xi|^3/2)} \le \mathrm{e}^{\frac{2\sqrt{2}}{\sqrt{27}} t'}, \qquad \forall \xi \in \mathbb{R}.$$

Therefore, we get from (12), (13) and a change of the variable that

$$\begin{aligned} &\left\| \int_0^t U(t-t')\partial_x(uv)(t')\mathrm{d}t' \right\|_{H^s(\mathbb{R})} \\ &\quad \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} |T|^{(2s+3)/6} \left(\int_0^1 |t'|^{-(2s+3)/6} |1-t'|^{2s/3}\, \mathrm{d}t' \right) \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s} \\ &\quad \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} |T|^{(2s+3)/6} \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s}, \end{aligned} \tag{14}$$

for all $0 \le t \le T$. On the other hand, a similar argument allows us to deduce for all $0 \le t \le T$ that

$$\begin{aligned} &|t|^{|s|/3} \left\| \int_0^t U(t-t')\partial_x(uv)(t')\mathrm{d}t' \right\|_{L^2(\mathbb{R})} \\ &\quad \le t^{|s|/3} \int_0^t \left\| \xi\, \mathrm{e}^{(t-t')\left(|\xi|-|\xi^3|\right)} \right\|_{L^2(\mathbb{R})} \left\| \widehat{u(t')} * \widehat{v(t')}(\xi) \right\|_{L^\infty(\mathbb{R})} \mathrm{d}t' \\ &\quad \le t^{|s|/3} \left(\int_0^t \left\| |\xi|\, \mathrm{e}^{t'(|\xi|-|\xi^3|)} \right\|_{L^2(\mathbb{R})} \frac{1}{|t-t'|^{2|s|/3}} \mathrm{d}t' \right) \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s} \\ &\quad \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} T^{(2s+3)/6} \left(\int_0^1 |t'|^{-1/2} |1-t'|^{-2|s|/3}\, \mathrm{d}t' \right) \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s} \\ &\quad \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} T^{(2s+3)/6} \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s}. \end{aligned}$$

This completes the proof. □

Remark 1. *If we consider $s' > s > -3/2$, then after modifying the space $\mathscr{X}_T^{s'}$ by*

$$\tilde{\mathscr{X}}_T^{s'} = \left\{ u \in \mathscr{X}_T^{s'};\ \|u\|_{\tilde{\mathscr{X}}_T^{s'}} < \infty \right\}$$

with

$$\|u\|_{\tilde{\mathscr{X}}_T^{s'}} = \|u\|_{\mathscr{X}_T^{s'}} + \sup_{t\in[0,T]} t^{|s|/3} \left\| (1-\partial_x^2)^{(s'-s)/2} u(t) \right\|_{L^2(\mathbb{R})}$$

and using

$$\left(1+\xi^2\right)^{s'/2} \lesssim \left(1+\xi^2\right)^{s/2} \left(1+\xi_1^2\right)^{(s'-s)/2} + \left(1+\xi^2\right)^{s/2} \left(1+(\xi-\xi_1)^2\right)^{(s'-s)/2}$$

and Proposition 2 we can deduce that for $s > s' > -3/2$, we have (see (10)*)*

$$\left\| \int_0^t U(t-t')\partial_x(uv)(t')\,\mathrm{d}t' \right\|_{\tilde{\mathscr{X}}_T^{s'}} \lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} T^{\theta(s)} \left(\|u\|_{\tilde{\mathscr{X}}_T^{s'}} \|v\|_{\mathscr{X}_T^s} + \|v\|_{\tilde{\mathscr{X}}_T^{s'}} \|u\|_{\mathscr{X}_T^s} \right).$$

