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Parameterized codes over some embedded sets and their applications to complete graphs*

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Abstract. Let K be a finite field, let $X \subset \mathbb{P}^{m-1}$ and $X' \subset \mathbb{P}^{r-1}$, with $r < m$, be two algebraic toric sets parameterized by some monomials in such a way that X' is embedded in X. We describe the relations among the main parameters of the corresponding parameterized linear codes of order d associated to X and X' by using some tools from commutative algebra and algebraic geometry. We also find the regularity index in the case of toric sets parameterized by the edges of a complete graph. Finally, we give some bounds for the minimum distance of the linear codes associated to complete graphs.

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1. Introduction

Let $K = \mathbb{F}_q$ be a finite field with q elements. Recall that a linear code C is a subspace of K^s. The main parameters of the code C are the following.

1. Length: s.
2. Dimension: k. It is the dimension as a vector space over K.
3. Minimum distance: δ. It is given by

$$\delta = \min\{\|v\| : 0 \neq v \in C\},$$

where $\|v\|$ is the norm of the Hamming distance, and it means that $\|v\|$ is the number of non-zero entries of v.

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In this case we say that C is an $[s,k,\delta]$-linear code. These basic parameters are related by the Singleton bound

$$k+\delta \leq s+1.$$

The main objective of this paper is the comparison of the parameters of some codes known as parameterized codes when they arise from some embedded sets and their implications when we work with the edges of a complete graph.

Parameterized codes were introduced in [14]. Projective parameterized codes are important because in some cases their main parameters have the best behavior. For example, in [4] the resulting codes are MDS. It is worth saying that projective parameterized codes are, in general, strictly different to toric codes which were defined in [9] and generalized for example in [11] and [16]. They evaluate over the complete torus, meanwhile we do it over specific subsets of the projective space.

Let $L = K[Z_1,\ldots,Z_n]$ be a polynomial ring over the field K and let $Z^{a_1},\ldots,Z^{a_m}$ be a finite set of monomials. As usual, if $a_i = (a_{i1},\ldots,a_{in}) \in \mathbb{N}^n$, where $\mathbb{N}$ stands for the non-negative integers, then we set $Z^{a_i} = Z_1^{a_{i1}} \cdots Z_n^{a_{in}}$ for all $i = 1,\ldots,m$. In this situation we say that the following set X, which is a multiplicative group under componentwise multiplication, is the toric set parameterized by these monomials.

$$X = \{[(t_1^{a_{11}} \cdots t_n^{a_{1n}}, t_1^{a_{21}} \cdots t_n^{a_{2n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})] \in \mathbb{P}^{m-1} : t_i \in K^*\}, \qquad (1)$$

where $K^* = K \setminus \{0\}$ and $\mathbb{P}^{m-1}$ is a projective space over the field K. We denote the equivalence class of any $\alpha \in K^m \setminus \{0\}$ by $[\alpha]$. The toric set parameterized by the monomials $Z_1,\ldots,Z_m$ is called a projective torus and it is given by

$$\mathbb{T}_{m-1} = \{[(t_1,t_2,\ldots,t_m)] \in \mathbb{P}^{m-1} : t_i \in K^*\}. \qquad (2)$$

Let $S = K[X_1,\ldots,X_m] = \bigoplus_{d=0}^{\infty} S_d$ be a polynomial ring over the field K with the standard grading and let $X = \{[P_1],\ldots,[P_{|X|}]\}$. The evaluation map

$$\begin{aligned} \mathrm{ev}_d : S_d &\to K^{|X|}, \\ f &\to \left(\tfrac{f(P_1)}{X_1^d(P_1)},\ldots,\tfrac{f(P_{|X|})}{X_1^d(P_{|X|})}\right) \end{aligned} \qquad (3)$$

defines a linear map of K-linear spaces. The image of this map is denoted by $C_X(d)$ and it will be called a parameterized code of order d associated to the toric set (1).

$$C_X(d) := \left\{ \left(\frac{f(P_1)}{X_1^d(P_1)},\ldots,\frac{f(P_{|X|})}{X_1^d(P_{|X|})} \right) \in K^{|X|} : f \in S_d \right\}. \qquad (4)$$

The vanishing ideal of X, denoted by I_X, is the ideal of S generated by the homogeneous polynomials of S that vanish on X.

The contents of this paper are as follows. In Section 2, we introduce the preliminaries and connect the algebraic invariants of S/I_X with the main parameters of the code $C_X(d)$ (see Propositions 1, 3, 4). In Section 3, we consider the case when X' is embedded in X and recall some results that were first stated in [18]. In fact, the basic parameters of $C_X(d)$ and $C_{X'}(d)$ were related in [18], but we find new relations among these parameters when $|X| \neq |X'|$, and generalize some of these results (see Theorem 1 and Corollary 3). In Section 4, we obtain a formula for the regularity index (see its definition in the next section) of S/I_X when X is the toric set parameterized by the edges of a complete graph (see Remark 3). This formula allows us to get an optimal lower bound for the regularity index in the case of the toric set parameterized by the edges of any connected non-bipartite graph (see Remark 4).

Actually, there are other families of graphs where the regularity index is known, for example for complete bipartite graphs [5, Corollary 5.4] (the regularity index is obtained adding 1 to the a-invariant), and for Hamiltonian bipartite graphs [18, Corollary 2.21], in particular for even cycles. Also, in [13, Corollary 5.6] the regularity index is computed in the case of connected bipartite graphs with pairwise disjoint even cycles.

