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Some inequalities for polynomials and transcendental entire functions of exponential type

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Abstract. Let p be a polynomial of degree n such that $|p(z)| \leq M$ $(|z| = 1)$. The Bernstein's inequality for polynomials states that $|p'(z)| \leq Mn$ $(|z| = 1)$. A polynomial p of degree n that satisfies the condition $p(z) \equiv z^n p(1/z)$ is called a self-reciprocal polynomial. If p is a self-reciprocal polynomial, then $f(z) = p(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type n such that $f(z) = \mathrm{e}^{\mathrm{i}nz} f(-z)$. Thus the class of entire functions of exponential type τ whose elements satisfy the condition $f(z) = \mathrm{e}^{\mathrm{i}\tau z} f(-z)$ is a natural generalization of the class of self-reciprocal polynomials. In this paper we present some Bernstein's type inequalities for self-reciprocal polynomials and related entire functions of exponential type under certain restrictions on the location of their zeros.
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1. Introduction and statement of results

1.1. Bernstein's inequality for polynomials

Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n and let $f \in \mathcal{P}_n$. An inequality for polynomials in $\mathcal{P}_n$, known as Bernstein's inequality, gives an estimate for $|f'(z)|$ on the unit circle in terms of the maximum of $|f(z)|$ on the same circle. It states (see [15], p. 508) that

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)|, \qquad f \in \mathcal{P}_n, \tag{1}$$

where the equality holds for polynomials of the form $cz^n, c \neq 0$.

It is known [13] that if f is as above and $f^*(z) := z^n \overline{f(1/\overline{z})}$, then on $|z| = 1$

$$|f'(z)| + |f^{*\prime}(z)| \leq n \max_{|z|=1} |f(z)|, \qquad f \in \mathcal{P}_n. \tag{2}$$

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Let $\mathcal{P}_n^{\sim}$ be the subclass of $\mathcal{P}_n$ consisting of all polynomials f which satisfy the condition $f(z) \equiv f^*(z)$. It follows from (2) that

$$\max_{|z|=1} |f'(z)| \le \frac{n}{2} \max_{|z|=1} |f(z)|, \qquad f \in \mathcal{P}_n^{\sim}. \tag{3}$$

Let $f \in \mathcal{P}_n$ and z_0 a point on the unit circle such that $|f(z_0)| = \max_{|z|=1} |f(z)|$. Clearly, $|f^{*\prime}(z_0)| = |nf(z_0) - z_0 f'(z_0)| \ge n|f(z_0)| - |f'(z_0)|$. Hence, if $f \in \mathcal{P}_n^{\sim}$, then

$$\max_{|z|=1} |f'(z)| \ge \frac{n}{2} |f'(z_0| = \frac{n}{2} \max_{|z|=1} |f(z)|$$

and so, in (3), the inequality sign "$\le$" may be replaced by "$=$". Thus, we have

$$\max_{|z|=1} |f'(z)| = \frac{n}{2} \max_{|z|=1} |f(z)|, \qquad f \in \mathcal{P}_n^{\sim}. \tag{4}$$

The subclass $\mathcal{P}_n^{\sim}$ of $\mathcal{P}_n$ is of considerable importance. There is another subclass of $\mathcal{P}_n$ which has proved itself to be equally significant, if not more. It consists of those polynomials f in $\mathcal{P}_n$ which satisfy the condition $f(z) \equiv z^n f(1/z)$. Let us denote it by $\mathcal{P}_n^{\vee}$. The condition defining the subclass $\mathcal{P}_n^{\vee}$ looks very similar to the one defining $\mathcal{P}_n^{\sim}$. As regards the distribution of their zeros, polynomials in $\mathcal{P}_n^{\sim}$ and those in $\mathcal{P}_n^{\vee}$, they all have at least half of their zeros outside the open unit disk (here it is understood that a polynomial f belonging to $\mathcal{P}_n$ but of degree $m < n$ has $n-m$ of its zeros at ∞).