Remark 2. *We should note that Proposition 2 holds for $s \geq 0$. Indeed since $H^s(\mathbb{R})$ is an algebra for $s > 1/2$, then bilinear estimate* (9) *holds easily. When $s \in [0, 1/2]$, we have*

$$\left\| \int_0^t U(t-t')\partial_x(uv)(t')\,\mathrm{d}t' \right\|_{H^s(\mathbb{R})} \lesssim \left\| \int_0^t \mathscr{V}(t-t') * \partial_x(uv)(t')\,\mathrm{d}t' \right\|_{H^s(\mathbb{R})}, \quad (15)$$

where

$$\mathscr{V}(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{t(\mathrm{i}\xi^3 - |\xi|^3 + |\xi|)} \mathrm{d}\xi.$$

Observe that for any $1 \leq p \leq \infty$ and $\nu \geq 0$, we have for some $K > 0$ that

$$\|D^\nu \mathscr{V}(t)\|_{L^p(\mathbb{R})} \lesssim \mathrm{e}^{Kt} t^{-\frac{1}{3}\left(\nu + \frac{1}{p'}\right)} \lesssim t^{-\frac{1}{3}\left(\nu + \frac{1}{p'}\right)}, \quad (16)$$

for $0 \leq t \leq T \leq 1$, where $\widehat{D^s \mathscr{V}} = |\xi|^s \hat{\mathscr{V}}$. Then by using the fractional Leibnitz rule, we get from (15), (16) *and the Sobolev embedding that*

$$\begin{aligned} &\left\| \int_0^t U(t-t')\partial_x(uv)(t')\,\mathrm{d}t' \right\|_{H^s(\mathbb{R})} \\ &\lesssim \int_0^t \|\partial_x \mathscr{V}(t-t')\|_{L^{2/(2s+1)}(\mathbb{R})} \|\langle D\rangle^s (uv)(t')\|_{L^{1/(1-s)}(\mathbb{R})} \mathrm{d}t' \\ &\lesssim \int_0^t (t-t')^{s/3-1/2} \|u(t')\|_{L^{2/(1-2s)}(\mathbb{R})} \|v(t')\|_{H^s(\mathbb{R})} \\ &\lesssim T^{\theta(s)} \|u\|_{\mathscr{X}_T^s} \|v\|_{\mathscr{X}_T^s}, \end{aligned}$$

where $\langle \cdot \rangle = 1 + |\cdot|$ and $\theta(s) > 0$ for any $s \geq 0$.

Next, we derive a regularity property which will be helpful in the regularity property in Theorem 1.

Proposition 3. *Let $0 \leq t \leq T \leq T^*$, $s \in (-3/2, 0)$ and $\kappa \in [0, s + 3/2)$; then*

$$\mathbb{V} : t \longmapsto \int_0^t U(t-t')\partial_x(u^2(t'))\,\mathrm{d}t',$$

is in $C([0,T]; H^{s+\kappa}(\mathbb{R}))$, for all $u \in \mathscr{X}_T^s$.

Proof. Let $t_0, t_1 \in [0, T]$ be fixed such that $t_0 < t_1$. Then by the Minkowski inequality, we have

$$\|\mathbb{V}(t_1) - \mathbb{V}(t_0)\|_{H^{s+\kappa}(\mathbb{R})} \leq \mathbb{V}_1(t_0, t_1) + \mathbb{V}_2(t_0, t_1),$$

where

$$\mathbb{V}_1(t_0,t_1) = \int_{t_0}^{t_1} \left\| U(t_1-t')\partial_x(u^2(t'))\right\|_{H^{s+\kappa}(\mathbb{R})} \, \mathrm{d}t',$$

and

$$\mathbb{V}_2(t_0,t_1) = \int_{0}^{t_0} \left\| \left(U(t_1-t') - U(t_0-t')\right)\partial_x(u^2(t'))\right\|_{H^{s+\kappa}(\mathbb{R})} \mathrm{d}t'.$$