Finally, in Section 5, we find some bounds for the minimum distance of the parameterized codes associated to the edges of a complete graph (see inequalities 16 and 18). We show an example in order to verify that some of the bounds are good enough.

2. Preliminaries

Let $X \subset \mathbb{P}^{m-1}$ be the algebraic toric set defined in (1) and $C_X(d)$ the parameterized code of order d defined in (4). The kernel of the evaluation map ev_d defined in (3) is precisely $I_X(d)$, the degree d piece of I_X. Therefore, $C_X(d)$ is isomorphic, as a linear space, to $S_d/I_X(d)$. Recall that the Hilbert function of S/I_X is given by

$$H_X(d) := \dim_K (S/I_X)_d = \dim_K S_d/I_X(d).$$

Then we obtain the following well known result.

Proposition 1. *The dimension of the linear parameterized code of order d, $C_X(d)$, is given by*

$$dim_K C_X(d) = H_X(d).$$

On the other hand, let $h_X(t) = \sum_{i=0}^{l-1} c_i t^i \in \mathbb{Z}[t]$ be the unique polynomial of degree $l-1 = \dim(S/I_X)-1$, where $\dim(S/I_X)$ means the Krull dimension of S/I_X such that $h_X(d) = H_X(d)$ for $d \gg 0$ (recall that this polynomial is called the Hilbert polynomial of S/I_X). The integer $c_{l-1}(l-1)!$, denoted by $\deg(S/I_X)$, is called the degree or multiplicity of S/I_X. In our situation $h_X(t)$ is a non-zero constant because S/I_X has dimension 1. Furthermore

Proposition 2. $h_X(d) = |X|$ *for* $d \geq |X| - 1$.

Proof. See [10, Lecture 13]. □

This result means that $|X| = \deg(S/I_X)$ and then we obtain that

Proposition 3. *The length of the linear parameterized code of order d, $C_X(d)$, is the degree of S/I_X.*

There are algebraic methods based on elimination theory and Grobner bases to compute the dimension and the length of $C_X(d)$ [14].

The regularity index of S/I_X denoted by $\operatorname{reg}(S/I_X)$ is the least integer $p \geq 0$ such that $h_X(d) = H_X(d)$ for $d \geq p$. In fact, the Hilbert function is strictly increasing until it stabilizes (see [1], [2]).

$$1 = H_X(0) < H_X(1) < \cdots < H_X(\operatorname{reg}(S/I_X)-1) < H_X(d) = |X| \text{ for } d \geq \operatorname{reg}(S/I_X).$$

In terms of applications, a good parameterized code should have large $|X|$ and at the same time $\dim_K C_X(d)/|X|$ and $\delta_X(d)/|X|$ as large as possible. The following result gives an indication of where to look for non-trivial parameterized codes. Only the codes $C_X(d)$ with $1 \leq d < \operatorname{reg}(S/I_X)$ have the potential to be good linear codes.

Proposition 4. *$\delta_X(d) = 1$ for $d \geq reg\,(S/I_X)$.*

Proof. Since $H_X(d)$ is equal to the dimension of $C_X(d)$ and $H_X(d) = |X|$ for $d \geq \operatorname{reg}(S/I_X)$, we obtain that $C_X(d) = K^{|X|}$, and the claim follows. $\square$

For more information about the basic parameters of evaluation codes, which are a generalization of parameterized codes, we refer to the introduction of [15].

3. First results

We continue using the notations and definitions used in Sections 1 and 2. Let

$$\mathcal{A} := \{a_1, \ldots, a_r, \ldots, a_m\}$$

be a subset of $\mathbb{Z}^n$, with $1 < r < m$. Let X be the toric set defined in (1) and let $\mathcal{A}'$ be a subset of $\mathcal{A}$ given by $\mathcal{A}' := \{a_1, \ldots, a_r\}$. Thus X' is the algebraic toric set

$$X' := \{[(t_1^{a_{11}} \cdots t_n^{a_{1n}}, \ldots, t_1^{a_{r1}} \cdots t_n^{a_{rn}})] \in \mathbb{P}^{r-1} : t_i \in K^* \text{ for all } i\}.$$

In this case we say that X' is embedded in X. The main goal of the following results is to describe the relations among the main parameters of the corresponding parameterized codes arising from X and X'. Note that X and X' are groups under componentwise multiplication.

Lemma 1. *$|X| = \eta\,|X'|$ for some positive integer η.*

Proof. Let π be the projection map (defined with a different notation in the proof of [18, Lemma 3.5])

$$\begin{gathered} \pi : X \to X', \\ [(t^{a_1}, \ldots, t^{a_r}, \ldots, t^{a_m})] \to [(t^{a_1}, \ldots, t^{a_r})]. \end{gathered}$$

We know that π is a well-defined map and in fact it is an epimorphism between multiplicative groups and then

$$|X'| = \frac{|X|}{|\ker \pi|}.$$

If we take $\eta = |\ker \pi|$, then the claim follows. $\square$

The following two propositions were first observed in the proof of [18, Lemma 2.13].