Frappier, Rahman and Ruscheweyh ([6], p. 97) showed that for the polynomial $f(z) := \{(1 - \mathrm{i}z)^2 + z^{n-2}(z - \mathrm{i})^2\}/4$, which clearly belongs to $\mathcal{P}_n^{\vee}$, we have

$$\max_{|z|=1} |f(z)| = 1 = |f(\mathrm{i})| \ \text{ whereas } \ |f'(-\mathrm{i})| = n - 1,$$

thus exhibiting a polynomial f in $\mathcal{P}_n^{\vee}$ for which

$$\max_{|z|=1} |f'(z)| \ge (n-1) \max_{|z|=1} |f(z)|. \tag{5}$$

Later Frappier, Rahman and Ruscheweyh ([7, Theorem 2]) proved that for polynomials $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$, whose constant term a_0 is equal to a_n (the coefficient of the leading term $a_n z^n$), we have

$$\max_{|z|=1} |f'(z)| \le \left(n - \frac{1}{2} + \frac{1}{2(n+1)}\right) \max_{|z|=1} |f(z)|. \tag{6}$$

Since f belongs to $\mathcal{P}_n^{\vee}$ if and only if $a_k = a_{n-k}$ for each k $(k = 0 \ldots n)$, the above inequality certainly holds for polynomials in $\mathcal{P}_n^{\vee}$. Inequalities (5) and (6) show that by restricting ourselves to the subclass $\mathcal{P}_n^{\vee}$, we do not obtain a meaningful improvement on the Bernstein's inequality (1). This is quite surprising since the two classes $\mathcal{P}_n^{\sim}$ and $\mathcal{P}_n^{\vee}$ look similar; for $\mathcal{P}_n^{\sim}$ holds formula (4) by which $|f'(z)|$ at a point of the unit circle cannot be larger than $n/2$ times $M := \max_{|z|=1} |f(z)|$ if $f \in \mathcal{P}_n^{\sim}$ while it can be as large as $n - 1$ times M if f belongs to $\mathcal{P}_n^{\vee}$, as (5) says.

However, under some additional restrictions, either on the location of the zeros or on the coefficients of polynomials in $\mathcal{P}_n^\vee$, the bound in (6) can be improved. For example, Rahman and Tariq [16] (see also [11]) proved that for a polynomial $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ in $\mathcal{P}_n^\vee$, whose coefficients lie in a sector of opening $0 \le \gamma < \pi$ with the vertex at the origin, we have

$$\max_{|z|=1} |f'(z)| \le \frac{n}{2\cos(\gamma/2)} |f(1)|. \tag{7}$$

In the case when n is an even integer, the equality holds in (7) for the polynomial $f(z) = z^n + 2\mathrm{e}^{\mathrm{i}\gamma} z^{n/2} + 1$.

On the other hand, if we assume that all the zeros of f are in the left half plane or in the right half plane [9], then

$$\max_{|z|=1} |f'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1} |f(z)|. \tag{8}$$

Very few sharp results are known about the class $\mathcal{P}_n^\vee$ although many papers have been written on the subject since 1976 (see for example, [9, 11, 16]). In fact, the sharp inequality analogous to (1) is still unknown even for $n = 3$.

The Bernstein's inequality has been generalized in many ways. For example, if f is a polynomial in $\mathcal{P}_n$, then by Zygmund [19] for any $p \ge 1$, we have

$$\int_{-\pi}^{\pi} |f'(e^{i\theta})|^p \, d\theta \le n^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta, \qquad f \in \mathcal{P}_n. \tag{9}$$

If we assume that f belongs to $\mathcal{P}_n^\sim$, the above inequality can be improved. In this case Dewan and Govil [5] proved the following result

$$\int_{-\pi}^{\pi} |f'(e^{i\theta})|^p \, d\theta \le n^p \, C_p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta, \qquad f \in \mathcal{P}_n^\sim, \tag{10}$$

where

$$C_p = \frac{2\pi}{\int_{-\pi}^{\pi} |1 + e^{i\alpha}|^p d\alpha} = 2^{-p} \, \frac{\sqrt{\pi}\, \Gamma(p/2+1)}{\Gamma(p/2+1/2)}. \tag{11}$$

In this paper, we present a property of polynomials in $\mathcal{P}_n^\vee$ which have all their zeros in the left half plane. More precisely, we have the following

Theorem 1. *Let f be a polynomial in $\mathcal{P}_n^\vee$ having all its zeros in the left half plane. Suppose in addition that its zeros which lie in the second quadrant are of modulus at most* 1. *Then*

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| \le |f'(\mathrm{e}^{\mathrm{i}\theta})|, \qquad 0 \le \theta \le \pi. \tag{12}$$

As the first application of Theorem 1, we will prove the following L^p inequality for the subclass $\mathcal{P}_n^\vee$. We do not know if it is sharp.

Corollary 1. *Let f, which has all its zeros in the left half plane, belong to $\mathcal{P}_n^\vee$. Furthermore, the zeros in the second quadrant are in the unit disk $\{z : |z| \le 1\}$. Then, for $p \ge 1$*

$$\int_{-\pi}^{0} |f'(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta \le n^p \, C_p \int_{-\pi}^{0} |f(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta, \tag{13}$$

where C_p is as given in (11).

