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The extensibility of D(4)-pairs

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Abstract. A set of m positive integers with the property that the product of any two of them increased by 4 is a perfect square is called a $D(4)$-m-tuple. In this paper, we consider the extensibility of a general $D(4)$-pair $\{a, b\}$ and prove some results supporting the conjecture that there does not exist a $D(4)$-quintuple.

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1. Introduction

Let $n \neq 0$ be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$, or simply a $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The most famous and most studied case is $n = 1$. There is a folklore conjecture that there does not exist a $D(1)$-quintuple. In 2004, Dujella [4] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples. In recent years, the second author [7, 8, 9, 10] proved analogous results in the case $n = 4$. Recently, there has been a lot of work done on $D(n)$-m-tuples, especially in the cases $n = 1$, $n = -1$ and $n = 4$. To see all details the reader can visit the webpage http://web.math.pmf.unizg.hr/˜duje/dtuples.html. Here, we consider only the case $n = 4$.

In the case $n = 4$, there is a conjecture (see [6, Conjecture 1]), that if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then

$$d = d_+ = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s and t are positive integers defined by $ab+4 = r^2$, $ac+4 = s^2$ and $bc+4 = t^2$. Notice that this immediately implies that there does not exist a $D(4)$-quintuple. The $D(4)$-quadruple $\{a, b, c, d\}$, where $d > \max\{a, b, c\}$ is called a regular quadruple if $d = d_+$. We also define $d_- = a + b + c + \frac{1}{2}(abc - rst)$. The set $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple if $d_- \neq 0$, but then $d_- < c$.

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There are many results which support the Conjecture (see for example [1, 2, 4, 7, 8, 9, 10, 12, 13, 14, 16, 17]). Recently, the authors [2] have proved that a $D(4)$-triple of the form $\{k-2, k+2, c\}$, where $k \geq 3$ is an integer, has a unique extension to a quadruple with a larger element. In this paper, we try to generalize that result by proving the following theorem.

Theorem 1. *Let $\{a, b, c\}$ be a $D(4)$-triple with $a < b$. Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $d > d_+$ and that $\{a, b, c', c\}$ is not a $D(4)$-quadruple for any c' with $0 < c' < d_-$.*

(1) If $b < 1.5a$, then $c < b^6$.

(2) If $1.5a \leq b < 5a$, then $c < b^5$.

(3) If $b \geq 5a$, then $c < 6b^5$.

Unfortunately we were not able to generalize the result from [2] completely, because the congruence method does not work for a general triple $\{a, b, c\}$, or at least it is not so obvious. But if we take a and b to be fixed and we furthermore know all possible c's which extend the $D(4)$-pair $\{a, b\}$, by proving this Theorem we know exactly which triples are left to be examined. For example, considering the extension of a parametric family of $D(4)$-pairs $\{F_{2k}, 5F_{2k}\}$ (which will be treated in a subsequent paper), by using this Theorem we can save a lot of time proving non-extendibility to a $D(4)$-quintuple, actually proving the uniqueness of extension of $D(4)$-triples $\{F_{2k}, 5F_{2k}, c\}$ to a quadruple. Although there we can use the version of Rickert's theorem from [6], the problem is with congruences, because it is not so obvious which congruences to look at for general c as it was done in [2] for the $D(4)$-pairs $\{k-2, k+2\}$. And with this Theorem we are only left with a few exact values of c (depending only on k) to consider.

In the proof of Theorem we use standard methods which are used in a solving similar kind of problems. So we firstly transform our problem of extensibility of $D(4)$-triples to solving systems of simultaneous Pellian equations. And it furthermore leads to finding the intersection of binary recurrence sequences. It is eventually solved by combining the congruence method firstly introduced by Dujella and Pethö in [5] with the hypergeometric method and Baker's theory of linear forms in logarithms. The main part in our proof is to improve Rickert's theorem in our special case similarly to what we have done in [2]. Let us also mention that the analogous result in the case $n = 1$ was recently proven by the second author, Fujita and Togbé in [11]. Here we use few more technical tricks to make our proof slightly faster. Those 'tricks' are that we make Baker-Davenport reduction at the beginning and in the case of sequences with odd indices we do not have to consider all subcases depending on the sign of initial values. Also, because some proofs are the same as in [2] and [11] we will not give all details here but we will cite the exact reference where they can be found.

