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Some notes on PD-operator pairs*

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Abstract. This paper points out several remarks on the paper of Pathak and Rai

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In fact, under contractive conditions (assumed in the above paper), proving the existence of a common fixed point by assuming the notion of a PD-operator is equivalent to proving the existence of common fixed point by assuming the existence of a common fixed point.
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1. Introduction

Pathak and Rai [15] define a PD-operator pair of single valued mappings and obtain some common fixed point theorems for this class of maps under relaxed conditions. Their theorem generalizes results of Bhatt et al. [3], Jungck and Rhoades [11], Hussain et al. [7] and others. As applications, they also study the solution of functional equations arising from dynamic programming and variational inequalities arising in the two point obstacle problem.

In 1982, Sessa [16] gave the first weaker version of commutativity condition [8], namely weakly commuting mappings. In recent years Jungck [9, 10], Pant [13], Pathak et al. [15], Al-Thangafi and Shahzad [2] and many others [12] have considered several generalizations of commuting mappings or weaker notions of commutativity. Now, it has been shown that weak compatibility is the minimal noncommuting condition for the existence of common fixed points of contractive type mapping pairs. In recent work, several authors have claimed to introduce some weaker noncommuting notions and pretended to show weak compatibility as a proper subclass of their weaker notions. This is, however, not true. In view of the results of Alghamdi et al. ([1] see also, [4, 14]), most of the generalized commutativity notions fall into

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the subclass of weak compatibility in the setting of a unique common fixed point (or unique point of coincidence). If there are just two maps involved, and they only have one coincidence point (which turns out to be the unique fixed point), then of course a PD-operator pair and all of the generalizations of commutativity coincide. If there are no coincidence points, then there cannot be any fixed points and PD-operator points either. Those generalizations of commuting mappings are novel but for their actual applications one should go beyond contractive conditions since contractive conditions do not allow more than one point of coincidence or fixed point. In fact, under contractive conditions proving the existence of common fixed points by assuming several weaker noncommuting notions is often equivalent to proving the existence of common fixed points by assuming the existence of common fixed points [14].

For completeness of the paper and convenience of the reader, in Section 2 we collect some basic definitions and facts which are applied in subsequent sections.

2. Preliminaries

Following Pathak and Rai [15] (see also [1]) we assume (X, d) denotes a metric space, and for $x \in X$ and $A \subset X$, set $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Let f and g be self mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence (POC) of f and g. The set of coincidence points of f and g will be denoted by $C(f, g)$. Let $PC(f, g)$ represent the set of points of coincidence of f and g. A point $x \in X$ is a common fixed point of f and g if $x = fx = gx$. The set of all common fixed points of f and g is denoted by $F(f, g)$.

Definition 1. *A pair of selfmaps (f, g) of a metric space (X, d) is said to be*

(i) *commuting [8], if $fgx = gfx$ for all x in X;*

(ii) *weakly commuting [16], if $d(fgx, gfx) \le d(fx, gx)$ for all x in X;*

(iii) *$R-$weakly commuting [13], if $d(fgx, gfx) \le Rd(fx, gx)$ for all x in X and $R > 0$;*

(iv) *compatible [9], if and only if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X;*

(v) *weakly compatible (WC) [10], if the pair commutes on the set of coincidence points, i.e., $fgx = gfx$ whenever $fx = gx$ for some $x \in X$;*

(vi) *occasionally weakly compatible (OWC) [2], if there exists a coincidence point x in X such that $fx = gx$ implies $fgx = gfx$;*

(vii) *a PD-operator pair [15], if there is a point $x \in X$ such that $x \in C(f, g)$ and*

$$d(fgx, gfx) \le diam(PC(f, g)), \text{ for some } x \in C(f, g).$$

Definition 2. *Let X be a non-empty set and d a function $d : X \times X \to [0, \infty)$ such that*

$$d(x, y) = 0 \text{ if and only if } x = y, \text{ for each } x, y \in X. \tag{1}$$

For a space (X, d) satisfying (1) and $A \subset X$, the diameter of A is defined by

$$diam(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Let us recall the following recent result.

