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Centrally symmetric convex polyhedra with regular polygonal faces

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Abstract. First we prove that the class C_I of centrally symmetric convex polyhedra with regular polygonal faces consists of 4 of the 5 Platonic, 9 of the 13 Archimedean, 13 of the 92 Johnson solids and two infinite families of $2n$-prisms and $(2n+1)$-antiprisms. Then we show how the presented maps of their halves (obtained by identification of all pairs of antipodal points) in the projective plane can be used for obtaining their flag graphs and symmetry-type graphs. Finally, we study some linear dependence relations between polyhedra of the class C_I.

AMS subject classifications: 37F20, 57M10

Key words: map, Platonic solid, Archimedean solid, Johnson solid, flag graph, convex polyhedron, projective plane

1. Introduction

Any centrally symmetric convex (hence: spherical) polyhedron $\mathcal{C}$ admits identification of pairs of antipodal points x and x^*; thus the *map* (i.e. embedding of a graph in a compact surface) of its *half* $\mathcal{C}/2 = \mathcal{C}/_{x \equiv x^*}$ has the Euler characteristic $E = v - e + f = 1$ (where v, e and f are the numbers of the vertices, edges and faces of the map, respectively) and can be drawn in a *projective plane* (represented as a disc with identified antipodal points [8]). Thus the flag graph of $\mathcal{C}/2$ can be easily constructed in the projective plane, too, while the flag graph of $\mathcal{C}$ is exactly a 2-sheet cover space [1, 2] over $\mathcal{C}/2$.

It is well known that the class C of convex polyhedra with regular polygonal faces consists of 5 *Platonic solids*, 13 *Archimedean solids* [9], the class of 92 non-uniform (i.e. having at least two orbits of vertices) *Johnson solids* [3] and two infinite families of prisms and antiprisms. Among these solids we will find a subset $C_I \subset C$ of centrally symmetric solids and present the maps of their halves $\mathcal{C}/2$ (obtained by identification of all pairs of antipodal points) in the projective plane modelled as a disk with identified antipodal points. From these maps we can deduce the corresponding *flag graphs* and *symmetry-type graphs* which can be used for the classification of *maps*, tilings and polyhedra, too [5, 4, 6, 7]. The maps of the halves of four Platonic solids (hemi-cube, hemi-octahedron, hemi-dodecahedron, hemi-icosahedron) and the maps of regular and semi-regular spherical polyhedra can

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also be found in Wikipedia (Regular polyhedron, Spherical polyhedron). For convex uniform polyhedra we use the usual notation $(p.q.r.\dots)$, describing the cyclical sequence of regular n-gonal faces $(n = 3, 4, 5, \dots)$ around any vertex.

STRUCTURE: First some general propositions about centrally symmetric polyhedra are given (Section 2), and then the solids $\mathcal{C} \in C_I$ are identified (Section 3). After that the maps of their halves in the projective plane are presented and we show how to construct the corresponding flag graph and symmetry-type graph (Section 4). From these maps the number of faces 3,4,5,6,8,10 for each $\mathcal{C}/2$ can be easily found, too. The corresponding vectors $n = (n_{10}, n_8, n_6, n_5, n_4, n_3)$ can be used to solve the following problem (Section 5): Is it possible to take a copies of a polyhedron $A \in C_I$ and b copies of polyhedron $B \in C_I$ and by dissecting their boundary into faces construct a polyhedron $C \in C_I$ so that no faces are left unused?

2. Centrally symmetric solids

The sets of vertices, edges and faces of a polyhedron $\mathcal{C}$ are denoted by $V(\mathcal{C})$, $E(\mathcal{C})$, $F(\mathcal{C})$, respectively. The *central point* (or the centre) of a polyhedron $\mathcal{C} \in C_I$ is defined as the point O fixed by the central inversion c preserving $\mathcal{C}$. The *antipodal elements* (vertices, edges, faces) of a vertex $v \in V(\mathcal{C})$, an edge $e \in E(\mathcal{C})$ and a face $f \in F(\mathcal{C})$ are denoted by $c(v) = v^*$, $c(e) = e^*$, $c(f) = f^*$, respectively. Here are some necessary (although not sufficient) conditions for $\mathcal{C}$ belonging to the class C_I:

Proposition 1. *Let $\mathcal{C} \in C_I$. Then:*

(i) *Any pair of antipodal edges $e, e^* \in E(\mathcal{C})$ is parallel; likewise, any pair of antipodal faces $f, f^* \in F(\mathcal{C})$ is parallel, too.*

(ii) *The numbers $\#v$, $\#e$, $\#f$ of vertices, edges and faces of C must be even.*

(iii) *The numbers of each class of faces with the same number $(3, 4, 5, \dots)$ of edges must be even.*

Proof. (i): Let O be the central point of $\mathcal{C} \in C_I$. For any vertex v let $\vec{OV} = \vec{v}$ be the vector with the starting point O and the ending point in v. Let $e(u, v) \in E(\mathcal{C})$. Then $\vec{e}^* = \vec{u}^* - \vec{v}^* = -\vec{u} - (-\vec{v}) = -\overrightarrow{(u - v)} = -\vec{e}$ for any $u, v \in V(\mathcal{C})$, hence the vectors $\vec{e}^*$ and $\vec{e}$ are parallel. Consequently, all the edges of faces f and f^* are parallel, hence f and f^* must be parallel, too.

(ii): Obviously $v \neq v^*$, $e \neq e^*$,$f \neq f^*$ for each $v \in V(\mathcal{C})$, $e \in E(\mathcal{C})$, $f \in F(\mathcal{C})$.

(iii): Faces f and f^* have the same number of edges. □

Corollary 1. *Tetrahedron (3.3.3) and truncated tetrahedron (6.6.3) are not in the class C_I.*

Proof. None of the four faces of (3.3.3) has a parallel face. The same holds for the four hexagonal faces of (6.6.3). Hence, by Proposition 1(i), these two solids cannot be in C_I. □

We shall say that a polyhedron has a rotation R if it is symmetric by rotation R. Similarly, it is symmetric by a reflection, we shall say it has a reflection.

Proposition 2.

(i) *If* $\mathcal{C} \in C$ *has two orthogonal reflection planes* Π *and* Ω*, but it is not preserved by the reflection over a plane orthogonal both to* Π *and* Ω*, then* $\mathcal{C} \notin C_I$*.*

(ii) *If* $\mathcal{C} \in C$ *has a rotation* R *for the angle* π*, but it has not a reflection plane orthogonal to the axis of* R*, then* $\mathcal{C} \notin C_I$*.*

(iii) *If* $\mathcal{C} \in C$ *has a a reflection plane* Π *but it has not a rotation* R *for the angle* π *with an axis orthogonal to* Π*, then* $\mathcal{C} \notin C_I$*.*

Proof. (i): Let the central point O of $\mathcal{C}$ be the origin of the Cartesian coordinate system with axes (x) and (y) in the plane Π and (y) and (z) in Ω. Then the reflections Z_Π and Z_Ω transform a vertex v with coordinates $v(x,y,z)$ into $v_\Pi = v(-x,y,z)$ and $v_\Omega = v(x,-y,z)$, respectively. If there is also the central inversion c, then $c(v) = v(-x,-y,-z)$. Hence there should also be the reflection $v(x,y,z) \to v(x,y,-z)$.

(ii) and (iii) are proved similarly as (i), using the fact that the rotation R sends the point (x,y,z) into the point $(-x,-y,z)$. □

Proposition 3.

(i) *If a polyhedron* $\mathcal{C} \in C$ *is symmetrical by the following two operations:*

- a) *reflection* Z *over a plane* Π*;*
- b) *rotation* R_π *for the angle* π *around an axis* a*, orthogonal to* Π*;*

then $\mathcal{C} \in C_I$*.*

(ii) *If a polyhedron* $\mathcal{C} \in C$ *is preserved by the reflections over three mutually orthogonal planes, then* $\mathcal{C} \in C_I$*.*

Proof. (i): The composition of reflection Z and rotation R_π sends any point (x,y,z) into its antipodal point $(-x,-y,-z)$: $ZR_\pi = R_\pi Z = c$.