By performing some straightforward computations, analogously to the proof of Proposition 2, we obtain that

$$\begin{aligned}
\mathbb{V}_1(t_0,t_1) &\le \left(\int_{t_0}^{t_1} \left\|(1+\xi^2)^{(1+s+\kappa)/2}\, \mathrm{e}^{(t_1-t')\left(|\xi|-|\xi^3|\right)}\right\|_{L^2(\mathbb{R})} |t'|^{-2|s|/3}\, \mathrm{d}t'\right) \|u\|^2_{\mathscr{X}_T^s} \\
&\lesssim \left(\int_{t_0}^{t_1} |t_1-t'|^{-(2s+2\kappa+3)/6} \mathrm{e}^{2\sqrt{2}(t_1-t')/\sqrt{27}} |t'-t_0|^{-2|s|/3}\, \mathrm{d}t'\right) \|u\|^2_{\mathscr{X}_T^s} \\
&\lesssim \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} (t_1-t_0)^{(2s-2\kappa+3)/6} \left[\int_0^1 |1-t'|^{-(2s+2\kappa+3)/6} |t'|^{-2|s|/3}\, \mathrm{d}t'\right] \|u\|^2_{\mathscr{X}_T^s}.
\end{aligned}$$

Now, by using the hypotheses, we get that

$$\lim_{t_1\to t_0} \mathbb{V}_1(t_0,t_1) = 0.$$

On the other hand, we have

$$\mathbb{V}_2(t_0,t_1) \le \left(\int_0^{t_0} \|g(t_0,t_1,t',\xi)\|_{L^2(\mathbb{R})} |t'|^{-2|s|/3}\, \mathrm{d}t'\right) \|u\|^2_{\mathscr{X}_T^s},$$

where

$$\begin{aligned}
g(t_0,t_1,t',\xi) =& |\xi|^{s+\kappa+1} \left[e^{(t_1-t')\left(|\xi|-|\xi^3|\right)}\, \mathrm{e}^{\mathrm{i}(t_1-t')\xi^3}\right] \\
&- |\xi|^{s+\kappa+1} \left[\mathrm{e}^{(t_0-t')\left(|\xi|-|\xi^3|\right)}\, \mathrm{e}^{\mathrm{i}(t_0-t')\xi^3}\right].
\end{aligned}$$

It is clear that $g(t_0,t_1,t',\xi)$ tends to zero pointwise for almost every $\xi \in \mathbb{R}$ and $t' \in [0,t_0]$ when $|t_1-t_0| \to 0$. Hence

$$|g(t_0,t_1,t',\xi)| \lesssim \chi_{\{|\xi|\le 1\}}(\xi) \mathrm{e}^{2\sqrt{2}T/\sqrt{27}} + |\xi|^{s+\kappa+1} \mathrm{e}^{(t_0-t')\left(|\xi|-|\xi^3|\right)}.$$

Thus, we deduce from the Lebesgue dominated convergence theorem that

$$\|g(t_0,t_1,t',\xi)\|_{L^2(\mathbb{R})} \longrightarrow 0,$$

as $t_1 \to t_0$. Using again the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{t_1\to t_0} \mathbb{V}_2(t_0,t_1) = 0.$$

This completes the proof. □

3. Local existence and ill-posedness

All the elements are now in place to mount a proof of the local well-posedness result in Theorem 1.

Proof of Theorem 1. Let $s > -3/2$ and $u_0 \in H^s(\mathbb{R})$. We are going to show that the operator Φ defined in (5) is a contraction in some closed ball of $\mathscr{X}_T^s$. By Propositions 1 and 2, there exist two positive constant $C = C(s)$ and $\theta = \theta(s)$ such that

$$\|\Phi(u)\|_{\mathscr{X}_T^s} \leq C \left(\|u_0\|_{H^s(\mathbb{R})} + T^\theta \|u\|^2_{\mathscr{X}_T^s} \right), \tag{17}$$

and

$$\|\Phi(u) - \Phi(v)\|_{\mathscr{X}_T^s} \leq CT^\theta \|u - v\|_{\mathscr{X}_T^s} \|u + v\|_{\mathscr{X}_T^s}, \tag{18}$$