Proposition 5. *If*

$$I' := I_{X'} \subset S' = K[X_1, \ldots, X_r]$$

and

$$I := I_X \subset S = K[X_1, \ldots, X_r, \ldots, X_m],$$

then

$$I \cap S' = I'.$$

Proof. If $f \in I'$, then $f \in I$ since $0 = f(t^{a_1}, \ldots, t^{a_r}) = f(t^{a_1}, \ldots, t^{a_m})$. On the other hand, if $h \in I \cap S'$, then $h(t^{a_1}, \ldots, t^{a_m}) = 0$, but $h \in S'$ and therefore $h(t^{a_1}, \ldots, t^{a_r}) = 0$. □

Proposition 6. *Let H' and H be the respective Hilbert functions of S'/I' and S/I. Then*

$$H'(d) \leq H(d) \text{ for all } d \in \mathbb{N}.$$

Corollary 1 ([18, Lemma 2.13])**.** *If $|X| = |X'|$, then $reg(S/I) \leq reg(S'/I')$.*

In order to generalize Corollary 1 to the case $|X| \neq |X'|$ we analyze how the minimum distances of the parameterized codes associated to X and X' are related. Let $C_X(d)$ and $C_{X'}(d)$ be the parameterized codes of order d associated to X and X', respectively. By $\delta_X(d)$ we denote the minimum distance of $C_X(d)$ and by $\delta_{X'}(d)$ the minimum distance of $C_{X'}(d)$.

For $G \in S$, we denote the zero set of G in $X \subset \mathbb{P}^{m-1}$ by $Z_G(X)$. It means that

$$Z_G(X) := \{P \in X : G(P) = 0\}.$$

Also, if

$$M_d := \max\{|Z_F(X)| : F \in S_d \setminus I_X(d)\},$$

then we have the next equality

$$\delta_X(d) = \min\{\|\mathrm{ev}_d(F)\| : \mathrm{ev}_d(F) \neq 0,\, F \in S_d\} = |X| - M_d.$$

In the same way, if $H \in S'$, we denote the zero set of H in $X' \subset \mathbb{P}^{r-1}$ by $Z_H(X')$. If M'_d is defined similarly as before, then we have

$$\delta_{X'}(d) = \min\{\|\mathrm{ev}_d(H)\| : \mathrm{ev}_d(H) \neq 0,\, H \in S'_d\} = |X'| - M'_d.$$

Theorem 1. *The minimum distances of $C_X(d)$ and $C_{X'}(d)$ are related in the following way*

$$\delta_X(d) \leq \eta\, \delta_{X'}(d),$$

where η is the integer that appears in Lemma 1.

Proof. Using the definitions above, we know that there exists some $f \in S'_d$ such that $M'_d = |Z_f(X')|$. If we take $P' \in Z_f(X')$, then $f(P) = 0$ for all $P \in \pi^{-1}(P')$ and, as $|\pi^{-1}(P')| = \eta$ (because $\pi^{-1}(P')$ is a coset of $X/\ker\pi$), thus

$$|Z_f(X)| = \eta\,|Z_f(X')|.$$

But $\delta_{X'}(d) = |X'| - M'_d = |X'| - |Z_f(X')|$ and it means that $|Z_f(X')| = |X'| - \delta_{X'}(d)$. Finally, we deduce the result

$$\begin{aligned}\delta_X(d) &= |X| - M_d \leq |X| - |Z_f(X)| = |X| - \eta\,|Z_f(X')| = |X| - \eta\,(|X'| - \delta_{X'}(d)) \\ &= |X| - \eta\,|X'| + \eta\,\delta_{X'}(d) \;= \eta\,\delta_{X'}(d),\end{aligned}$$

and the claim follows. □

Corollary 2 ([18, Lemma 3.5])**.** *If $|X| = |X'|$, then $\delta_X(d) \leq \delta_{X'}(d)$.*

Corollary 3. *The regularity indexes of S/I and S'/I' are related in the following way*

$$\operatorname{reg}(S/I) \leq \operatorname{reg}(S'/I') + \eta - 1. \tag{5}$$

Proof. If u is the regularity index of S'/I', then $\delta_{X'}(u) = 1$. By using Theorem 1 we get that

$$\delta_X(u) \leq \eta\,\delta_{X'}(u) = \eta.$$

If $\operatorname{reg}(S/I) > u + \eta - 1$, then, due to the fact that δ_X is a strictly decreasing function, we obtain that

$$\delta_X(u + \eta - 1) < \cdots < \delta_X(u+1) < \delta_X(u) \leq \eta.$$

Therefore $\delta_X(u + \eta - 1) = 1$, and this contradiction proves inequality (5). □

Corollary 3 is a generalization of Corollary 1 and it plays an important role in the next section where we find the regularity index when the toric set is parameterized by the edges of a complete graph.

4. Regularity index and complete graphs

Let X' be a toric set parameterized by a subset of monomials $M' \subset K[Z_1, \ldots, Z_n]$ in such a way that $Z_1Z_2, Z_3Z_4 \in M'$ and $Z_1Z_3, Z_2Z_4 \notin M'$. Let X be a toric set parameterized by a subset of monomials $M \subset K[Z_1, \ldots, Z_n]$ in such a way that X' is embedded in X and $Z_1Z_3, Z_2Z_4 \in M$. Thus without loss of generality we can write

$$X' = \{[(1, t_1^{-1}t_2^{-1}t_3t_4, t_1^{-1}t_2^{-1}t^{a_3}, \ldots, t_1^{-1}t_2^{-1}t^{a_r})] \in \mathbb{P}^{r-1} : t_i \in K^*\} \tag{6}$$

and

$$X = \{[(1, t_1^{-1}t_2^{-1}t_3t_4, \ldots, t_1^{-1}t_2^{-1}t^{a_r}, t_2^{-1}t_3, t_1^{-1}t_4, \ldots, t_1^{-1}t_2^{-1}t^{a_m})] \in \mathbb{P}^{m-1} : t_i \in K^*\}. \tag{7}$$

The following theorem gives an inequality that is very useful to describe the regularity index in the case of toric sets parameterized by the edges of a complete graph.