(ii): The composition Z_1Z_2 of two reflections Z_1 and Z_2 over two orthogonal planes produces a rotation for the angle π around the axis a, which is orthogonal to the third plane, and we can use (i). □

Corollary 2. *The cube (4.4.4), the octahedron (3.3.3.3), the dodecahedron (5.5.5) and the icosahedron (3.3.3.3.3) are in the class* C_I*.*

Proof. For (4.4.4) and (3.3.3.3) this is true by Proposition 3(i), while for (5.5.5) and (3.3.3.3.3) this is true by Proposition 3(ii). □

All Platonic and Archimedean solids can be obtained from a tetrahedron using the operations *medial* $Me(\mathcal{C})$, *truncation* $Tr(\mathcal{C})$, *dual* $Du(\mathcal{C})$ and *snub* $Sn(\mathcal{C})$ [9].

Proposition 4.

(i) *If the solid* $\mathcal{C}$ *belongs to the class* C_I*, the same holds for its truncation* $Tr(\mathcal{C})$*,* $Me(\mathcal{C})$ *and dual* $Du(\mathcal{C})$*.*

(ii) *However, there are solids such that* $\mathcal{C} \notin C_I$ *and* $Me(C) \in C_I$*.*

(*iii*) *Likewise, there are solids $\mathcal{C} \in C_I$ such that $Me(C) \notin C_I$ or $Sn(\mathcal{C}) \notin C_I$.*

Proof. (i): From the definitions of operations Tr, Me and Du it follows that they do not have any impact on the central symmetry of the solid.

(ii): We already know that the tetrahedron (3.3.3) is not in C_I, while its medial-the octahedron (3.3.3.3)-is.

(iii): The cube (4.4.4) and the dodecahedron (5.5.5) are in C_I, while the snub cube (4.3.3.3.3) and the snub dodecahedron (5.3.3.3.3) are not, as we can conclude by Proposition 2(ii). □

Proposition 5. *The 4-antiprism is not centrally symmetric.*

Proof. Suppose the 4-antiprism is centrally symmetric. Then the central inversion c sends the vertices 1,2,3,4 of one square face into vertices $c(1) = 1^*$, $c(2) = 2^*$, $c(3) = 3^*$, $c(4) = 4^*$ of the other square face (Figure 1 left).

For any face f the face $c(f) = f^*$ has no common point with the face f, hence $v \neq v^*$ for any vertex v. Since v and v^* do not belong to the same face, they cannot be adjacent vertices. Therefore $A = 3^*$ or $A = 4^*$. Likewise $B = 4^*$ or $B = 1^*$. Likewise $C = 1^*$ or $C = 2^*$ and $D = 2^*$ or $D = 3^*$. As soon as we choose one of the possibilities for A, then B, C and D are determined by the above relations.

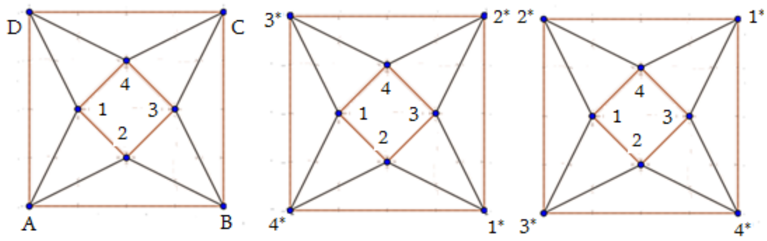

Figure 1: *The 4-antiprism and why it is not centrally symmetric*

In the first case (Figure 1 in the middle), the antipod of triangle $\Delta(124^*)$ cannot be the triangle $\Delta(1^*2^*4)$, since the antipod of the edge 14^* is not the edge 1^*4. In the second case (Figure 1 right), the antipod of the triangle $\Delta(123^*)$ is not the triangle $\Delta(1^*2^*3)$, since the antipod of the edge 23^* is not the edge 2^*3. □

Proposition 6. *The n-antiprism belongs to the class C_I if and only if n is odd.*