2. Preliminaries

We first prove one useful result showing how to get the third element c in a $D(4)$-triple $\{a, b, c\}$ if we know the first two and if they satisfy $a < b < 5a$.

Let $\{a, b, c\}$ be a $D(4)$-triple and s, t positive integers satisfying $ac + 4 = s^2$, $bc + 4 = t^2$. Eliminating c, we get

$$at^2 - bs^2 = 4(a - b). \tag{1}$$

Now we can generate all solutions of this equation if we know the fundamental ones. If (t, s) belongs to the same class as either of the solutions $(\pm 2, 2)$, which means that

$$t\sqrt{a} + s\sqrt{b} = (\pm 2\sqrt{a} + 2\sqrt{b})\left(\frac{r + \sqrt{ab}}{2}\right)^{\nu},$$

then s can be expressed as $s = s_\nu^\pm$, where

$$s_0^\pm = 2,\ s_1^\pm = r \pm a,\ s_{\nu+2}^\pm = rs_{\nu+1}^\pm - s_\nu^\pm \tag{2}$$

with r being a positive integer satisfying $ab + 4 = r^2$. Define $c_\nu^\pm = ((s_\nu^\pm)^2 - 4)/a$. Now we have the following lemma.

Lemma 1. *Let $\{a, b, c\}$ be a $D(4)$-triple. Assume that $a < b < 5a$. Then $c = c_\nu^\pm$ for some ν.*

Proof. The proof uses the same idea which was used in proving the analogous result in [15] for the case $n = 1$ and $a < b < 4a$.

Let us define s', t' by $s' = \frac{rs - at}{2}$ and $t' = \frac{rt - bs}{2}$. Then, it is easy to see that (t, s) and (t', s') belong to the same class of solutions of Pellian equation (1), because we have

$$t\sqrt{a} + s\sqrt{b} = (t'\sqrt{a} + s'\sqrt{b})\left(\frac{r + \sqrt{ab}}{2}\right).$$

Also define $c' = ((s')^2 - 4)/a$. Notice that we may assume that $c' < r^2$ because we can continue with the same procedure defining now $c = c'$ until we get $c' < r^2$. We can do this because in every step we have

$$ac' = (s')^2 - 4 < \frac{s^2}{r^2} - 4 = \frac{ac + 4}{r^2} - 4 < \frac{ac}{r^2},$$

which implies $0 \le c' < \frac{c}{r^2}$. In this way we always remain in the same class of solutions.

Now, if $c' > b$, then in the same way as in [15, Theorem 2], using $c' < r^2$, we get $c' = a + b + 2r = c_1^+$ and hence $c = c_\nu^+$ for some ν.

If $c' = b$, then $a + b + c + \frac{1}{2}(abc - rst) = b$ implies

$$c^2 - (2a + 4b + ab^2)c + a^2 - 4ab - 16 = 0.$$

The discriminant of this equation with respect to c is equal to $(ab + 4)^2(b^2 + 4)$, so we must have $b = 0$, a contradiction.

Suppose now that $0 < c' < b$. Let us define $r' = \frac{s'r - at'}{2}$ and $b' = ((r')^2 - 4)/a$. Then, $b' = a + b + c' + \frac{1}{2}(abc' - rs't')$, and because $\{a, b', c', b\}$ is a regular $D(4)$-quadruple $b < 5a$ implies that

$$b' < \frac{b}{ac'} < \frac{5a}{ac'} = \frac{5}{c'}$$

and $b'c' < 5$. But with such defined numbers $\{a, b', c'\}$ is a $D(4)$-triple (with a possibility that some element is equal to 0), so it implies that $b'c' + 4$ is a perfect square. This furthermore implies that $b' = 0$ which yields $c' = a + b - 2r = c_1^-$, whence $c = c_\nu^-$ for some ν.