As the next application we state the following corollary.

Corollary 2. *Let f, which has all its zeros in the left half plane, belong to $\mathcal{P}_n^{\vee}$. Furthermore, the zeros in the second quadrant are in the unit disk $\{z : |z| \le 1\}$. Suppose that $|f(\mathrm{e}^{-\mathrm{i}\theta})| \le M$ for $0 \le \theta \le \pi$. Then*

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| \le M\frac{n}{2}, \qquad 0 \le \theta \le \pi. \tag{14}$$

The example $f(z) = (z^2+1)^{\frac{n}{2}}$ shows that the estimate is sharp when n is even. For odd n, the equality holds for $f(z) = (z+1)^n$.

1.2. Transcendental entire functions of exponential type

For an entire function f and a real number $r > 0$, let $M(r) = M_f(r) := \max_{|z|=r} |f(z)|$. Unless f is a constant of modulus less than or equal to 1, its order, which is denoted by ρ, is defined to be $\limsup_{r\to\infty} (\log r)^{-1} \log\log M(r)$. Constants of modulus less than or equal to 1 are of order 0 by convention.

If f is of finite positive order ρ, then $T := \limsup_{r\to\infty} r^{-\rho} \log M(r)$ is called its type.

An entire function f is said to be of exponential type τ if for any $\varepsilon > 0$ there exists a constant $k(\varepsilon)$ such that $|f(z)| \le k(\varepsilon)\mathrm{e}^{(\tau+\varepsilon)|z|}$ for all $z \in \mathbb{C}$. Any entire function of order less than 1 is of exponential type τ, where τ can be taken to be any number greater than or equal to 0. Functions of order 1 type $T \le \tau$ are also of exponential type τ.

If f is an entire function of exponential type, then its indicator function $h_f(\theta)$ is defined by $h_f(\theta) := \limsup_{r\to\infty} r^{-1} \log |f(r\mathrm{e}^{\mathrm{i}\theta})|$. It describes the growth of f along the ray $\{z | \arg z = \theta\}$. $h_f(\theta)$ is either finite or $-\infty$ and is a continuous function of θ unless it is identically $-\infty$.

For a detailed discussion on entire functions of exponential type, we refer the reader to Boas [4].

Bernstein [2], (see also [3], p. 102) extended inequality (1) to arbitrary entire functions of exponential type bounded on the real line.

Theorem 2. *Let f be an entire function of exponential type $\tau > 0$ such that $|f(x)| \le M$ on the real axis. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \le M\tau. \tag{15}$$

The equality in (15) holds if and only if $f(z) \equiv a\mathrm{e}^{\mathrm{i}\tau z} + b\mathrm{e}^{-\mathrm{i}\tau z}$, where $a, b \in \mathbb{C}$.

If $f \in \mathcal{P}_n^{\vee}$, then $g(z) := f(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type which satisfies the condition $g(z) \equiv \mathrm{e}^{\mathrm{i}nz} g(-z)$. Moreover, its type is n. This suggests that the class of entire functions of exponential type that generalizes $\mathcal{P}_n^{\vee}$ consists of entire functions of exponential type f such that $f(z) \equiv \mathrm{e}^{\mathrm{i}\tau z} f(-z)$. Let us denote this class by $\mathcal{F}_\tau^{\vee}$ which has been studied by Govil [8], Rahman and Tariq [17, 18].

Rahman and Tariq ([17, Theorem 2]) proved the following Theorem which is akin to (5), a result proved by Frappier, Rahman and Ruscheweyh [7] for polynomials.

Theorem 3. *For a given positive number ε, as small as we please, there exists an entire function $f_\varepsilon \in \mathcal{F}_\tau^\vee$ such that*

$$\sup_{-\infty<x<\infty} |f_\varepsilon'(x)| \ \geq\ (\tau - \varepsilon) \sup_{-\infty<x<\infty} |f_\varepsilon(x)|. \tag{16}$$

Like polynomials, improved inequalities for $\mathcal{F}_\tau^\vee$ can be obtained if we impose some additional restriction on it. For example, Rahman and Tariq ([17, Theorem 1]) proved the following theorem for functions in $\mathcal{F}_\tau^\vee$ which are uniformly almost periodic on the real line. It is clearly an extension of (7) for entire functions of exponential type.