Proof. If the antiprism $(N.3.3)$, $N \geq 3$ has the central symmetry c, then c sends the vertices of the upper n-gon $1, 2, 3, 4, \ldots, n$ into vertices $1^*, 2^*, 3^*, 4^*, \ldots, N^*$ of the upper N-gon (Figure 2). The antipod of the triangle $\Delta(1, 2, X^*)$ must be the triangle $\Delta(1^*, 2^*, X)$. The vertex X belongs to the upper N-gon. Along the upper N-gon we have $X - 1$ (grey) triangles $\Delta(1, 2, X^*)$, $\Delta(2, 3, (X + 1)^*)$,..., $\Delta(X - 1, X, 1^*)$. The same number of (white) triangles is between vertices X^* and 1 along the lower N-gon: $\Delta(X^*, (X + 1)^*, 2)$, $\Delta((X + 1)^*, (X + 2)^*, 3)$,..., $\Delta(1^*, 2^*, X)$. Therefore it is $X - 1 \equiv 2 - X \pmod N$, hence $2X \equiv 3 \pmod N$. But this is possible only if N is an odd number, since the remainder of $2X$ modulo $2n$ is always an even number. Therefore such labeling of the triangles as shown in Figure 2 is possible only if $N = 2n + 1$, and it is not possible if $N = 2n$. □

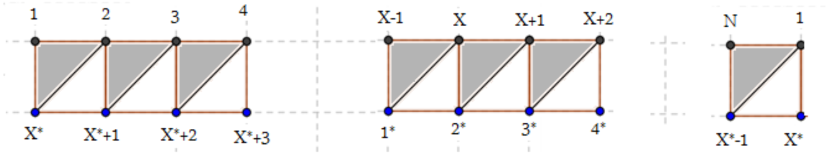

Figure 2: *Triangles of a centrally symmetric antiprism*

3. Determination of the class C_I

Theorem 1. *The class C_I consists of the following solids:*

- (i) *four of the five Platonic solids: Cube (4.4.4), Octahedron (3.3.3.3), Dodecahedron (5.5.5) and Icosahedron (3.3.3.3.3);*
- (ii) *nine of the 13 Archimedean solids: Truncated Cube (8.8.3), Truncated Dodecahedron (10.10.3), Truncated Octahedron (4.6.6), Truncated Icosahedron (5.6.6), Truncated Cuboctahedron (8.4.6), Cuboctahedron (4.3.4.3), Icosidodecahedron (5.3.5.3), Rhombicuboctahedron (4.4.3.4), Rhombicosidodecahedron (5.4.3.4);*
- (iii) *the infinite families of $2n$-prisms and $(2n+1)$-antiprisms;*
- (iv) *13 Johnson solids: J15, J28, J31, J36, J39, J43, J55, J59, J67, J69, J73, J80, J91.*

All these solids satisfy the condition of Proposition 3(i) (this will help us to draw the maps of their halves in Section 4).

Proof. (i): These solids are in the class C_I by Corolary 2.

(ii): All these solids are duals, medials or truncations of solids being in C_I, hence by Proposition 4(i) they are in C_I, too.

(iii): The $2n$-prisms are in C_I by Proposition 3(i). The $(2n+1)$-prisms have odd number of faces 4. The result on antiprisms is given in Proposition 6.

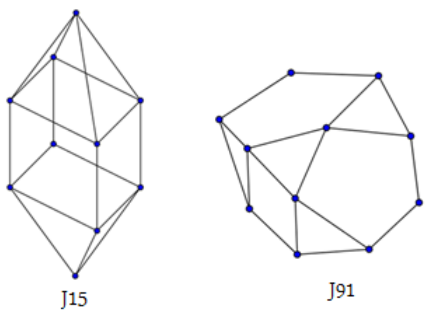

Figure 3: *Two centrally symmetric Johnson solids:J15 and J91*

(iv): Either using computer programs for polyhedra (like Great Stella) or with the help of 3D-models of Johnson solids it is easy to see that all these 13 solids satisfy one or both of the conditions of Proposition 3 (see Table 1). The arguments why the other 79 Johnson solids are not in the class C_I are given in Table 2. $\square$