Theorem 2. *Let X and X' be the toric sets defined in (6) and (7) and we suppose that $|X| = (q-1)|X'|$. If $u = \mathrm{reg}\,(S'/I')$ and $1 \leq l \leq \frac{q-\rho}{2}$, where $\gcd(q,2) = \rho$, $q > 3$, then*

$$\delta_X(u+l) \leq q-1-2l. \tag{8}$$

Proof. We notice that $C_{X'}(u) = K^{|X'|}$ and we know that there exists $f' \in S'_u$ such that $f'(1,\ldots,1) \neq 0$ and $f'(P) = 0$ for all $P \in X' \setminus \{[(1,\ldots,1)]\}$. By using the same notation of Section 3, we notice that

$$|Z_{f'}(X)| = |X| - |C|,$$

where $C = \ker\pi$ and π is the projection map defined in the proof of Lemma 1 when X' and X are the sets defined in (6) and (7). In fact,

$$C := \{[(1, t^{a_2-a_1}, \ldots, t^{a_r-a_1}, t^{a_{r+1}-a_1}, \ldots, t^{a_m-a_1})] \in X : t^{a_i-a_1} = 1,\ i = 2,\ldots,r\}.$$

But it means that for the elements of C, $t_1^{-1}t_2^{-1}t_3t_4 = 1$. Thus

$$t^{a_{r+1}-a_1}t^{a_{r+2}-a_1} = t_1^{-1}t_2^{-1}t_3t_4 = 1.$$

Therefore $t^{a_{r+1}-a_1} = (t^{a_{r+2}-a_1})^{-1}$ and then

$$C = \{[(1,\ldots,1,\beta^j,\beta^{-j},\ldots,t^{a_m-a_1})] \in X :\ j = 1,\ldots,q-1\},$$

where β is a generator of the cyclic group K^*. We will separate the proof in two cases.

Case (I): We consider $\rho = 1$. If $g_l := X_{r+1}\beta^{2l} - X_{r+2} \in S$, then it is easy to see that

$$g_l(1,\ldots,1,\beta^j,\beta^{-j},\ldots,t^{a_m-a_1}) = \beta^{j+2l} - \beta^{-j} = 0 \Leftrightarrow \beta^{2(j+l)} = 1.$$

Therefore the following two elements of C are different roots of g_l.

$$[(1,\ldots,1,\beta^{\frac{q-1}{2}-l},\beta^{l-\frac{q-1}{2}},\ldots,t^{a_m-a_1})],\ \ [(1,\ldots,1,\beta^{q-1-l},\beta^{l-q+1},\ldots,t^{a_m-a_1})],$$

where $1 \leq l \leq \frac{q-1}{2}$. Hence $|Z_{g_l}(C)| \geq 2$. Moreover, if $G_l = g_1g_2\cdots g_l$, then $|Z_{f'G_l}(X)| \geq |X| - |C| + 2l$, with $f'G_l \in S_{u+l}$. Then

$$\delta_X(u+l) = |X| - M_{u+l} \leq |X| - |Z_{f'G_l}(X)| \leq |C| - 2l = q-1-2l,$$

where we recall that $M_{u+l} = \max\{|Z_F(X)| : F \in S_{u+l} \setminus I_X(u+l)\}$.

Case (II): We consider $\rho = 2$. Let $\mathcal{I}$ be the index set given by

$$\mathcal{I} = \{1,2,\ldots,q-2\}.$$

Let $i_1 \in \mathcal{I}$. Then it is easy to see that there exists $j_1 \in \mathcal{I}\setminus\{i_1\}$ such that $1+\beta^{i_1} = \beta^{j_1}$ and then $1+\beta^{i_1}+\beta^{j_1} = 0$. We define the polynomial

$$h_1 := X_r + X_{r+1} + \beta^{k_1}X_{r+2},$$

with $k_1 = i_1 + j_1 - q + 1$, and we notice that

$$h_1(1,\ldots,1,\beta^{i_1},\beta^{q-1-i_1},\ldots,t^{a_m-a_1}) = 0 = h_1(1,\ldots,1,\beta^{j_1},\beta^{q-1-j_1},\ldots,t^{a_m-a_1}).$$

Thus $|Z_{h_1}(C)| \geq 2$. In the same way, let $i_2 \in \mathcal{I} \setminus \{i_1, j_1\}$. We can find $j_2 \in \mathcal{I} \setminus \{i_1, j_1, i_2\}$ such that $1 + \beta^{i_2} + \beta^{j_2} = 0$. Then we define the polynomial

$$h_2 := X_r + X_{r+1} + \beta^{k_2} X_{r+2},$$

with $k_2 = i_2 + j_2 - q + 1$, and we notice that

$$h_2(1, \ldots, 1, \beta^{i_2}, \beta^{q-1-i_2}, \ldots, t^{a_m - a_1}) = 0 = h_2(1, \ldots, 1, \beta^{j_2}, \beta^{q-1-j_2}, \ldots, t^{a_m - a_1}).$$

