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# On the fine spectrum of the upper triangle double band matrix $\Delta^+$ on the sequence space $c_0$

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**Abstract.** In this study, we determine the fine spectrum of the matrix operator $\Delta^+$ defined by an upper triangle double band matrix acting on the sequence space $c_0$ with respect to the Goldberg's classification. As a new development, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $\Delta^+$ on $c_0$.

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**Key words**: spectrum, fine spectrum, Goldberg's classification, approximate point spectrum, defect spectrum, compression spectrum

## 1. Introduction

Let $X$ and $Y$ be Banach spaces, and $T : X \to Y$ also be a bounded linear operator. By $R(T)$ we denote the range of $T$, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$ we also denote the set of all bounded linear operators on $X$ into itself. If $X$ is any Banach space and $T \in B(X)$, then the adjoint $T^*$ of $T$ is a bounded linear operator on the dual $X^*$ of $X$ defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $T \in B(X)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is a bijection}\}$$

is called the *resolvent set* of $T$ and its complement with respect to the complex plain

$$\sigma(T) := \mathbb{C}\backslash\rho(T)$$

is called the *spectrum* of $T$. By the closed graph theorem, the inverse operator

$$R(\lambda; T) := (\lambda I - T)^{-1},\ (\lambda \in \rho(T)) \tag{1}$$

is always bounded and is usually called *resolvent operator* of $T$ at $\lambda$.

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## 2. Subdivisions of the spectrum

In this section, we give the definitions of the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

### 2.1. The point spectrum, continuous spectrum and residual spectrum

The name *resolvent* is appropriate, since $T_\lambda^{-1}$ helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1}y$ provided $T_\lambda^{-1}$ exists. More important, the investigation of properties of $T_\lambda^{-1}$ will be basic for an understanding of the operator $T$ itself. Naturally, many properties of $T_\lambda$ and $T_\lambda^{-1}$ depend on $\lambda$, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all $\lambda$'s in the complex plane such that $T_\lambda^{-1}$ exists. Boundedness of $T_\lambda^{-1}$ is another property that will be essential. We shall also ask for what $\lambda$'s the domain of $T_\lambda^{-1}$ is dense in $X$, to name just a few aspects. A *regular value* $\lambda$ of $T$ is a complex number such that $T_\lambda^{-1}$ exists and bounded and whose domain is dense in $X$. For our investigation of $T$, $T_\lambda$ and $T_\lambda^{-1}$, we need some basic concepts in spectral theory which are given as follows (see [22, pp. 370-371]):

The *resolvent set* $\rho(T,X)$ of $T$ is the set of all regular values $\lambda$ of $T$. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that $T_\lambda^{-1}$ does not exist. An $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of $T$.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that $T_\lambda^{-1}$ exists and is unbounded and the domain of $T_\lambda^{-1}$ is dense in $X$.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that $T_\lambda^{-1}$ exists (and may be bounded or not) but the domain of $T_\lambda^{-1}$ is not dense in $X$.

Therefore, these three subspectras form disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

### 2.2. The approximate point spectrum, defect spectrum and compression spectrum

In this subsection, following Appell et al. [7], we give the definitions of the three more subdivisions of the spectrum called the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator $T$ in a Banach space $X$, we call a sequence $(x_k)$ in $X$ a *Weyl sequence* for $T$ if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{3}$$

the *approximate point spectrum of* $T$. Moreover, the subspectrum

$$\sigma_\delta(T,X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{4}$$

is called the *defect spectrum* of $T$.

The two subspectra given by (3) and (4) form (not necessarily disjoint) subdivisions

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}$$

which is often called the *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in (2) we note that

$$\sigma_r(T,X) = \sigma_{co}(T,X) \backslash \sigma_p(T,X),$$
$$\sigma_c(T,X) = \sigma(T,X) \backslash [\sigma_p(T,X) \cup \sigma_{co}(T,X)].$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

**Proposition 1** ([7, Proposition 1.3, p. 28]). *Spectra and subspectra of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations:*

(*a*) $\sigma(T^*,X^*) = \sigma(T,X)$;

(*b*) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$;

(*c*) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$;

(*d*) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$;

(*e*) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$;

(*f*) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$;

(*g*) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

Relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum, and the point spectrum dual to the compression spectrum.

Equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if $X$ is a Hilbert space and $T$ is normal. Roughly speaking, this shows that normal (in particular, selfadjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [7]).

## 2.3. Goldberg's classification of spectrum

If $X$ is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$;

(II) $R(T) \neq \overline{R(T)} = X$;

(III) $\overline{R(T)} \neq X$;

and

(1) $T^{-1}$ exists and is continuous,

(2) $T^{-1}$ exists but is discontinuous,

(3) $T^{-1}$ does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1$, $I_2$, $I_3$, $II_1$, $II_2$, $II_3$, $III_1$, $III_2$, $III_3$. If an operator is e.g. in state $III_2$, then $\overline{R(T)} \neq X$ and $T^{-1}$ exists but is discontinuous (see [17]).

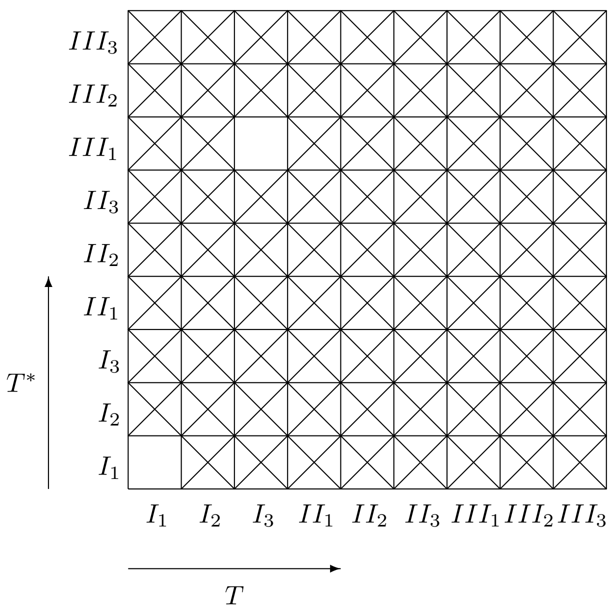


Table 1: *State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space $X$*

If $\lambda$ is a complex number such that $T_\lambda = \lambda I - T \in I_1$ or $T_\lambda = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of $\lambda$ not in $\rho(T, X)$ comprise the spectrum of $T$. Further classification of $\sigma(T, X)$ gives rise to the fine spectrum of $T$. That is, $\sigma(T, X)$ can be divided into subsets $I_2\sigma(T, X) = \emptyset$, $I_3\sigma(T, X)$, $II_2\sigma(T, X)$, $II_3\sigma(T, X)$,

$III_1\sigma(T,X)$, $III_2\sigma(T,X)$, $III_3\sigma(T,X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, say $III_2$, then we write $\lambda \in III_2\sigma(T,X)$.

By the definitions given above, we can illustrate subdivisions (2) in the following table:

| | | 1 | 2 | 3 |
|---|---|---|---|---|
| | | $T_\lambda^{-1}$ exists<br>and is bounded | $T_\lambda^{-1}$ exists<br>and is unbounded | $T_\lambda^{-1}$<br>does not exist |
| I | $R(\lambda I - T) = X$ | $\lambda \in \rho(T,X)$ | – | $\lambda \in \sigma_p(T,X)$<br>$\lambda \in \sigma_{ap}(T,X)$ |
| II | $\overline{R(\lambda I - T)} = X$ | $\lambda \in \rho(T,X)$ | $\lambda \in \sigma_c(T,X)$<br>$\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$ | $\lambda \in \sigma_p(T,X)$<br>$\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$ |
| III | $\overline{R(\lambda I - T)} \neq X$ | $\lambda \in \sigma_r(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$<br><br>$\lambda \in \sigma_{co}(T,X)$ | $\lambda \in \sigma_r(T,X)$<br>$\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$<br>$\lambda \in \sigma_{co}(T,X)$ | $\lambda \in \sigma_p(T,X)$<br>$\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$<br>$\lambda \in \sigma_{co}(T,X)$ |