Theorem 4. *Let $f \in \mathcal{F}_\tau^\vee$ be uniformly almost periodic on the real line. Furthermore, suppose that the coefficients $A_1, A_2, \dots$ of the Fourier series $\sum_{n=1}^\infty A_n \mathrm{e}^{\mathrm{i}\Lambda_n x}$ of f lie in a sector of opening $0 \leq \gamma < \pi$ with the vertex at the origin. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{\tau}{2\cos(\gamma/2)} |f(0)|. \tag{17}$$

The result is best possible as the equality holds for $f(z) = \mathrm{e}^{\mathrm{i}\tau z} + 2\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}\tau z/2} + 1$.

Let $p > 0$ be a real number. We say that a function f belongs to L^p on the real line if, $\int_{-\infty}^\infty |f(x)|^p \, dx < \infty$. Inequalities (9) and (10) have been generalized for entire functions of exponential type as well. For example, as a generalization of (9) we have

Theorem 5. *Let f be an entire function of exponential type τ that belongs to L^p on the real line, where $p > 0$ is a real number. Then*

$$\int_{-\infty}^\infty |f'(x)|^p \, dx \leq \tau^p \int_{-\infty}^\infty |f(x)|^p \, dx. \tag{18}$$

For various refinement and detailed information we refer the reader to the paper of Rahman and Schemeisser [14].

For functions f in $\mathcal{F}_\tau^\vee$ that belong to L^2 on the real line, Rahman and Tariq ([18, Theorem 3]) proved that

$$\int_{-\infty}^\infty |f'(x)|^2 \, dx \leq \frac{\tau^2}{2} \int_{-\infty}^\infty |f(x)|^2 \, dx, \tag{19}$$

where the coefficient $\tau^2/2$ of $\int_{-\infty}^\infty |f(x)|^2 \, dx$ cannot be replaced by a smaller number.

In this paper, we present the following theorem for functions in $\mathcal{F}_\tau^\vee$ that have all their zeros in the first and the third quadrants. It is clearly an extension of Theorem 1 for entire functions of exponential type.

Theorem 6. *Let f, which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Then*

$$|f'(-x)| \leq |f'(x)|, \qquad x > 0. \tag{20}$$

As applications of Theorem 6, we state the following inequality about functions in $\mathcal{F}_\tau^\vee$. We do not know if it is sharp.

Corollary 3. *Let f, which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further suppose that $f \in L^p$ on $(-\infty, 0)$. Then, for $p \geq 1$*

$$\int_{-\infty}^{0} |f'(x)|^p \, dx \leq \tau^p \, C_p \int_{-\infty}^{0} |f(x)|^p \, dx, \tag{21}$$

where C_p is as given in (11).

Corollary 4. *Let f, which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further assume that $|f(x)| \leq M$ on $(-\infty, 0)$. Then*

$$|f'(x)| \leq \frac{M\tau}{2}, \qquad x \leq 0. \tag{22}$$

The estimate is sharp as the example $M(1 + \mathrm{e}^{\mathrm{i}\tau z})/2$ shows.

Corollary 5. *Let f, which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further assume that $|f(x)| \leq M$ on $(-\infty, 0)$. Then*

$$|f'(x + \mathrm{i}y)| \leq \frac{M\tau}{2} \mathrm{e}^{-\tau y}, \qquad x < 0, y < 0. \tag{23}$$

The estimate is sharp as the example $M(1 + \mathrm{e}^{\mathrm{i}\tau z})/2$ shows.

1.3. Mean value of entire functions of exponential type

Let $p > 0$ be a real number. For a function f, the mean of order p on the real line is defined by

$$M^p f(x) = \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx. \tag{24}$$

We say that f has a bounded mean of order p, if $M^p f(x) < \infty$. It can be easily seen that a function bounded on the real axis will always have a bounded mean. However, there are functions which have a bounded mean but not bounded on the real line. Harvey [12] considered the problems of the mean value of entire functions of exponential type. Here is one of his results.

Theorem 7. *If f is an entire function of exponential type τ, then*

$$M^p f'(x) \leq \frac{(p+2)2^{p+2}}{\pi \tau p \, \delta^{p+1}} (\mathrm{e}^{\tau \delta p} - 1) M^p f(x), \qquad p > 0, \tag{25}$$

where δ is an arbitrary positive number.

However, when p is greater than one, the constant in the above theorem can be replaced by τ^p. More precisely, Harvey [12] proved that

Theorem 8. *If f is an entire function of exponential type τ, then*

$$M^p f'(x) \leq \tau^p M^p f(x), \qquad p > 1. \tag{26}$$

As to the mean value of functions in $\mathcal{F}_\tau^\vee$, Rahman and Tariq [18] considered the case when $p = 2$ and obtained the following inequality.