Thus $|Z_{h_1 h_2}(C)| \geq 4$. Following this construction, for $1 \leq l \leq \frac{q-2}{2}$, we can find a polynomial

$$h_l := X_r + X_{r+1} + \beta^{k_l} X_{r+2},$$

with $i_l \in \mathcal{I} \setminus \{i_1, \ldots, i_{l-1}, j_1, \ldots, j_{l-1}\}$, $j_1 \in \mathcal{I} \setminus \{i_1, \ldots, i_{l-1}, i_l, j_1, \ldots, j_{l-1}\}$, $k_l = i_l + j_l - q + 1$ and in such a way that

$$h_l(1, \ldots, 1, \beta^{i_l}, \beta^{q-1-i_l}, \ldots, t^{a_m - a_1}) = 0 = h_l(1, \ldots, 1, \beta^{j_l}, \beta^{q-1-j_l}, \ldots, t^{a_m - a_1}).$$

Thus $|Z_{h_1 \cdots h_l}(C)| \geq 2l$ and $|Z_{f' h_1 \cdots h_l}(X)| \geq |X| - |C| + 2l$, with $f' h_1 \cdots h_l \in S_{u+l}$. If $1 \leq l \leq \frac{q-2}{2}$ then

$$\delta_X(u + l) = |X| - M_{u+l} \leq |X| - |Z_{f' h_1 \cdots h_l}(X)| \leq |C| - 2l = q - 1 - 2l,$$

where we recall that $M_{u+l} = \max\{|Z_F(X)| : F \in S_{u+l} \setminus I_X(u + l)\}$. Therefore the claim follows in both cases. $\square$

The following corollary gives a relation between the regularity indexes of the toric sets defined in (6) and (7). We continue using the notation introduced in Section 3.

Corollary 4. *Let X and X' be the toric sets defined in (6) and (7) and we suppose that $|X| = (q - 1)|X'|$. Then*

$$\operatorname{reg}(\mathrm{S/I}) \leq \operatorname{reg}(S'/I') + \frac{q - \rho}{2}, \tag{9}$$

where $\rho = \gcd(q, 2)$.

Proof. If $q = 2$ or $q = 3$, inequality (9) is exactly inequality (5). Thus we can take $q > 3$.

Let $\rho = \gcd(q, 2) = 1$. By using (8) with $l = \frac{q-3}{2}$, we obtain that

$$\delta_X(u + \frac{q-3}{2}) \leq q - 1 - q + 3 = 2.$$

Therefore $\delta_X(u + \frac{q-1}{2}) = 1$ and then

$$\operatorname{reg}(\mathrm{S/I}) \leq u + \frac{q - 1}{2}.$$

Thus inequality (9) is proved when $\rho = 1$.

Now let $\rho = \gcd(q,2) = 2$. If we use (8) with $l = \frac{q-2}{2}$, then

$$\delta_X(u + \frac{q-2}{2}) \leq q - 1 - q + 2 = 1.$$

Therefore

$$\mathrm{reg}\,(\mathrm{S/I}) \leq u + \frac{q-2}{2},$$

and the claim follows. $\square$

Now we are able to find the regularity index in the case of toric sets parameterized by the edges of complete graphs with an even number of vertices.

Corollary 5. *Let X be the toric set parameterized by the edges of a complete graph $\mathcal{K}_n$, with $n = 2n'$ and $n' \geq 2$. Then*

$$reg\,(S/I) = \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil, \tag{10}$$

where $\lceil x \rceil$ is the ceiling function of x, and it means that $\lceil x \rceil = \min\{y \in \mathbb{Z} : y \geq x\}$.

Proof. From [6, Corollary 3.13] we get the inequality

$$\mathrm{reg}\,(S/I) \geq \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil.$$

Let X' be the toric set parameterized by the edges of the complete bipartite graph $\mathcal{K}_{n',n'}$. In [5], it was proved that $\mathrm{reg}\,(S'/I') = (n'-1)(q-1)-n'+1 = (q-2)(n'-1)$. If we use (9) with $\rho = 1$, we obtain that

$$\mathrm{reg}\,(S/I) \leq (q-2)(n'-1) + \frac{q-1}{2} = \frac{(q-2)(2n'-1)+1}{2} = \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil.$$

If we use (9) with $\rho = 2$, we obtain that

$$\mathrm{reg}\,(S/I) \leq (q-2)(n'-1) + \frac{q-2}{2} = \frac{(q-2)(2n'-1)}{2} = \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil.$$

In any case the claim follows. $\square$

Remark 1. *In the case $\mathcal{K}_2$ the set X becomes $\mathbb{T}_0$ (see Eq. (2)) and then* $\mathrm{reg}(S/I) = 0$ *[3, Lemma 1].*

In a similar way, in order to find the regularity index in the case of toric sets parameterized by the edges of complete graphs with an odd number of vertices, we will make the following construction.