for all $u, v \in \mathscr{X}_T^s$ and $0 < T \leq T^*$. Now we define

$$\mathscr{X}_T^s(b) = \left\{ u \in \mathscr{X}_T^s \ : \ \|u\|_{\mathscr{X}_T^s} \leq b \right\} \quad \text{with} \quad b = 2C\|u_0\|_{H^s(\mathbb{R})}$$

and we choose

$$0 < T < \min \left\{ 1, (2Cb)^{-1/\theta} \right\}.$$

Estimates (17) and (18) imply that Φ is a contraction on the Banach space $\mathscr{X}_T^s(b)$; so that we deduce by the fixed point theorem, the existence of a unique solution u of the integral equation (5) in $\mathscr{X}_T^s(b)$ with the initial data $u(0) = u_0$. Note that Proposition 3 assures that $\Phi(u) \in C\left([0, T]; H^s(\mathbb{R})\right)$.

The uniqueness of the solution of (5) on the whole space $\mathscr{X}_T^s$ and the smoothness of the flow map solution follow by standard arguments (see for example [13]).

Note that a similar contraction argument shows that the existence result holds for any $s' > s > -3/2$, in the time interval $[0, T]$ with $T = T(\|u_0\|_{H^s(\mathbb{R})})$ (see Remark 1). Finally, we know that the map $t \longmapsto U(t)u_0$ is continuous in the time interval $(0, T]$ with respect to the topology of $H^\infty(\mathbb{R})$. Since our solution u belongs to $\mathscr{X}_T^s$, we deduce from Proposition 3 that there exists $\kappa > 0$ such that the map $\mathbb{V}$ belongs to $C\left([0, T]; H^{s+\kappa}(\mathbb{R})\right)$, so that

$$u \in C\left((0, T]; H^{s+\kappa}(\mathbb{R})\right).$$

Therefore, by a standard bootstrapping argument, using the uniqueness result and the fact that the time interval of the existence of the solutions depends only on the $H^s(\mathbb{R})$-norm of the initial data, we deduce that

$$u \in C\left((0, T]; H^\infty(\mathbb{R})\right).$$

□

Remark 3. *A standard argument similar to [3], one can observe that if* $u_0 \in H^s(\mathbb{R})$, *for* $s \geq 0$, *the corresponding local solution of* (1) *extends globally in time. More precisely, since the solution* u *of* (1) *is in* $C((0, T]; H^\infty(\mathbb{R}))$, *one only needs to prove an a priori estimate for* u. *So* u *solves the Cauchy problem* (1) *in the classical sense. Recall that* $T = T\left(\|u_0\|_{H^s(\mathbb{R})}\right)$. *This allows us to take the* L^2*-scalar product of* (1) *with* u, *integrate by parts and use the properties of the Hilbert transform (see for* *example [10, 11]), the Gagliardo-Nierenberg inequality and the Young inequality to obtain*

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R})} &= \|D^{1/2}u\|^2_{L^2(\mathbb{R})} - \|D^{3/2}u\|^2_{L^2(\mathbb{R})}\\ &\le C\|D^{3/2}u\|^{2/3}_{L^2(\mathbb{R})}\|u\|^{4/3}_{L^2(\mathbb{R})} - \|D^{3/2}u\|^2_{L^2(\mathbb{R})} \le C\|u\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where $C>0$ is independent of t. Then by the Gronwall inequality, it yields

$$\|u(t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}\mathrm{e}^{CT}, \qquad \text{for all } t\in[0,T].$$

Next, we are going to show that our well-posedness result is sharp. We will first prove that we cannot solve the Cauchy problem (1) in $H^s(\mathbb{R})$ using the fixed point theorem when $s<-3/2$. Then we show that this fact implies Theorem 2.