Proposition 1 (see [4])**.** *Let a pair of mappings (f, g) have a unique POC. Then it is WC if and only if it is OWC.*

3. Main results

We start with an auxiliary result.

Proposition 2. *Let $d : X \times X \to [0, \infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Let a pair of mappings (f, g) have a unique POC. If it is a pair of PD-operators, then it is WC.*

Proof. First we have that $C(f, g) \neq \phi$ because $PC(f, g) \neq \phi$ ($PC(f, g)$ is a singleton). Since (f, g) is a PD-operator, then there exists some $x \in C(f, g)$ such that $d(fgx, gfx) \leq diam(PC(f, g)) = 0$. Hence $d(fgx, gfx) = 0$, i.e., there exists $fx = gx$ with $fgx = gfx$. Therefore the pair (f, g) is OWC. According to [4], (f, g) is WC. □

Proposition 3. *Let $\phi : R_+ \to R_+$ be a nondecreasing function satisfying the condition $\phi(t) < t$, for each $t > 0$, and let $d : X \times X \to [0, \infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Suppose that (f, g) is a PD-operator pair and satisfying the following condition:*

$$d(fx, fy) \leq \phi(max\{d(gx, gy), d(gx, fy), d(fx, gy), d(gy, fy)\}), \tag{2}$$

for each $x, y \in X$. Then f and g are WC.

Proof. By hypothesis, there exists some $x \in X$ such that $w = fx = gx$. It remains to show that (f, g) has a unique POC. Suppose there exists another point $w_1 = fy = gy$ with $w \neq w_1$. Then, we have

$$\begin{aligned} d(w, w_1) &= d(fx, fy) \\ &\leq \phi(max\{d(gx, gy), d(gx, fy), d(fx, gy), d(gy, fy)\}) \\ &< d(w, w_1), \end{aligned}$$

a contradiction. Thus (f, g) has a unique POC. By Proposition 2, the pair (f, g) is WC. □

In a recent paper, Pathak and Rai [15] proved the following theorem:

Theorem 1 (see [15])**.** *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function satisfying condition* (1)*. Suppose that the pair (f, g) is a PD-operator satisfying condition* (2)*, then (f, g) have a unique common fixed point.*

Now we prove our main result.

Theorem 2. *Under contractive condition* (2) *assumed in Theorem 1, the assumption of PD-operators and the existence of a unique common fixed point are equivalent conditions.*

Proof. We first observe that under contractive condition (2), the assumption of PD-operators and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that f and g satisfy the contractive condition (2). If f and g have a common fixed point, say z, then $z = fz = gz$, $fgz = gfz = z$. Thus, f and g are PD-operators, since contractive condition (2) excludes the existence of two points of coincidence or common fixed points.

On the other hand, suppose that f and g are PD-operators such that $fx = gx$ and

$$d(fgx, gfx) \leq diam(PC(f, g))$$

for some $x \in C(f, g)$. Now, in view of condition (2), we get $diam(PC(f, g)) = 0$, (since contractive condition (2) excludes the existence of two points of coincidence). Hence $fgx = gfx$. If $fx \neq ffx$, using (2) we get

$$\begin{aligned} d(ffx, fx) &\leq \phi(\max(d(gfx, gx), d(gfx, fx), d(ffx, gx), d(gx, fx))) \\ &< d(ffx, fx), \end{aligned}$$

a contradiction. Hence $fx = ffx$ and $fx = ffx = gfx$. This means that fx is a common fixed point of f and g. Uniqueness of the fixed point follows from contractive condition (2). □

Remark 1. *Let us remark that the same results in Theorem 2 will also be true for many contractive conditions assumed in paper [15], e.g.,*

$$d(fx, fy) < \max(d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)). \tag{3}$$

Therefore, under contractive conditions (2) *and* (3) *the existence of a common fixed point and PD-operators are equivalent conditions. In order to find actual applications of PD-operators one should go beyond contractive conditions, since contractive conditions do not allow more than one point of coincidence or fixed point.*

Problem 1. *It would be interesting to know if the related results of Theorem 2 are true in nonself cases ([5, 6])?*

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