Table 2: *Subdivisions of spectrum of a linear operator*

Observe that the case in the first row and second column cannot occur in a Banach space $X$, by the closed graph theorem. If we are not in the third column, i.e., if $\lambda$ is not an eigenvalue of $T$, we may always consider the resolvent operator $T_\lambda^{-1}$ (on a possibly “thin”domain of definition) as an “algebraic” inverse of $\lambda I - T$.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains $\phi$, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write $\ell_\infty$, $c$, $c_0$ and $bv$ for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$, while $\phi$ is not a Banach space with respect to any norm, respectively. Also, by $\ell_p$ we denote the space of all $p$–absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$, where $1 \leq p < \infty$.

We give a short survey concerning the spectrum and the fine spectrum of linear operators defined by some particular limitation matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space $\ell_p$ has been studied by Gonzàlez [18], where $1 < p < \infty$. Also, the spectrum of the weighted mean matrices of operators on $\ell_p$ have been investigated by Cartlidge [12]. The spectrum of the Cesàro operator of order one on the sequence spaces $bv_0$ and $bv$ has also been investigated by Okutoyi [24, 25]. The fine spectrum of the difference operator $\Delta$ over the sequence spaces $c_0$ and $c$ has been studied by Altay and Başar [3]. The same authors have studied the fine spectrum of the generalized difference operator $B(r,s)$ over $c_0$ and $c$, in [4]. The fine spectra of $\Delta$ over $\ell_1$ and $bv$ have been studied by Kayaduman and Furkan [21]. Recently, the fine spectra of the difference operator $\Delta$ over the sequence spaces $\ell_p$ and $bv_p$ have been studied by Akhmedov and Başar [1, 2], where $bv_p$ is the space of $p$-bounded variation sequences and introduced by Başar and Altay [8], and $1 \le p < \infty$. Also, the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_1$ and $bv$ has been studied by Furkan et al. [16]. Recently, the fine spectrum of $B(r,s,t)$ over the sequence spaces $c_0$ and $c$ has been studied by Furkan et al. [14]. Quite recently, de Malafosse [23]; Altay and Başar [8] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces $s_r$ and $c_0$, $c$; where $s_r$ denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}$ , $(r > 0)$. Durna and Yildirim [13] have examined the subdivision of the spectrum of factorable matrices on the sequence spaces $c$ and $\ell^p$. Altay and Karakuş [5] have determined the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces $\ell_1$ and $bv$. Also, the fine spectrum of the same operator over $\ell_1$ and $bv$ has been studied by Bilgiç and Furkan [11]. More recently, the fine spectrum of the operator $B(r,s)$ over $\ell_p$ and $bv_p$ has been studied by Bilgiç and Furkan [10]. The fine spectra of lacunary matrices have been studied by Altun and Karakaya [6]. In 2010, Srivastava and Kumar [26] determined the spectra and the fine spectra of generalized difference operator $\Delta_\nu$ on $\ell_1$, where $\Delta_\nu$ is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_k)$. Finally, the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces $\ell_p$ and $bv_p$ with $1 < p < \infty$ has been studied by Furkan et al. [15]. Quite recently, the fine spectra of the operator $U(s,r,s)$ defined by symmetric triple band matrix over the sequence spaces $\ell_p$ and $bv_p$ has been determined by Karakaya et al. [20].

In this paper, we study the fine spectrum of the matrix operator $\Delta^+$ defined by an upper triangle double band matrix acting on the sequence space $c_0$ with respect to the Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $\Delta^+$ on the space $c_0$.