Remark 4. *With a slight modification, the proofs of Theorems 2 and 3 (below) are very similar to Pastrán's results in his thesis [19]. The author should mention that he proved Theorems 2 and 3 independent of Pastrán's thesis in [19], and for the sake of completeness of this paper, the author gives the proofs in details here.*

Theorem 3. *Let $s<-3/2$ and $T>0$. Then, there does not exist any space $\mathscr{X}^s_T$ such that $\mathscr{X}^s_T$ is continuously embedded in $C([0,T];H^s(\mathbb{R}))$, i.e.*

$$\|u\|_{L^\infty_T H^s(\mathbb{R})} \lesssim \|u\|_{\mathscr{X}^s_T}, \quad \forall\, u\in\mathscr{X}^s_T \tag{19}$$

and such that

$$\|U(t)u_0\|_{\mathscr{X}^s_T} \lesssim \|u_0\|_{H^s(\mathbb{R})}, \quad \forall\, u_0\in H^s(\mathbb{R}) \tag{20}$$

and

$$\left\|\int_0^t U(t-t')(uv)_x(t')\,\mathrm{d}t'\right\|_{\mathscr{X}^s_T} \lesssim \|u\|_{\mathscr{X}^s_T}\|v\|_{\mathscr{X}^s_T}, \tag{21}$$

for all $u,v\in\mathscr{X}^s_T$.

Proof. Suppose that there exists a space $\mathscr{X}^s_T$ as in Theorem 3. Take $u_0\in H^s(\mathbb{R})$, $u(t)=U(t)u_0$, and fix $0<t<T$. Then by using relations (19), (20) and (21), we see that

$$\left\|\int_0^t U(t-t')\partial_x\left((U(t')u_0)^2\right)\,\mathrm{d}t'\right\|_{H^s(\mathbb{R})} \lesssim \|u_0\|^2_{H^s(\mathbb{R})}. \tag{22}$$

We will show that (22) fails for an appropriate choice of u_0, which would lead to a contradiction. Define u_0 by

$$\widehat{u_0}(\xi) = N^{-s}\gamma^{-1/2}(\chi_{I_1}(\xi)+\chi_{I_2}(\xi)),$$

where $N\gg 1$, $\gamma=N^{1-\epsilon_0}$ ($0<\epsilon_0\ll 1$ fixed) and

$$I_1=[N,N+2\gamma], \quad I_2=[-N-2\gamma,-N].$$

It is easy to see that

$$\|u_0\|_{H^s(\mathbb{R})}\sim 1.$$

Then, we use the definition of $U(t)$ and Fubini's theorem to get

$$\begin{aligned}\left|\widehat{h(\cdot,t)}(\xi)\right| &:= \left|\left(\int_0^t U(t-t')\partial_x\left((U(t')u_0)^2\right)\,\mathrm{d}t'\right)^{\wedge}(\xi)\right| \\ &= \left|\int_0^t \mathrm{i}\xi \mathrm{e}^{(t-t')\left(\mathrm{i}\xi^3-(|\xi|^3-|\xi|)\right)}\widehat{U(t')u_0} * \widehat{U(t')u_0}\,(\xi)\,\mathrm{d}t'\right| \\ &= \left|\mathrm{e}^{\mathrm{i}t\xi^3}\int_{\mathbb{R}} \mathrm{i}\xi\widehat{u}_0(\xi_1)\widehat{u}_0(\xi_2)f(t,\xi,\xi_1)\,\mathrm{d}\xi_1\right| \\ &\gtrsim \left|\frac{1}{\gamma N^{2s}}\int_{\mathscr{M}} \xi f(t,\xi,\xi_1)\,\mathrm{d}\xi_1\right|,\end{aligned}$$

where

$$f(t,\xi,\xi_1) = \frac{\mathrm{e}^{-t(|\xi_2^3|-|\xi_2|+|\xi_1^3|-|\xi_1|)}\mathrm{e}^{\mathrm{i}t(\xi_1^3+\xi_2^3-\xi^3)} - \mathrm{e}^{-t(|\xi^3|-|\xi|)}}{\omega(\xi,\xi_1)},$$