Johnson solid $\mathcal{C}$	Jxx	$\mathcal{C} \in C_I$
elongated square dipyramid	J15	by Proposition 3(ii)
square orthobicupola	J28	by Proposition 3(ii)
pentagonal gyrobicupola	J31	by Proposition 3(i)
elongated triangular gyrobicupola	J36	by Proposition 3(i)
elongated pentagonal gyrobicupola	J39	by Proposition 3(i)
elongated pentagonal gyrobirotunda	J43	by Proposition 3(i)
parabiaugmented hexagonal prism	J55	by Proposition 3(ii)
parabiaugmented dodecahedron	J59	by Proposition 3(i)
biaugmented truncated cube	J67	by Proposition 3(ii)
parabiaugmented truncated dodecahedron	J69	by Proposition 3(i)
parabigyrate rhombicosidodecahedron	J73	by Proposition 3(i)
paradiminished rhombicosidodecahedron	J80	by Proposition 3(i)
bilunabirotunda	J91	by Proposition 3(ii)

Table 1: *The 13 Johnson solids belonging to the class C_I*

And here are the Johnson solids not in the class C_I:

Johnsons solid $\mathcal{C}$	Jxx	eliminated since
square pyramid	J1	only one face 4
pentagonal pyramid	J2	only one face 5
triangular cupola	J3	only one face 6
square cupola	J4	only one face 8
pentagonal cupola	J5	only one face 10
pentagonal rotunda	J6	only one face 10
elongated triangular pyramid	J7	3 faces with 4 edges
elongated square pyramid	J8	5 faces with 4 edges
elongated pentagonal pyramid	J9	only one face 5
gyroelongated square pyramid	J10	only one face 4
gyroelongated pentagonal pyramid	J11	only one face 5
triangular dipyramid	J12	$v = 5$ odd number
pentagonal dipyramid	J13	$v = 7$ odd number
elongated triangular dipyramid	J14	$f = 9$ odd number
elongated pentagonal dipyramid	J16	5 faces 4
gyroelongated square dipyramid	J17	it contains a 4-antiprism
elongated triangular cupola	J18	only one face 6
elongated square cupola	J19	only one face 8
elongated pentagonal cupola	J20	only one face 10
elongated pentagonal rotunda	J21	only one face 10
gyroelongated triangular cupola	J22	only one face 6
gyroelongated square cupola	J23	only one face 8

gyroelongated pentagonal cupola	J24	only one face 10
gyroelongated pentagonal rotunda	J25	only one face 10
gyrobifastigium	J26	by Proposition 2(ii)
triangular orthobicupola	J27	by Proposition 2(ii)
square gyrobicupola	J29	by Proposition 2(iii)
pentagonal orthobicupola	J30	by Proposition 2(iii)
pentagonal gyrobicupola	J32	7 faces with 5 edges
pentagonal gyrocupolarotunda	J33	7 faces with 5 edges
pentagonal orthobirotunda	J34	by Proposition 2(ii)
elongated triangular orthobicupola	J35	by Proposition 2(ii)
elongated square gyrobicupola	J37	by Proposition 2(ii)
elongated pentagonal orthobicupola	J38	by Proposition 2(ii)
elongated pentagonal orthocupolarotunda	J40	7 faces with 5 edges
elongated pentagonal gyrocupolarotunda	J41	7 faces with 5 edges
elongated pentagonal orthobirotunda	J42	by Proposition 2(ii)
gyroelongated triangular bicupola	J44	by Proposition 2(ii)
gyroelongated square bicupola	J45	by Proposition 2(ii)
gyroelongated pentagonal bicupola	J46	by Proposition 2(ii)
gyroelongated pentagonal cupolarotunda	J47	7 faces 5
gyroelongated pentagonal birotunda	J48	by Proposition 2(ii)
augmented triangular prism	J49	$v = 7$ odd number
biaugmented triangular prism	J50	only one face 4
triaugmented triangular prism	J51	$v = 9$ odd number
augmented pentagonal prism	J52	$v = 11$ odd number
biaugmented pentagonal prism	J53	3 faces 4
augmented hexagonal prism	J54	$f = 11$ odd number
parabiaugmented hexagonal prism	J56	by Proposition 2(ii)
triaugmented hexagonal prism	J57	3 faces 4
augmented dodecahedron	J58	$v = 21$ odd number
metabiaugmented dodecahedron	J60	by Proposition 2(ii)
triaugmented dodecahedron	J61	$v = 23$ odd number
metadiminished dodecahedron	J62	by Proposition 2(ii)
tridiminished icosahedron	J63	$v = 9$ odd number
augmented tridiminished icosahedron	J64	3 faces 5
augmented truncated tetrahedron	J65	3 faces 6
augmented truncated cube	J66	5 faces 8
augmented truncated dodecahedron	J68	$v = 65$ odd number
metabiaugmented truncated dodecahedron	J70	by Proposition 2(i)
triaugmented truncated dodecahedron	J71	$v = 75$ odd number
gyrate rhombicosidodecahedron	J72	by Proposition 2(iii)
metabigyrate rhombicosidodecahedron	J74	by Proposition 2(i)
trigyrate rhombicosidodecahedron	J75	by Proposition 2(iii)
diminished rhombicosidodecahedron	J76	by Proposition 2(iii)