Now, we may quote the following lemmas which are needed in proving the next theorems:

**Lemma 1** ([27, Theorem 1.3.6, p. 6]). *The matrix $A = (a_{nk})$ gives raise to a bounded linear operator $T \in B(c_0)$ from $c_0$ to itself if and only if the following statements hold:*

1. *The rows of $A$ are in $\ell_1$ and their $\ell_1$ norms are bounded.*

2. *The columns of $A$ are in $c_0$.*

*The operator norm of $T$ is the supremum of $\ell_1$ norms of the rows.*

**Lemma 2** ([17, p. 59]). *$T$ has a dense range if and only if $T^*$ is one to one, where $T^*$ denotes the adjoint operator of the operator $T$.*

**Lemma 3** ([21, Theorem 2.6]). $\sigma_p(\Delta, c_0) = \emptyset$.

**Lemma 4** ([3, Theorem 2.6]). $\sigma_c(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$.

## 3. The spectrum and the fine spectrum of the upper triangle double band matrix $\Delta^+$ on the sequence space $c_0$

In this section, we study the spectrum and the fine spectrum of the operator represented by the upper triangle double band matrix $\Delta^+$ on the sequence space $c_0$. The operator $\Delta^+$ is represented by the matrix

$$\Delta^+ = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

First, we give the following theorem which presents the null space $\mathcal{N}(\Delta^+ - \lambda I)$ of the operator $\Delta^+ - \lambda I$ on the sequence space $c_0$.

**Theorem 1.** *Let $|\lambda - 1| \geq 1$. Then, the null space $\mathcal{N}(\Delta^+ - \lambda I)$ of the operator $\Delta^+ - \lambda I$ on the sequence space $c_0$ is $\{\theta\}$.*

**Proof**. By solving the system of linear equations

$$\left.\begin{aligned} x_0 - x_1 &= \lambda x_0 \\ x_1 - x_2 &= \lambda x_1 \\ x_2 - x_3 &= \lambda x_2 \\ &\vdots \end{aligned}\right\}$$

we obtain that

$$x_n = (1-\lambda)^{n-n_0} x_{n_0},\ (n > n_0)$$

for $x_{n_0} \neq 0$ which leads us to the fact that

$$\mathcal{N}(\Delta^+ - \lambda I) = \{x = (x_n) \in c_0 : x_n = (1-\lambda)^{n-n_0} x_{n_0},\ (n > n_0)\}.$$

Therefore, it is clear that the null space $\mathcal{N}(\Delta^+ - \lambda I)$ of the operator $\Delta^+ - \lambda I$ is $\{\theta\}$ for $|\lambda - 1| \geq 1$, as asserted. □

**Theorem 2.** $\sigma(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.

**Proof**. Define $D = \{\lambda \in \mathbb{C} : |\lambda - 1| > 1\}$. It is enough to prove that $(\Delta^+ - \lambda I)^{-1}$ exists and is in $(c_0 : c_0)$ for $\lambda \in D$ and $(\Delta^+ - \lambda I)^{-1} \notin (c_0 : c_0)$ for $\lambda \notin D$. Let $\lambda \in D$. Since $\Delta^+ - \lambda I$ is triangle, so $(\Delta^+ - \lambda I)^{-1}$ exists and solving $(\Delta^+ - \lambda I)x = y$ for $x$ in terms of $y$ gives the matrix of $B = (b_{nk})$ of the equation $x = By$. Therefore the matrix $B = (b_{nk})$ is given by

$$b_{nk} = \begin{cases} (1-\lambda)^{n-k-1} & , k \geq n, \\ 0 & , k < n. \end{cases}$$

Then, we have

$$\begin{aligned}\|(\Delta^+ - \lambda I)^{-1}\|_{(c_0:c_0)} &= \sup_{n\in\mathbb{N}} \sum_{k=n}^{\infty} \frac{|1-\lambda|^n}{|1-\lambda|^{k+1}} \\ &= \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \frac{1}{|1-\lambda|^{k+1}} < \infty. \end{aligned} \tag{5}$$