Theorem 9. *If f, which is a uniformly almost periodic function on the real line, belongs to $\mathcal{F}_\tau^\vee$, then*

$$\limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{0} |f'(x)|^2 \, dx \le \frac{\tau^2}{2} \limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{0} |f(x)|^2 \, dx. \tag{27}$$

Inequality (27) is sharp as the example $f(z) = (1 + \mathrm{e}^{\mathrm{i}\tau z})/2$ shows.

Here, we will prove the following inequality about the mean value theorem for functions in $\mathcal{F}_\tau^\vee$. We do not know if it is sharp.

Theorem 10. *Let f, which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Assume further that f has a bounded mean of order p where $p \ge 1$. Then*

$$\limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{0} |f'(x)|^p \, dx \le \tau^p \, C_p \limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{0} |f(x)|^p \, dx, \tag{28}$$

where C_p is as given in (11).

The rest of the paper is organized as follows. In Section 2, we list all the lemmas needed in our proofs. Section 3 deals with the proofs of Theorem 1, Theorem 6 and Theorem 10 and their corollaries discussed above.

2. Lemmas

The first two Lemmas have been proved by Rahman and Tariq [18].

Lemma 1. *Let f belong to $\mathcal{F}_\tau^\vee$ such that $|f(x)|$ is bounded on the real line. Then, for any real γ and $s = -\gamma/\tau$, we have*

$$-\mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\gamma} f'(x) + \mathrm{e}^{\mathrm{i}\tau x} f'(-x)\right\} = \sum_{n=-\infty}^{\infty} c_n f\left(x - s + \frac{n\pi}{\tau}\right), \qquad x \in \mathbb{R}, \tag{29}$$

where

$$c_n = \frac{1}{(s\tau - n\pi)^2}\left\{1 + (-1)^n\right\}\left\{1 - (-1)^n \cos\gamma\right\}\tau, \qquad n = 0, \pm 1, \pm 2, \ldots$$

and $\sum_{n=-\infty}^{\infty} |c_n| = \tau$.

Lemma 2. *Let f belong to $\mathcal{F}_\tau^\vee$ such that $|f(x)| \le M$ on the real line. Then*

$$|f'(x)| + |f'(-x)| \le M\tau, \qquad x \in \mathbb{R}. \tag{30}$$

We will make use of the following interpolation formula due to Aziz and Mohammad [1].

Lemma 3. *Let f belong to $\mathcal{P}_n$ and let $\xi_1, \xi_2, \cdots \xi_n$ be the zeros of $z^n + a$, where $a \neq -1$ is an arbitrary complex number. Then for any complex number z we have*

$$zf'(z) = \frac{n}{1+a} f(z) + \frac{1+a}{na} \sum_{\nu=1}^{n} c_\nu(a) f(z\xi_\nu), \tag{31}$$

where

$$\sum_{\nu=1}^{n} c_\nu(a) = \sum_{\nu=1}^{n} \frac{\xi_\nu}{(\xi_\nu - 1)^2} = -\frac{n^2 a}{(1+a)^2}.$$

The next inequality, that can be found in Malik [13] (also see, Govil and Rahman ([10], Inequality (3.2)) where this inequality is given for any order derivatives) is well-known and widely used in the study of polynomials.

Lemma 4. *Let f belong to $\mathcal{P}_n$. Define $g(z) \equiv z^n \overline{f(1/\bar{z})}$, a polynomial in $\mathcal{P}_n$. Then*

$$|f'(z)| + |g'(z)| \leq n \max_{|z|=1} |f(z)|, \qquad |z| = 1. \tag{32}$$

Lemma 5. *Let us denote $\omega_\nu = z_\nu + 1/z_\nu$ and $\omega_\mu = z_\mu + 1/z_\mu$, where $z_\nu,\ z_\mu$ are complex numbers such that $\pi/2 \leq arg\ z_\nu,\ arg\ z_\mu \leq \pi$ and $|z_\nu| \leq 1,\ |z_\mu| \leq 1$. Define*

$$\begin{aligned} F(x;\ \omega_\nu,\ \omega_\mu) &= -4x^2 \Im(\omega_\nu - \bar{\omega}_\mu) + 4x \Im(\omega_\nu \omega_\mu) - (|\omega_\mu|^2 \Im\omega_\nu + |\omega_\nu|^2 \Im\omega_\mu), \\ G(x;\ \omega_\nu) &= -2(x+1)\Im\omega_\nu. \end{aligned}$$

Then for $-1 \leq x \leq 1$,

$$\begin{aligned} F(x;\ \omega_\nu,\ \omega_\mu) &\geq 0, \\ G(x;\ \omega_\nu) &\geq 0. \end{aligned}$$