paragyrate diminished rhombicosidodecahedron	J77	by Proposition 2(iii)
metagyrate diminished rhombicosidodecahedron	J78	by Proposition 2(iii)
bigyrate diminished rhombicosidodecahedron	J79	by Proposition 2(iii)
metadiminished rhombicosidodekahedron	J81	by Proposition 2(i)
gyrate bidiminished rhombicosidodekahedron	J82	by Proposition 2(iii)
tridiminished rhombicosidodekahedron	J83	$v = 45$ odd number
snub disphenoid	J84	by Proposition 2(i)
snub square antiprism	J85	by Proposition 2(ii)
sphenocorona	J86	by Proposition 2(i)
augmented sphenocorona	J87	by Proposition 2(iii)
augmented sphenocorona	J88	by Proposition 2(i)
hebesphenomegacorona	J89	3 faces 4
disphenocingulum	J90	by Proposition 2(i)
triangular hebesphenorotunda	J92	only one face 6

Table 2: *The 79 Johnson solids not being in the class C_I*

4. Maps of $\mathcal{C}/2$, $\mathcal{C} \in C_I$ in the projective plane

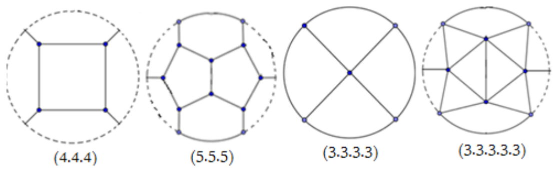

Figure 4: *The halves of Platonic solids in the projective plane*

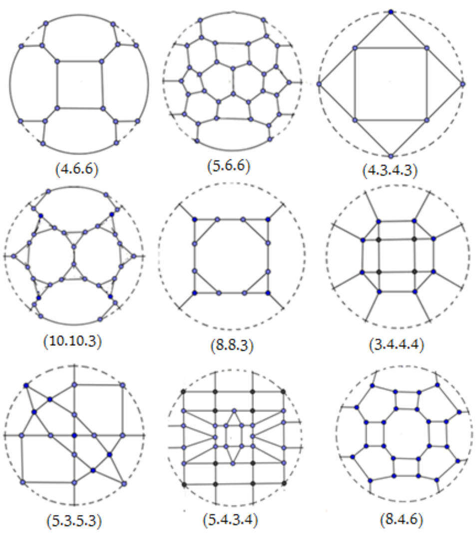

Figure 5: *The halves of Archimedean solids in the projective plane*

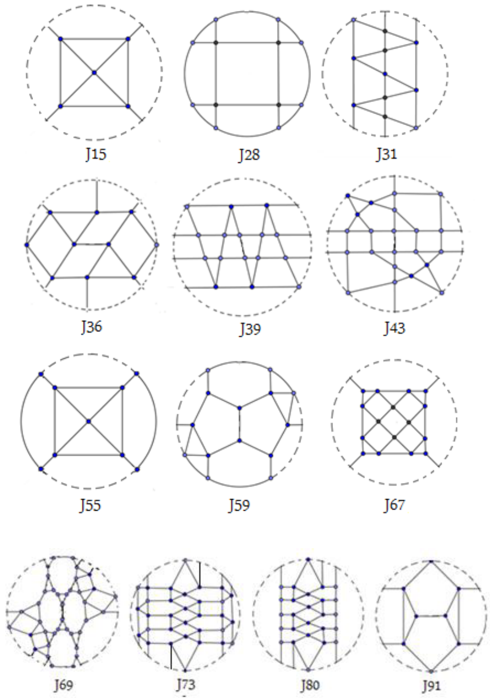

Figure 6: *The halves of Johnson solids in the projective plane*

The flag graphs of halves of solids $\mathcal{C} \in C_I$ can now be deduced from Figures 4, 5 and 6. For J31 this is done in Figure 8. Now it is easy to obtain the symmetry-type graphs of any $\mathcal{C} \in C_I$. For J15 this is done in Figure 9.

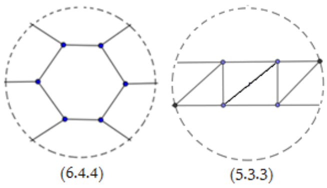

Figure 7: *The halves of 6-prism and 5-antiprism in the projective plane*

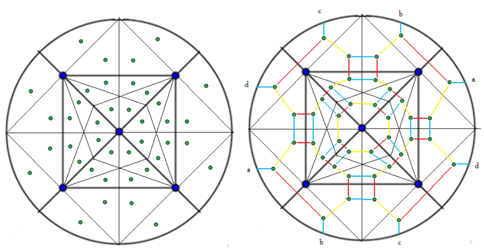