Let Y' be a toric set parameterized by a subset of monomials N' in such a way that

$$Z_1Z_2, Z_2Z_3 \in N' \text{ and } Z_1Z_3, Z_1Z_{n+1}, Z_2Z_{n+1} \notin N',$$

with $n \geq 3$. Let Y be a toric set parameterized by a subset of monomials N in such a way that Y' is embedded in Y and $Z_1Z_3, Z_1Z_{n+1}, Z_2Z_{n+1} \in N$. Thus without loss of generality we can write

$$Y' = \{[(1, t_1^{-1}t_3, t_1^{-1}t_2^{-1}t^{a_3}, \ldots, t_1^{-1}t_2^{-1}t^{a_r})] \in \mathbb{P}^{r-1} : t_i \in K^*\} \tag{11}$$

and

$$Y = \{[(1, t_1^{-1}t_3, \ldots, t_1^{-1}t_2^{-1}t^{a_r}, t_2^{-1}t_3, t_2^{-1}t_{n+1}, t_1^{-1}t_{n+1}, \ldots, t_1^{-1}t_2^{-1}t^{a_m})] : t_i \in K^*\}. \tag{12}$$

Theorem 3. *Let Y' and Y be the toric sets defined in (11) and (12) and we suppose that $|Y| = (q-1)^2|Y'|$. If $w = \operatorname{reg}(S'/I')$ and $1 \leq l \leq q-2$, where* $\gcd(q,2) = 1$, *then*

$$\delta_Y(w + l) \leq (q - 1 - l)^2. \tag{13}$$

Proof. We notice that $C_{Y'}(w) = K^{|Y'|}$ and then there exists $g' \in S'_w$ such that $g'(1, \ldots, 1) \neq 0$ and $g'(Q) = 0$ for all $Q \in Y' \setminus \{[(1, \ldots, 1)]\}$. We observe that

$$|Z_{g'}(Y)| = |Y| - |D|,$$

where

$$D := \{[(1, t^{a_2 - a_1}, \ldots, t^{a_r - a_1}, \ldots, t^{a_m - a_1})] \in Y : t^{a_i - a_1} = 1 \text{ for } i = 2, \ldots, r\}.$$

But it means that, for the elements of D, $t_1^{-1}t_3 = 1$. Thus

$$t^{a_{r+1} - a_1}t^{a_{r+3} - a_1} = t_2^{-1}t_3t_1^{-1}t_{n+1} = t_2^{-1}t_{n+1} = t^{a_{r+2} - a_1},$$

and due to the fact that $|Y| = (q-1)^2|Y'|$ then $|D| = (q-1)^2$ and

$$D = \{[(1, \ldots, 1, \beta^i, \beta^{i+j}, \beta^j, \ldots, t^{a_m - a_1})] \in Y : i, j \in \{1, \ldots, q-1\}\},$$

where β is a generator of the cyclic group K^*. Moreover, we can write

$$D = \{[(1, \ldots, 1, x, xy, y, \ldots, t^{a_m - a_1})] \in Y : x, y \in K^*\}.$$

Notice that there is a natural bijection between D and the set $\mathcal{T} = K^* \times K^*$. If $1 \leq v \leq l \leq q-2$ we define the polynomial

$$F_v := \beta^{2v-2}X_r - \beta^{v-1}X_{r+1} + X_{r+2} - \beta^{v-1}X_{r+3},$$

and we take $F := F_1 \cdots F_l \in S_l$. We observe that the points in $\mathcal{T}$ that correspond to the roots of F in D are:

$$\begin{aligned} \{1\} &\times \{\beta^j\}_{j=1}^{q-1}, \\ \{\beta^i\}_{i=1}^{q-1} &\times \{1\}, \\ \{\beta\} &\times \{\beta^j\}_{j=1}^{q-1}, \\ \{\beta^i\}_{i=1}^{q-1} &\times \{\beta\}, \\ &\cdots\cdots\cdots \\ \{\beta^{l-1}\} &\times \{\beta^j\}_{j=1}^{q-1}, \\ \{\beta^i\}_{i=1}^{q-1} &\times \{\beta^{l-1}\}, \end{aligned}$$

and then $|Z_F(D)| = 2l(q-1) - l^2$. Therefore $|Z_{g'F}(Y)| = |Y| - |D| + 2l(q-1) - l^2$, with $g'F \in S_{w+l}$. Thus, by taking $M_{w+l} = \max\{|Z_G(Y)| : G \in S_{w+l} \setminus I_Y(w+l)\}$,

$$\delta_Y(w+l) = |Y| - M_{w+l} \leq |Y| - |Z_{g'F}(Y)| = (q-1)^2 - 2l(q-1) + l^2 = (q-1-l)^2,$$

and the result follows. □

Corollary 6. *Let Y' and Y be the toric sets given in (11) and (12) with $|Y| = (q-1)^2|Y'|$. If $\gcd(q,2) = 1$, then*

$$\operatorname{reg}(S/I) \leq \operatorname{reg}(S'/I') + q - 2. \tag{14}$$

Proof. By using $l = q-2$ in (13) we get

$$\delta_Y(w+l) = 1,$$

and then

$$\operatorname{reg}(S/I) \leq w + q - 2.$$

□

Corollary 7. *Let Y be the toric set parameterized by the edges of the complete graph $\mathcal{K}_n$ with $n = 2n'+1$ and S/I its coordinate ring, $n' \geq 3$. Therefore*

$$\mathit{reg}\,(S/I) = n'(q-2). \tag{15}$$

Proof. From [6, Corollary 3.13] we obtain that

$$\operatorname{reg}(S/I) \geq n'(q-2).$$

In order to prove the reverse inequality we will consider two cases.