If $c' = 0$, then $s' = t' = 2$ and therefore $c = s + t$ which is possible only if $s = a + r$, $t = b + r$ and $c = a + b + 2r = c_1^+$, so again $c = c_\nu^+$ for some ν. □

Let $\{a, b, c\}$ be a $D(4)$-triple and r, s, t positive integers satisfying $ab + 4 = r^2$, $ac + 4 = s^2$, $bc + 4 = t^2$. Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $d_+ < d$. Then, there exist positive integers x, y, z such that $ad + 4 = x^2$, $bd + 4 = y^2$, $cd + 4 = z^2$, from which we obtain

$$az^2 - cx^2 = 4(a - c), \tag{3}$$

$$bz^2 - cy^2 = 4(b - c). \tag{4}$$

Positive solutions of Pellian equations (3) and (4) respectively have the forms:

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})\left(\frac{s + \sqrt{ac}}{2}\right)^m, \tag{5}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})\left(\frac{t + \sqrt{bc}}{2}\right)^n, \tag{6}$$

where m, n are non-negative integers and (z_0, x_0), (z_1, y_1) are fundamental solutions of (3), (4), respectively, satisfying

$$2 \le x_0 < \sqrt{s + 2}, 2 \le |z_0| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}},$$

$$2 \le y_1 < \sqrt{t + 2},\ 2 \le |z_1| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}}.$$

Thus, we have $z = v_m = w_n$, where

$$v_0 = z_0, \quad v_1 = \frac{sz_0 + cx_0}{2}, \quad v_{m+2} = sv_{m+1} - v_m,$$

$$w_0 = z_1, \quad w_1 = \frac{tz_1 + cy_1}{2}, \quad w_{n+2} = tw_{n+1} - w_n. \tag{7}$$

In what follows, we assume that

$$\{a, b, c', c\} \text{ is not a } D(4)\text{-quadruple for any } c' \text{ with } 0 < c' < d_-, \tag{8}$$

where $d_- = a + b + c + \frac{1}{2}(abc - rst)$, in order to furthermore narrow the possibilities for fundamental solutions (z_0, x_0) and (z_1, y_1).

Lemma 2. *Assume* (8) *and* $c \ge b^5$. *Then,* $v_{2m+1} \ne w_{2n}$ *and* $v_{2m} \ne w_{2n+1}$. *Moreover, we obtain the following.*

(i) If $v_{2m} = w_{2n}$, *then* $z_0 = z_1$ *and* $|z_0| = |z_1| = 2$.

(ii) If $v_{2m+1} = w_{2n+1}$, *then* $|z_0| = t$, $|z_1| = s$ *and* $z_0 z_1 > 0$.

Proof. The proof of this lemma is the same as the proof of [2, Lemma 3] where we have considered the case of extendibility of $D(4)$-pairs $\{k-2, k+2\}$, the novelty being the use of the fact that $c \geq b^5$. □

3. Congruences

In this section we firstly prove the following lemma.

Lemma 3. *Let* $\{a, b, c\}$ *be a* $D(4)$*-triple with* $a < b$. *If* $b \leq 10^4$, *then the only extension of* $\{a, b, c\}$ *to a* $D(4)$*-quadruple is with* $d = d_+$ *and* $d = d_-$.

Proof. To prove this we use the Baker-Davenport reduction method which is explained in detail in [7, Section 5]. Notice that we can use the exact values for fundamental solutions from Lemma 2. We have done this in Mathematica 8 package where we start with fixed a and b for which we have upper bounds, and then we start with smallest possible c to prove the uniqueness of the extension of a $D(4)$-triple $\{a, b, c\}$. On that way we can always assume (8). Remember that from [7, Lemma 8] we have an upper bound for c and the first upper bound for indices m and n using the same arguments as in [7, Section 5]. The only difference is that there it was done for a general triple $\{a, b, c\}$.