Figure 8: *Flags and flag graph of (J15)/2*

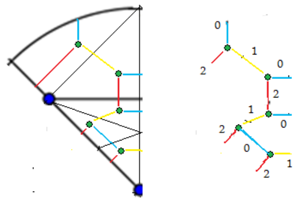

Figure 9: *Representative flags of orbits and symmetry-type graph of J15*

5. Spectral analysis of faces of $\mathcal{C}/2$ for $\mathcal{C} \in C_I$

Definition 1. *For any polyhedron $\mathcal{P}$ with regular faces having at most n vertices let the vector $\boldsymbol{s}(\mathcal{C}) = (f_n, \dots, f_6, f_5, f_4, f_3)$ denote its spectral vector, counting the numbers f_i of its faces with i vertices. In the corresponding spectral codes $S(\mathcal{P})$ (see the right column of Table 3) only the nonzero numbers f_i are given.*

$\mathcal{C}$	n_{10}	n_8	n_6	n_5	n_4	n_3	$S(\mathcal{C}/2)$
(4.4.4)					3		4_3
(5.5.5)				6			5_6
(3.3.3.3)						4	3_4
(3.3.3.3.3)						10	3_{10}
(4.6.6)			4		3		$6_4 4_3$
(5.6.6)			10	6			$6_{10} 5_6$
(8.8.3)		3				4	$8_3 3_4$
(10.10.3)	6					10	$10_6 3_{10}$
(8.4.6)		3	4		6		$8_3 6_4 4_6$
(3.4.4.4)					9	4	$4_9 3_4$
(4.3.4.3)					3	4	$4_3 3_4$
(5.3.5.3)				6		10	$5_6 3_{10}$
(5.4.3.4)				6	15	10	$5_6 4_{15} 3_{10}$
J15					2	4	$4_2 3_4$
J28					5	4	$4_5 3_4$
J31				1	5	5	$5_1 4_5 3_5$
J36					6	4	$4_6 3_4$
J39				1	10	5	$5_1 4_{10} 3_5$
J43				6	5	10	$5_6 4_5 3_{10}$
J55					4	4	$4_4 3_4$
J59				5		5	$5_5 3_5$
J67		2			5	8	$8_2 4_5 3_8$
J69	5				5	15	$10_5 4_5 3_{15}$
J73				6	15	10	$5_6 4_{15} 3_{10}$
J80	1			5	10	5	$10_1 5_5 4_{10} 3_5$
J91				2	1	4	$5_2 4_1 3_4$

Table 3: *Spectral vectors of $\mathcal{C}/2$ for $\mathcal{C} \in C_I$*

5.1. Linear dependence relations between polyhedra

The concept of the spectral vector paves the way to the study of linear dependence relations between any polyhedra.