Case (A): $\gcd(q,2) = 1$. Let Y' be the toric set parameterized by the edges of an even cycle $C_{2n'}$, which is a subgraph of $\mathcal{K}_n$, and let S'/I' be its coordinate ring. Thus

$$Y' = \{[(t_1t_2, t_2t_3, \ldots, t_{2n'-1}t_{2n'}, t_{2n'}t_1)] \in \mathbb{P}^{2n'-1} : t_i \in K^*\}.$$

We know that $|Y'| = (q-1)^{2n'-2}$ and $|Y| = (q-1)^{2n'}$ (see [14, Corollary 3.8]). Moreover, $\operatorname{reg}(S'/I') = (q-2)(n'-1)$ (see [13, Theorem 5.2] or [18, Theorem 2.20]). Therefore, by using (14), we obtain that

$$\operatorname{reg}(\mathrm{S/I}) \leq (q-2)(n'-1) + q - 2 = n'(q-2),$$

and the result is proved in the case $\gcd(q,2) = 1$.

Case (B): $\gcd(q,2) = 2$. We will continue using the notation of case (A). Now, let G'' be the graph with the same vertex set that $C_{2n'}$ but with two more edges in such a way that the toric set associated to G'' is given by

$$Y'' = \{[(t_1t_2, \ldots, t_{2n'}t_1, t_1t_5, t_2t_4)] \in \mathbb{P}^n : t_i \in K^*\}.$$

G'' is a non-bipartite graph because it has an odd cycle and then $|Y''| = (q-1)^{2n'-1}$. We note that $|Y''| = (q-1)|Y'|$. By Corollary 4 we obtain that

$$\operatorname{reg}(S''/I'') \leq \operatorname{reg}(S'/I') + \frac{q-2}{2} = (q-2)(n'-1) + \frac{q-2}{2},$$

where S''/I'' is the coordinate ring associated to Y''. Moreover,

$$\operatorname{reg}(S/I) \leq \operatorname{reg}(S''/I'') + \frac{q-2}{2} \leq (q-2)(n'-1) + q - 2 = n'(q-2),$$

and the claim follows. □

Remark 2. *In Corollary 7 we consider $n' \geq 3$ (and then $n \geq 7$) in order to satisfy the conditions of the previous results. However, in the two remaining cases $n = 3$ (see [3, Lemma 1]) and $n = 5$, we know that* $\operatorname{reg}(S/I) = 2(q-2)$.

Remark 3. *Corollaries (5) and (7) describe the regularity index in the case of toric sets parameterized by the edges of complete graphs. Equations (10) and (15) can be joined in the following result: Let S/I be the coordinate ring associated to the edges of a complete graph $\mathcal{K}_n$. Then*

$$\operatorname{reg}(S/I) = \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil.$$

Remark 4. *Let G be any connected non-bipartite graph with n vertices and X' the toric set parameterized by its edges. If X is the toric set parameterized by the edges of the complete graph $\mathcal{K}_n$ then, by using Corollary 1 and Remark 3, we get an optimal lower bound for the regularity index corresponding to X':*

$$\operatorname{reg}(S'/I') \geq \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil.$$

5. Minimum distance and complete graphs

In this section we find some bounds for the minimum distance of the parameterized codes associated to the edges of a complete graph.

Corollary 8. *Let X be the toric set associated to the edges of a complete graph $\mathcal{K}_n$, $n = 2n'$, $n' \geq 2$. Then the minimum distance of the corresponding parameterized code of order d, $\delta_X(d)$, is bounded by*

$$\delta_{\mathbb{T}_{n-1}}(2d) \leq \delta_X(d) \leq \xi_{n'}(d), \tag{16}$$

where $\delta_{\mathbb{T}_{n-1}}(2d)$ is the minimum distance of the parameterized code of order $2d$ associated to the projective torus $\mathbb{T}_{n-1}$ and $\xi_{n'}(d)$ is given by

$$\xi_{n'}(d) = \begin{cases} (q-1)^{n-2k'-3}(q-1-\ell)^2, & \text{if } d \leq \alpha_{n'} - 1, \\ q - 1 - 2\ell', & \text{if } \alpha_{n'} \leq d \leq \alpha_{n'} + \frac{q-\rho}{2} - 1, \\ 1, & \text{if } d \geq \alpha_{n'} + \frac{q-\rho}{2}, \end{cases} \tag{17}$$

where $\alpha_{n'} = (q-2)(n'-1)$ is the regularity index corresponding to the projective torus $\mathbb{T}_{n'-1}$, $\ell' = d - \alpha_{n'}$ when $\alpha_{n'} \leq d \leq \alpha_{n'} + \frac{q-\rho}{2} - 1$, $\rho = \gcd(q,2)$ and k', ℓ are the unique integers such that $k' \geq 0$, $1 \leq \ell \leq q-2$ and $d = k'(q-2) + \ell$.

Proof. The result $\delta_{\mathbb{T}_{n-1}}(2d) \le \delta_X(d)$ was proved in [6, Corollary 3.12]. In order to prove the second inequality, let X' be the toric set associated to the complete bipartite graph $\mathcal{K}_{n',n'}$. We know that $|X| = (q-1)|X'|$ and thus, by using Theorem 1, we get

$$\delta_X(d) \le (q-1)\delta_{X'}(d).$$

But in [5] it was proved that $\delta_{X'}(d) = (\delta_{\mathbb{T}_{n'-1}}(d))^2$. Moreover, it is known that

$$\delta_{\mathbb{T}_{n'-1}}(d) = \begin{cases} (q-1)^{n'-(k'+2)}(q-1-\ell), & \text{if } d \le \alpha_{n'}-1, \\ 1, & \text{if } d \ge \alpha_{n'}, \end{cases}$$

where k' and ℓ are unique integers such that $k' \ge 0$, $1 \le \ell \le q-2$ and $d = k'(q-2)+\ell$. Therefore,

$$\delta_X(d) \le \begin{cases} (q-1)^{n-2k'-3}(q-1-\ell)^2, & \text{if } d \le \alpha_{n'}-1, \\ q-1, & \text{if } d \ge \alpha_{n'}. \end{cases}$$

where k' and ℓ are unique integers such that $k' \ge 0$, $1 \le \ell \le q-2$ and $d = k'(q-2)+\ell$.