In the case $b < 5a$ we use that we know exactly all possible c's which are given by formula

$$c = c_\nu^{\pm} = \frac{4}{ab}\left\{\left(\frac{\sqrt{b} \pm \sqrt{a}}{2}\right)^2 \left(\frac{r + \sqrt{ab}}{2}\right)^{2\nu} + \left(\frac{\sqrt{b} \mp \sqrt{a}}{2}\right)^2 \left(\frac{r - \sqrt{ab}}{2}\right)^{2\nu} - \frac{a+b}{2}\right\},$$

while in the case $b \geq 5a$ we find all possible c's by solving equation (1) in s and t knowing the upper bounds for fundamental solutions. It takes us slightly more time in that way, but the interesting thing is that in most of the cases we get only the fundamental solutions $(s_0, t_0) = (2, \pm 2)$ so $c = c_\nu^{\pm}$.

To run all of this in Mathematica it took us around 400 hours with the processor Intel(R) Core(TM) i3 CPU 550 @3.20GHz.

□

Now using this we get the following Lemma.

Lemma 4. *Assume* (8) *and* $c \geq b^5$.

(i) If $v_{2m} = w_{2n}$, *then* $m < \begin{cases} 1.008n & \text{if } b < 1.5a, \\ 1.031n & \text{if } 1.5a \leq b < 5a, \\ 1.21n & \text{if } b \geq 5a. \end{cases}$

(ii) If $v_{2m+1} = w_{2n+1}$, *then* $m < \begin{cases} 1.008n & \text{if } b < 1.5a, \\ 1.031n & \text{if } 1.5a \leq b < 5a, \\ 1.21n & \text{if } b \geq 5a. \end{cases}$

Proof. (i) Suppose that $v_{2m} = w_{2n}$. Since

$$v_{2m} > v_1(s-1)^{2m-1},$$
$$w_{2n} < w_1(t)^{2n-1},$$

and $c \geq b^5$ in any case, we have $(s-1)^{2m-1} < \frac{c+t}{c+s}(t)^{2n-1}$. We now have

$$\frac{c+t}{c+s} \cdot (t)^{2n-1} < 1.001 \cdot (bc+4)^{n-1/2} < 1.001^{2n}(bc)^{n-1/2}$$

and

$$(s-1)^{2m-1} > 0.999^{2m-1}(ac)^{m-1/2}.$$

The first two statements now follow easily from this, using that $b > 10^4$ and the upper bound for b in terms of a.

In the case $b \geq 5a$, we have $0.999^{2m-1}c^m < 1.001^{2n}c^{(12n-1)/10}$, which implies $m < 1.21n$. Because if we suppose $m \geq 1.21n$, using the fact that $c \geq b^5 > 10^{20}$, we have

$$20(0.01n + 0.1)\log(10) < 2n\log(1.001) - (2m-1)\log(0.999).$$

It furthermore yields, using that we always have $n \leq m < 2n$,

$$0.46n + 4.6 < 0.007n,$$

which is a contradiction for any positive integer n.

(ii) Suppose that $v_{2m+1} = w_{2n+1}$. The proof of this case is the same, but here we also use that $v_1{=}w_1$. □

Lemma 5. *Assume* (8).

(i) If $v_{2m} = w_{2n}$, *then*

$$n > \begin{cases} 1.414a^{-1/2}c^{1/8} & \text{if } b < 1.5a \text{ and } c \geq b^6, \\ 0.183a^{1/2}b^{-1}c^{1/2} & \text{if } b \geq 1.5a \text{ and } c \geq b^5, \\ 0.863a^{1/2}b^{-1}c^{1/2} & \text{if } b \geq 5a \text{ and } c \geq 6b^5. \end{cases}$$

(ii) If $v_{2m+1} = w_{2n+1}$ *and* $c \geq b^5$, *then* $n > 0.433b^{-3/4}c^{1/4}$.