$\xi_2 = \xi - \xi_1$,

$$\omega(\xi,\xi_1) = |\xi_1| - |\xi_1^3| - |\xi_2^3| + |\xi_2| + |\xi^3| + |\xi| + 3\mathrm{i}\xi\xi_1\xi_2.$$

and

$$\mathscr{M} = \{\xi_1 \;:\; \xi_1 \in I_1,\ \xi_2 \in I_2\}.$$

When $\xi_1 \in I_1$ and $\xi_2 \in I_2$, we deduce that $\xi \in [2N, 2N+4\gamma]$ and $\omega(\xi,\xi_1) \lesssim N^3$. Now we choose a sequence of times $t_N = N^{-3-\epsilon_0}$, so that $\mathrm{e}^{-(|\xi^3|-|\xi|)t_N} \sim \mathrm{e}^{-N^3 t_N} \sim \mathrm{e}^{-N^{-\epsilon_0}} > C > 0$. Hence

$$\left|\frac{\mathrm{e}^{-t(|\xi_2^3|-|\xi_2|+|\xi_1^3|-|\xi_1|-|\xi^3|+|\xi|)}\mathrm{e}^{\mathrm{i}t(\xi_1^3+\xi_2^3-\xi^3)} - 1}{\omega(\xi,\xi_1)}\right| = \frac{1}{N^{3+\epsilon_0}} + O\left(\frac{1}{N^{3+2\epsilon_0}}\right).$$

Therefore,

$$\|h(\cdot,t)\|_{H^s(\mathbb{R})} \gtrsim N^{-s-3/2-3\epsilon_0/2}.$$

Hence, we obtain that

$$N^{-s-3/2-3\epsilon_0/2} \lesssim 1, \quad \forall\, N \gg 1;$$

which contradicts the assumption $s < -3/2$. □

A proof of Theorem 2 is now in sight.

Proof of Theorem 2. Let $s < -3/2$, suppose that there exists $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$ in the time interval $[0,T]$ and that the flow map solution $\mathscr{F} : H^s(\mathbb{R}) \longrightarrow C([0,T]; H^s(\mathbb{R}))$ is C^2 at the origin. When $u_0 \in H^s(\mathbb{R})$, we will denote $u_{u_0}(t) = \mathscr{F}(u_0)(t)$ the solution of equation (1) with initial datum u_0. This means that u_{u_0} is a solution of the integral equation

$$u_{u_0}(t) = \mathscr{F}(u_0)(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-t')\partial_x\left(u_{u_0}^2\right)(t')\,\mathrm{d}t'.$$

By computing the Fréchet derivative of $\mathscr{F}$ at φ in the direction u_0, we obtain that

$$d_\varphi \mathscr{F}(u_0)(t) = U(t)u_0 - \int_0^t U(t-t')\mathscr{B}\left[u_\varphi(t'), d_\varphi \mathscr{F}(u_0)(t')\right]\ \mathrm{d}t', \tag{23}$$

where $\mathscr{B}[\varphi,\psi] = (\varphi\psi)_x$. Since the Cauchy problem (1) is supposed to be well-posed, we know by using the uniqueness that $\mathscr{F}(0)(t) = u_0(t) = 0$ and then we deduce from (23) that

$$d_0 \mathscr{F}(u_0)(t) = U(t)u_0. \tag{24}$$

Using (23), we compute the second Fréchet derivative at the origin in the direction (u_0,ψ) and using (24), we deduce that

$$d_0^2 \mathscr{F}(u_0,\psi)(t) = -\int_0^t U(t-t')\mathscr{B}\left[U(t')\psi, U(t')u_0\right]\ \mathrm{d}t'.$$

The assumption of C^2 regularity of $\mathscr{F}$ at the origin would imply that

$$d_0^2 \mathscr{F} \in \mathscr{L}\left(H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R})\right),$$

which would lead to the following inequality

$$\left\| d_0^2 \mathscr{F}(u_0,\psi)(t)\right\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})}\|\psi\|_{H^s(\mathbb{R})}, \tag{25}$$

for all $u_0,\psi \in H^s(\mathbb{R})$. But (25) is equivalent to (22) which has been shown to fail in the proof of Theorem 3. □

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