On the other hand, if $\alpha_{n'} \le d \le \alpha_{n'} + \frac{q-\rho}{2} - 1$, and we take $\ell' = d - \alpha_{n'}$, then, by using (8),

$$\delta_X(d) = \delta_X(u+\ell') \le q-1-2\ell',$$

where $u = \alpha_{n'} = (q-2)(n'-1) = \operatorname{reg}(S'/I_{X'})$.

Finally, we notice that

$$\alpha_{n'} + \frac{q-\rho}{2} = \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil = \operatorname{reg}(S/I).$$

Then $\delta_X(d) = 1$ if $d \ge \alpha_{n'} + \frac{q-\rho}{2}$, and the claim follows. $\square$

In the case of complete graphs with an odd number of vertices we obtain the following result.

Corollary 9. *Let Y be the toric set associated to the edges of a complete graph $\mathcal{K}_n$, $n = 2n'+1$, $n' \ge 3$. Then the minimum distance of the corresponding parameterized code of order d, $\delta_Y(d)$ is bounded by*

$$\delta_{\mathbb{T}_{n-1}}(2d) \le \delta_Y(d) \le (q-1)^{n'+1}\delta_{\mathbb{T}_{n'-1}}(d), \tag{18}$$

where $\delta_{\mathbb{T}_{n-1}}(2d)$ is the minimum distance of the parameterized code of order $2d$ associated to the projective torus $\mathbb{T}_{n-1}$ and $\delta_{\mathbb{T}_{n'-1}}(d)$ is the minimum distance of the parameterized code of order d associated to the projective torus $\mathbb{T}_{n'-1}$. Moreover, if $\gcd(q,2) = 1$, *then*

$$\delta_Y(d) \le \begin{cases} (q-1)^{n-k'-2}(q-1-\ell), & \text{if } d \le \alpha_{n'}-1, \\ (q-1-\ell')^2, & \text{if } \alpha_{n'} \le d \le \alpha_{n'}+q-3, \\ 1, & \text{if } d \ge \alpha_{n'}+q-2, \end{cases}$$

where $\alpha_{n'} = (q-2)(n'-1)$ is the regularity index corresponding to the projective torus $\mathbb{T}_{n'-1}$, $\ell' = d - \alpha_{n'}$ when $\alpha_{n'} \le d \le \alpha_{n'}+q-3$ and k', ℓ are unique integers such that $k' \ge 0$, $1 \le \ell \le q-2$ and $d = k'(q-2)+\ell$.

Proof. The first inequality in (18) follows from [6, Corollary 3.12]. In order to prove the second inequality, let Y' be the toric set associated to the edges of an even cycle C_{n-1}, which is a subgraph of $\mathcal{K}_n$. Then $|Y| = (q-1)^2|Y'|$ and thus, by using Theorem 1, we get

$$\delta_Y(d) \leq (q-1)^2 \delta_{Y'}(d).$$

But in [7, Theorem 3.4] it was proved that $\delta_{Y'}(d) \leq (q-1)^{n'-1}\delta_{\mathbb{T}_{n'-1}}(d)$. Thus the second inequality in (18) follows.

Now let $\gcd(q,2) = 1$. The corresponding result follows directly from the formula of the minimum distance in the case of the projective torus given in [15, Theorem 3.4], inequality (13) by taking $w = \alpha_{n'}$, $l = \ell' = d - \alpha_{n'}$, and the fact that the regularity index associated to $\mathcal{K}_n$ is exactly $\alpha_{n'} + q - 2$. $\square$

Example 1. *In order to show that some of our bounds for the minimum distance in the case of parameterized codes arising from complete graphs are good enough we give the following example.*

Let $K = \mathbb{F}_{11}$ be a finite field with $q = 11$ elements and let X be the toric set associated to the edges of the complete graph $\mathcal{K}_4$. Let $C_X(d)$ be the parameterized code of order d arising from the set X. By using the notation appeared in Corollary 8 we notice that $n = 4$, $n' = 2$, $\alpha_{n'} = (q-2)(n'-1) = 9$ and the corresponding regularity index is $\left\lceil \frac{(9)(3)}{2} \right\rceil = 14$. With the help of Macaulay2 [8] we get the next information.

d	1	2	3	4	5	6	7	8	9	10
$\|X\|$	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
$\dim C_X(d)$	6	19	44	85	146	231	344	489	670	832
$\delta_X(d)$	810	640	490	360	250	160	90	40	10	8
$\xi_{n'}(d)$	810	640	490	360	250	160	90	40	10	8

d	11	12	13	14
$\|X\|$	1000	1000	1000	1000
$\dim C_X(d)$	930	980	998	1000
$\delta_X(d)$	6	4	2	1
$\xi_{n'}(d)$	6	4	2	1

The main conclusion of this example is that in this case the upper bound $\xi_{n'}(d)$, defined in (17) is attained for any $d \geq 1$.

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