Proof. We give the proof only for the first case in (i) because the proof, based on the well known congruence method introduced in [5], is the same as for the analogous lemma in [11]. Suppose that $v_{2m} = w_{2n}$. Note that we may assume that $m \geq 2$ and $n \geq 2$. We see with $z_0 = z_1 = \pm 2$ and $x_0 = y_1 = 2$ that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{9}$$

Consider now $b < 1.5a$ and $c \geq b^6$. Suppose that $n \leq 1.414a^{-1/2}c^{1/8}$. Since $m < 1.008n$ by Lemma 4 and $c \geq b^6 > 10^{24}$, it is easy to see that

$$am^2 < \frac{c}{2}, \ \ sm < \frac{c}{2}, \ \ bn^2 < \frac{c}{2}, \ \ tn < \frac{c}{2}. \tag{10}$$

Then, it follows from (9) that

$$\pm am^2 + sm = \pm bn^2 + tn. \tag{11}$$

Moreover, squaring both sides of (9) twice, we have

$$\left\{(am^2 - bn^2)^2 - 4(m^2 + n^2)\right\}^2 \equiv 64m^2n^2 \pmod{c}. \tag{12}$$

Since

$$\left\{(am^2 - bn^2)^2 - 4(m^2 + n^2)\right\}^2 < (0.5an^2)^4 < c,$$
$$64m^2n^2 \leq 64 \cdot 1.008^2 \cdot 1.414^4 a^{-2} c^{1/2} < c,$$

(12) in fact becomes an equation, and hence we have

$$am^2 - bn^2 = \pm 2(m \pm n). \tag{13}$$

If $am^2 - bn^2 = \pm 2(m + n)$, we have

$$\pm \left\{ a \left(\frac{t \pm 2}{s \mp 2} \right)^2 - b \right\} n = \pm 2 \left(\frac{t \pm 2}{s \mp 2} + 1 \right).$$

Then we have either

$$n = \frac{(s+2)(s+t)}{2(2b - 2a + at + bs)} > \frac{ac}{b(2s+4)} > 0.33a^{-1/2}c^{1/2} > 1.414a^{-1/2}c^{1/8},$$

which is a contradiction, or we have

$$n = \frac{(s-2)(s+t)}{2(2a - 2b + at + bs)} > \frac{s-2}{2b} > 0.33a^{-1/2}c^{1/2} > 1.414a^{-1/2}c^{1/8},$$

a contradiction again. Similarly, if $am^2 - bn^2 = \pm 2(m - n)$, we have

$$n = \frac{(t-s)(s \pm 2)}{2(bs - at \pm 2(b-a))} > \frac{t-s}{2(b-a)} = \frac{c}{2(s+t)} > 0.24b^{-1/2}c^{1/2} > 1.414a^{-1/2}c^{1/8},$$

which is a contradiction. Therefore, if $b < 1.5a$ and $c \geq b^6$, then $n > 1.414a^{-1/2}c^{1/8}$. □

4. Proof of Theorem 1

In this Section, we firstly give the improvement of [3, Theorem 3.2] and [18, Theorem] where we use the important fact that in our case N is divisible by ab similarly to what was done in [2].

Theorem 2. *Let a, b and N be integers with $0 < a \leq b - 7$, $b \geq 12$ and $N \geq 308.07a'b^2(b-a)^2$, where $a' = \max\{4(b-a), 4a\}$. Assume that N is divisible by ab. Then the numbers*

$$\theta_1 = \sqrt{1 + \frac{4b}{N}} \quad \textit{and} \quad \theta_2 = \sqrt{1 + \frac{4a}{N}}$$

satisfy

$$\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > \left(\frac{16.01a'bN}{a} \right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(8.01a^{-1}a'bN)}{\log(0.026a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2.$$

Proof. Proof of this Theorem is the same as the proof of [2, Theorem 2]. □

Now using the last Theorem for $N = abc$, $p_1 = sbx$, $p_2 = tay$ and $q = abz$ and combining it with [6, Lemma 6] we can prove the following.