This shows that $(\Delta^+ - \lambda I)^{-1} \in (c_0 : c_0)$. Also from (5) we can show that

$$\|(\Delta^+ - \lambda I)^{-1}\|_{(c_0:c_0)} = \infty$$

for $\lambda \notin D$. This completes the proof of the theorem. □

**Theorem 3.** $\sigma_p(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

**Proof**. Suppose $\Delta^+ x = \lambda x$ for $x \neq \theta = (0, 0, 0, ...)$ in $c_0$. Then, by solving the system of linear equations

$$\left.\begin{aligned} x_0 - x_1 &= \lambda x_0 \\ x_1 - x_2 &= \lambda x_1 \\ x_2 - x_3 &= \lambda x_2 \\ &\vdots \end{aligned}\right\}$$

we can find that, if $x_{n_0}$ is the first non-zero entry of the sequence $x = (x_n)$, then we have

$$x_n = (1-\lambda)^{n-n_0} x_{n_0},\ (n > n_0).$$

Hence, we show that $|1 - \lambda| < 1$ if and only if $x \in c_0$. □

Since the adjoint operator of matrix transformation on $c_0$ is a transpose of the matrix, then by Lemma 3 we have the following result:

**Corollary 1.** $\sigma_p(\Delta^{+^*}, c_0^*) = \emptyset$.

**Theorem 4.** $\sigma_c(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$.

**Proof**. From Theorem 3, $\lambda \notin \sigma_p(\Delta^+, c_0)$. Then $\lambda I - \Delta^+$ is one to one, and hence has an inverse. Also, by Corollary 1 $\lambda I - \Delta^{+^*}$ is one to one and by Lemma 2 we have

$$\overline{\mathcal{R}(\lambda I - \Delta^+)} = c_0.$$

This completes the proof of the theorem. □

**Theorem 5.** $\sigma_r(\Delta^+, c_0) = \emptyset$.

**Proof**. Since the set of the spectrum is a union of the point spectrum, the continuous spectrum and the residual spectrum, then from Theorems 2, 3 and 4 we observe that $\sigma_r(\Delta^+, c_0) = \emptyset$. This completes the proof of the theorem. □

**Theorem 6.** *If* $\lambda \neq 1$ *and* $\lambda \in \sigma_p(\Delta^+, c_0)$, *then* $\lambda \in II_3\sigma(\Delta^+, c_0)$.

**Proof**. By Theorem 3 we have $\lambda I - \Delta^+ \in II_3 \cup I_3$. Now, we show that the operator $\lambda I - \Delta^+$ is not onto. For the sequence $y = \{(1-\lambda)^n\} \in c_0$, since $x \in c_0$ is not present that supplies the equality $(\lambda I - \Delta^+)x = y$, so the transformation $\lambda I - \Delta$ is not onto which is what we wished to prove. □

**Theorem 7.** $1 \in I_3\sigma(\Delta^+, c_0)$.

**Proof**. For $\lambda = 1$, the matrix $\lambda I - \Delta^+ = I - \Delta^+$ is

$$\lambda I - \Delta^+ = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By Theorem 3 it is clear that $I - \Delta^+ \in 3$.

To show $I - \Delta^+ \in I$, we show that the transformation $I - \Delta^+$ is onto. For $x, y \in c_0$, from the equation $(I - \Delta^+)x = y$, we have

$$\left.\begin{aligned} x_1 &= y_0 \\ x_2 &= y_1 \\ x_3 &= y_2 \\ &\vdots \end{aligned}\right\}$$

such that for every $y \in c_0$ there is a sequence $x \in c_0$. Then, this shows that the transformation $I - \Delta^+$ is onto. This step concludes the proof. □