Definition 2. *Polyhedra $P_1, \ldots, P_m$ are linearly dependent, if their corresponding spectral vectors $\boldsymbol{s}(P_i)$ are linearly dependent.*

In other words: there are $a_1, \ldots, a_m \in \mathbb{Z}$ not all equal to zero such that

$$a_1\mathbf{s}(P_1) + \cdots + a_m\mathbf{s}(P_m) = \mathbf{0}.$$

Using spectral vectors we can also define such concepts as “collinearity” and “coplanarity” of polyhedra.

Definition 3. *Let A, B, C be any polyhedra. If it is possible to take a copies of A and b copies of B and by dissecting their boundary into faces construct c copies of a polyhedron C so that no faces are left unused, we say that the solids A, B, C are* coplanar *and we write this symbolically as $aA + bB = cC$. Similarly, we write $aA = bB$ and say that A and B are collinear, if the relation $a\boldsymbol{s}(A) = b\boldsymbol{s}(B)$ holds for their corresponding spectral vectors.*

Using the information gathered in Table 3 we can now easily solve questions about linear dependence relations polyhedra from C_I (since for the corresponding spectral vectors we obviously have the relation $\mathbf{s}(\mathcal{C}) = 2\mathbf{s}(\mathcal{C}/2)$.

Example 1. *Are the polyhedra J55, J59 and J73 coplanar? To answer this we have to solve the vector equation $a\boldsymbol{s}(J55/2) + b\boldsymbol{s}(J59/2) = c\boldsymbol{s}(J73/2)$, or, equivalently, $a(4_4 + 3_4) + b(5_5 + 3_5) = c(5_6 + 4_{10} + 3_{10})$. From this we obtain the following system of three linear equations: $5b = 6c$, $4a = 10c$, $4a + 5b = 10c$. Thus $b = 6c/5$, $a = 5c/2$ and $4(5c/2) + 5(6c/5) = 10c$, hence $10c + 6c = 10c$ and $c = 0$. Thus J55, J59 and J73 are not coplanar.*

Some examples of coplanar solids from C_I are:

J31, J59 and J59, since $5_1 4_5 3_5 + 5_5 3_5 = 5_6 4_5 3_{10}$, hence J31 + J59 = J43;

J31, (4.4.4) and J39, since $3 \cdot 5_1 4_5 3_5 + 5 \cdot 4_3 = 3 \cdot 5_1 4_{10} 3_5$, hence $3{\cdot}J31 + 5 \cdot (4.4.4) = 3{\cdot}J39$;

J15, (3.4.4.4) and J39, since $4 \cdot 4_2 3_4 + 3 \cdot 4_9 3_4 = 7 \cdot 4_5 3_4$, hence $4{\cdot}J15 + 3 \cdot (3.4.4.4) = 7{\cdot}J28$.

Other “linear polyhedral equations”, as $aA + bB = cC + dD$, may be treated in a similar way, too.

References

[1] W. FULTON, *Algebraic topology*, Springer-Verlag, New York, Berlin, Heidelberg, 1995.
[2] A. HATCHER, *Algebraic topology*, Cambridge University press, Cambridge, 2002.
[3] N. W. JOHNSON, *Convex polyhedra with regular faces*, Canad. J. Math. **18**(1966), 169–200.

[4] J. Kovič, *Symmetry-type graphs of Platonic and Archimedean solids*, Math. Commun. **16**(2011), 491–507.

[5] S. Lins, *Graph-encoded maps*, J. Combin. Theory Ser. B **32**(1982), 171–181.

[6] W. Massey, *Algebraic topology: An introduction*, Springer-Verlag, New York, 1967.

[7] A. Orbanić, *Edge-transitive maps*, PhD thesis, University of Ljubljana, Ljubljana, 2006.

[8] T. Pisanski, B. Servatius, *Configurations, incidence structures from a graphical viewpoint*, Birkhäuser, Boston, 2012.

[9] T. Pisanski, A. Žitnik, *Representations of graphs and maps*, Preprint series **42**(2004), 924.