Proof. First, we note that ω_ν may be written as $\omega_\nu := x_\nu R_\nu + \mathrm{i} y_\nu L_\nu$, where

$$R_\nu = \left(1 + \frac{1}{x_\nu^2 + y_\nu^2}\right) \geq 0, \quad L_\nu = \left(1 - \frac{1}{x_\nu^2 + y_\nu^2}\right) \leq 0,$$

$x_\nu = \Re z_\nu$ and $y_\nu = \Im z_\nu$. Since $arg z_\nu$ lies in $[\pi/2,\ \pi]$, it implies that $\Re\omega_\nu \leq 0$ and $\Im\omega_\nu \leq 0$. Similarly, $\omega_\mu := x_\mu R_\mu + \mathrm{i} y_\mu L_\mu$, where $R_\mu \geq 0, L_\mu \leq 0, \Re\omega_\mu \leq 0$ and $\Im\omega_\mu \leq 0$.

$F(x;\ \omega_\nu,\ \omega_\mu)$ is a quadratic function of the form ax^2+bx+c, where $a = -4\Im(\omega_\nu - \bar{\omega}_\mu) \geq 0, b = 4\Im(\omega_\nu\omega_\mu) \geq 0$, and $c = -(|\omega_\mu|^2\Im\omega_\nu + |\omega_\nu|^2\Im\omega_\mu) \geq 0$. Its vertex is $(-b/2a,\ F(-b/2a;\ \omega_\nu,\ \omega_\mu))$, where $-b/2a = \Im(\omega_\nu\omega_\mu)/2\Im(\omega_\nu - \overline{\omega}_\mu)$ and

$$\begin{aligned} F(-b/2a;\ \omega_\nu,\ \omega_\mu) &= \frac{\left(\Im(\omega_\nu\omega_\mu)\right)^2}{\Im(\omega_\nu - \overline{\omega}_\mu)} - (|\omega_\mu|^2\Im\omega_\nu + |\omega_\nu|^2\Im\omega_\mu) \\ &= \frac{\left(\Im(\omega_\nu\omega_\mu)\right)^2 - (|\omega_\mu|^2\Im\omega_\nu + |\omega_\nu|^2\Im\omega_\mu)\Im(\omega_\nu - \overline{\omega}_\mu)}{\Im(\omega_\nu - \overline{\omega}_\mu)} \end{aligned}$$

The numerator $(\Im(\omega_\nu\omega_\mu))^2-(|\omega_\mu|^2\Im\omega_\nu+|\omega_\nu|^2\Im\omega_\mu)\Im(\omega_\nu-\overline{\omega}_\mu)$ of the above expression is equal to

$$\begin{aligned}(x_\mu R_\mu y_\nu L_\nu+x_\nu R_\nu y_\mu L_\mu)^2&-\left\{y_\nu L_\nu(x_\mu^2R_\mu^2+y_\mu^2L_\mu^2)+y_\mu L_\mu(x_\nu^2R_\nu^2+y_\nu^2L_\nu)\right\}(y_\nu L_\nu+y_\mu L_\mu)\\ &=-\left[y_\mu L_\mu y_\nu L_\nu\left\{(x_\mu R_\mu-x_\nu R_\nu)^2+(y_\mu^2L_\mu^2+y_\nu^2L_\nu)^2\right\}+2y_\nu^2L_\nu^2y_\mu^2L_\mu^2\right]\\ &\leq 0.\end{aligned}\tag{33}$$

Since $\Im(\omega_\nu-\overline{\omega}_\mu)\leq 0$, we have $a\geq 0$ and $F(-b/2a;\omega_\nu,\omega_\mu)\geq 0$. Also, $F(x;\omega_\nu,\omega_\mu)$ will attain the minimum value at the vertex. Thus, for any real number x we have $F(x;\omega_\nu,\omega_\mu)\geq F(-b/2a;\omega_\nu,\omega_\mu)\geq 0$ and hence in particular for $-1\leq x\leq 1$.