**Theorem 8.** *If* $\lambda \in \sigma_c(\Delta^+, c_0)$ *then* $\lambda I - \Delta^+ \in II_2\sigma(\Delta^+, c_0)$ .

**Proof**. Let $|\lambda - 1| = 1$. By Theorem 3 the transformation $\lambda I - \Delta^+$ has an inverse, from Theorem 2 the transformation $(\lambda I - \Delta^+)^{-1}$ is discontinuous and by Theorem 4, since $\overline{\mathcal{R}(\lambda I - \Delta^+)} = c_0$, we have

$$\lambda I - \Delta^+ \in I_2 \cup II_2\sigma(\Delta^+, c_0).$$

Also, from Theorem 6 we can obtain that the transformation $\lambda I - \Delta^+$ is not onto. This step completes the proof. □

**Theorem 9.** *The following statements hold:*

(*a*) $\sigma_{ap}(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$;

(*b*) $\sigma_{\delta}(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$;

(*c*) $\sigma_{co}(\Delta^+, c_0) = \emptyset$.

**Proof**. Since the following equality

$$\sigma\left(\Delta^+, c_0\right) = I_3\sigma\left(\Delta^+, c_0\right) \cup II_2\sigma\left(\Delta^+, c_0\right) \cup II_3\sigma\left(\Delta^+, c_0\right)$$

holds by Theorems 2-7, and the subdivisions in the Goldberg's classification are disjoint, then we must have

$$III_1\sigma\left(\Delta^+, c_0\right) = III_2\sigma\left(\Delta^+, c_0\right) = III_3\sigma\left(\Delta^+, c_0\right) = \emptyset.$$

(a) $\sigma_{ap}(\Delta^+, c_0) = \sigma(\Delta^+, c_0) \setminus III_1\sigma(\Delta^+, c_0)$ is obtained from Table 2. Then, one can easily see that

$$\sigma_{ap}\left(\Delta^+, c_0\right) = \sigma\left(\Delta^+, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\},$$

as desired.

(b) Since the following equality

$$\sigma_\delta\left(\Delta^+, c_0\right) = \sigma\left(\Delta^+, c_0\right) \setminus I_3\sigma\left(\Delta^+, c_0\right)$$

holds from Table 2, we derive that $\sigma_\delta(\Delta^+, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.

(c) From Table 2,

$$\sigma_{co}\left(\Delta^+, c_0\right) = III_1\sigma\left(\Delta^+, c_0\right) \cup III_2\sigma\left(\Delta^+, c_0\right) \cup III_3\sigma\left(\Delta^+, c_0\right),$$

then we have $\sigma_{co}(\Delta^+, c_0) = \emptyset$. □

The following corollary can be obtained by Proposition 1.

**Corollary 2.** *The following statements hold:*

(*a*) $\sigma_{ap}((\Delta^+)^* = \Delta, c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$*;*

(*b*) $\sigma_\delta((\Delta^+)^* = \Delta, c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$*;*

(*c*) $\sigma_p((\Delta^+)^* = \Delta, c_0^* \cong \ell_1) = \emptyset$*, see [21, Theorem 2.6].*

## 4. Conclusion

Many researchers have determined the spectrum and the fine spectrum of a matrix operator in some sequence spaces. In addition to this, we add the definition of some new divisions of the spectrum called the approximate point spectrum, the defect spectrum and the compression spectrum of the matrix operator and give the related results for the matrix operator $\Delta^+$ on the space $c_0$ which is a new development for this type works giving the fine spectrum of a matrix operator on a sequence space with respect to the Goldberg's classification.

We should note here that although the continuous dual of the sequence spaces $c_0$ and $c$ is the space $\ell_1$ the adjoint of the matrix operator $\Delta^+$ on the spaces $c_0$ and $c$ are respectively defined by

$$(\Delta^+)^* = \Delta \quad \text{and} \quad (\Delta^+)^* = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix}.$$

So, although the corresponding results of the present study coincides with the results of Başar et al. [9], the adjoint operators are different.

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