Lemma 6. *Let $\{a, b, c, d\}$ be a $D(4)$-quadruple with $a < b < c < d$. Assume that $c > 308.07a'b(b-a)^2/a$, where $a' = \max\{4a, 4(b-a)\}$. Then,*

$$\log z < \frac{\log(32.02aa'b^4c^2)\log(0.026ab(b-a)^{-2}c^2)}{\log(0.00325a(a')^{-1}b^{-1}(b-a)^{-2}c)}.$$

And eventually we need one more lemma which is actually part of the proof of [7, Lemma 8].

Lemma 7. *Assume that $c \geq b^5$. If $z = v_{m'} = w_{n'}$ with $(m', n') \in \{(2m, 2n), (2m+1, 2n+1)\}$, then*

$$\log z > \frac{n'}{2}\log(bc).$$

At the end we prove the Proposition which will finish the proof of the main Theorem.

Proposition 1. *(1) If $b < 1.5a$ and $c \geq b^6$, then $b \leq 9940$.*

(2) If $1.5a \leq b < 5a$ and $c \geq b^5$, then $b \leq 3505$.

(3) If $b \geq 5a$ and $c \geq 6b^5$, then $b \leq 8877$.

Proof. Here we give only the proof of (1) because the proofs for other cases are the same using the previous results.

Since $c \geq b^5$ in any case, and $b > 10^4$, we have $c > 308.07a'b(b-a)^2/a$ and we may apply Lemma 6, which together with Lemma 7 implies that

$$\frac{n'}{2} < \frac{\log(32.02a(a')b^4c^2)\log(0.026ab(b-a)^{-2}c^2)}{\log(bc)\log(0.00325a(a')^{-1}b^{-1}(b-a)^{-2}c)}, \tag{14}$$

where $n' \in \{2n, 2n+1\}$ and $a' = \max\{4a, 4(b-a)\}$.

(1) Assume that $b < 1.5a$ and $c \geq b^6$. Since $4b/1.5 < a' = 4a < 4b$ and $8 < b - a < \frac{b}{3}$, we have

$$32.02a(a')b^4c^2 < 32.02 \cdot 4b^6c^2 = 128.08b^6c^2,$$
$$0.026ab(b-a)^{-2}c^2 < 0.026b^2 \cdot 8^{-2}c^2 < 0.00041b^2c^2,$$
$$0.00325a(a')^{-1}b^{-1}(b-a)^{-2}c > 0.00325 \cdot a(4a)^{-1}b^{-1}\left(\frac{b}{3}\right)^{-2} c > 0.00731b^{-3}c,$$

which together with (14) implies that

$$\frac{n'}{2} < \frac{\log(128.08b^6c^2)\log(0.00041b^2c^2)}{\log(bc)\log(0.00731b^{-3}c)} =: f(c).$$

Since $f(c)$ is a decreasing function with respect to c for $c \geq b^6$, we have $f(c) \leq f(b^6)$ and thus

$$\frac{n'}{2} < \frac{\log(128.08b^{18})\log(0.00041b^{14})}{\log(b^7)\log(0.00731b^3)}. \tag{15}$$

(i) If $v_{2m} = w_{2n}$, then we have

$$1.414b^{-0.5}(b^6)^{0.125} < \frac{\log(128.08b^{18})\log(0.00041b^{14})}{\log(b^7)\log(0.00731b^3)},$$

which implies $b \leq 9940$.

(ii) If $v_{2m+1} = w_{2n+1}$, then we have

$$0.433b^{-0.75}(b^6)^{0.25} + 0.5 < \frac{\log(128.08b^{18})\log(0.00041b^{14})}{\log(b^7)\log(0.00731b^3)},$$

which implies $b \leq 127$.

□

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