As far as the function G is concerned, we just have to note that $\Im\omega_\nu\leq 0$, which shows that $G(x;\omega_\nu)=-2(x+1)\Im\omega_\nu\geq 0$ for $-1\leq x\leq 1$. □

3. Proofs of Theorem 1, Theorem 6 and Theorem 10

3.1. Proof of Theorem 1 and its corollaries

Case 1. f has all its zeros on the unit circle

Let us assume that $z_\nu:=\mathrm{e}^{\mathrm{i}\theta_\nu}$, where $\pi/2\leq\theta_\nu\leq\pi$ ($\nu=1,\ 2,\ldots,l$) are l zeros of f. Since f belongs to $\mathcal{P}_n^\vee$, for every $\nu, 1/z_\nu=\mathrm{e}^{-\mathrm{i}\theta_\nu}$ is also a zeros of f. Assume further that f has a zero of multiplicity m at -1, where $m\geq 0$. Thus f may be written as

$$f(z)=(z+1)^m\prod_{\nu=1}^{l}(z-\mathrm{e}^{\mathrm{i}\theta_\nu})(z-\mathrm{e}^{-\mathrm{i}\theta_\nu}),$$

where $n=2l+m$. Let θ be a number in $[0,\pi]$ such that $\theta\neq\theta_\nu,(\nu=1,\ 2,\ldots,l)$. Then

$$\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}=\Re\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}+\mathrm{i}\Im\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})},$$

where

$$\Re\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}=\frac{m(\cos\theta+1)}{|1+\mathrm{e}^{\mathrm{i}\theta}|^2}+\sum_{\nu=1}^{l}\frac{\cos\theta-\cos\theta_\nu}{|\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta_\nu}|^2}+\frac{\cos\theta-\cos\theta_\nu}{|\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta_\nu}|^2}$$

and

$$\Im\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}=-\frac{m\sin\theta}{|1+\mathrm{e}^{\mathrm{i}\theta}|^2}+\sum_{\nu=1}^{l}\frac{\sin\theta_\nu-\sin\theta}{|\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta_\nu}|^2}-\frac{\sin\theta_\nu+\sin\theta}{|\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta_\nu}|^2}.$$

Note that $\Re\left(f'(\mathrm{e}^{\mathrm{i}\theta})/f(\mathrm{e}^{\mathrm{i}\theta})\right)$ is an even function of θ and $\Im\left(f'(\mathrm{e}^{\mathrm{i}\theta})/f(\mathrm{e}^{\mathrm{i}\theta})\right)$ is an odd function of θ. So $|f'(\mathrm{e}^{\mathrm{i}\theta})|/|f(\mathrm{e}^{\mathrm{i}\theta})|$ is equal to

$$\begin{aligned}\sqrt{\left(\Re\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}\right)^2+\left(\Im\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}\right)^2}&=\sqrt{\left(\Re\frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f(\mathrm{e}^{-\mathrm{i}\theta})}\right)^2+\left(-\Im\frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f(\mathrm{e}^{-\mathrm{i}\theta})}\right)^2}\\ &=\left|\frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f(\mathrm{e}^{-\mathrm{i}\theta})}\right|.\end{aligned}\tag{34}$$

For $f \in \mathcal{P}_n^{\vee}$ and $\theta \in [0,\ \pi]$, we have $\left|f(\mathrm{e}^{-\mathrm{i}\theta})\right| = \left|f(\mathrm{e}^{\mathrm{i}\theta})\right|$. So we conclude, from (34)

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| \le |f'(\mathrm{e}^{\mathrm{i}\theta})|, \qquad 0 \le \theta \le \pi, f(\mathrm{e}^{\mathrm{i}\theta}) \ne 0.$$

By continuity, the same must hold for those θ for which $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$.

Case 2. f is a second degree polynomial

Let z_ν be the zero of f such that $\pi/2 \le arg\, z_\nu \le \pi$ and $|z_\nu| \le 1$. The polynomial f and its derivative f' may be written as $f(z) = (z - z_\nu)(z - 1/z_\nu)$ and $f'(z) = 2z - \omega_\nu$, respectively, where $\omega_\nu := z_\nu + 1/z_\nu = x_\nu R_\nu + \mathrm{i}\, y_\nu L_\nu, -1 \le x_\nu = \Re z_\nu \le 0$,

$$R_\nu = \left(1 + \frac{1}{x_\nu^2 + y_\nu^2}\right) > 0, 0 \le y_\nu = \Im z_\nu \le 1 \text{ and } L_\nu = \left(1 - \frac{1}{x_\nu^2 + y_\nu^2}\right) < 0.$$

The conditions on $R_\nu, L_\nu, x_\nu, y_\nu$ ensure that ω_ν lies in the third quadrant. Thus, we have

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| = |2\mathrm{e}^{-\mathrm{i}\theta} - \omega_\nu| \le |2\mathrm{e}^{\mathrm{i}\theta} - \omega_\nu| = |f'(\mathrm{e}^{\mathrm{i}\theta})|, \qquad 0 \le \theta \le \pi.$$

This proves the theorem when f is a polynomial of degree 2. We also note that for $0 \le \theta \le \pi$, $|f(\mathrm{e}^{-\mathrm{i}\theta}| = |f(\mathrm{e}^{\mathrm{i}\theta}|$. Thus, for any f in $\mathcal{P}_2^{\vee}$ we have

$$\left|\frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f(\mathrm{e}^{-\mathrm{i}\theta})}\right| \le \left|\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}\right|, \qquad 0 \le \theta \le \pi. \tag{35}$$

Case 3. Not all the zeros of f are on the unit circle

Let z_ν $(\nu = 1, 2, \cdots, l)$ be the zeros f such that $\pi/2 \le arg\, z_\nu \le \pi$ and $|z_\nu| \le 1$. Also suppose that f has a zero of multiplicity m at -1 where $m \ge 0$. Then f can be represented as

$$f(z) = (z+1)^m \prod_{\nu=1}^{l} \; g_\nu(z),$$

where $g_\nu(z) = (z - z_\nu)(z - 1/z_\nu)$ is a second degree polynomial in $\mathcal{P}_2^{\vee}$ for each ν. For any z on the unit circle such that $f(z) \ne 0$ we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z+1} + \sum_{\nu=1}^{l} \frac{g'_\nu(z)}{g_\nu(z)}.$$

A straightforward calculation gives us

$$\begin{aligned} \left|\frac{f'(z)}{f(z)}\right|^2 &= \left|\frac{m}{z+1}\right|^2 \\ &+ \sum_{\nu=1}^{l} \left(\left|\frac{g'_\nu(z)}{g_\nu(z)}\right|^2 + 2\Re\left(\overline{\frac{m}{(z+1)}}\frac{g'_\nu(z)}{g_\nu(z)}\right) + 2\sum_{\mu=\nu+1}^{l} \Re\left(\frac{g'_\nu(z)}{g_\nu(z)}\overline{\frac{g'_\mu(z)}{g_\mu(z)}}\right)\right). \end{aligned} \tag{36}$$

There are four parts in the above equation. We will compare the value of each part at $\mathrm{e}^{-\mathrm{i}\theta}$ and $\mathrm{e}^{\mathrm{i}\theta}$, respectively.

The first part $|m/(z+1)|^2$ gives us

$$\left|\frac{m}{\mathrm{e}^{-\mathrm{i}\theta}+1}\right|^2 = \left|\frac{m}{\mathrm{e}^{\mathrm{i}\theta}+1}\right|^2, \qquad 0 \le \theta \le \pi. \tag{37}$$

Since $g_\nu(z)$ belongs to $\mathcal{P}_2^\vee$ for each ν, from (35) the second part $|g_\nu'(z)/g_\nu(z)|^2$ gives us

$$\left|\frac{g_\nu'(\mathrm{e}^{-\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{-\mathrm{i}\theta})}\right|^2 \le \left|\frac{g_\nu'(\mathrm{e}^{\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{\mathrm{i}\theta})}\right|^2, \qquad 0 \le \theta \le \pi. \tag{38}$$

Let $z = x + \mathrm{i}y$ be a point on the unit circle. From Case 2 again, it is easy to verify that

$$\begin{aligned}
\frac{m}{\overline{(z+1)}}\frac{g_\nu'(z)}{g_\nu(z)} &= \frac{m}{\overline{(z+1)}}\frac{2z-\omega_\nu}{z^2-\omega_\nu z+1} \\
&= \frac{m}{|z+1|^2}\frac{2z-\omega_\nu}{|z^2-\omega_\nu z+1|^2}(z+1)(\overline{z}^2-\overline{\omega_\nu z}+1) \\
&= \frac{m}{|z+1|^2}\frac{(Q_1(x;\omega_\nu)+y\ S_1(x;\omega_\nu))+\mathrm{i}\,(Q_2(x;\omega_\nu)+y\ S_2(x;\omega_\nu))}{|z^2-\omega_\nu z+1|^2},
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
Q_1(x;\omega_\nu) &= (x+1)\left(4x-2(x+1)\Re\omega_\nu+|\omega_\nu|^2\right); \\
S_1(x;\omega_\nu) &= -2(x+1)\Im\omega_\nu; \\
Q_2(x;\omega_\nu) &= 2(1-x^2)\Im\omega_\nu; \\
S_2(x;\omega_\nu) &= \left(4x+2(x-1)\Re\omega_\nu-|\omega_\nu|^2\right).
\end{aligned} \tag{40}$$

Thus from Lemma 5, (39) and the fact that

$$|\mathrm{e}^{-2\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\,\omega_\nu\,+1|^2 = |\mathrm{e}^{-2\mathrm{i}\theta}||\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\,\omega_\nu\,+1|^2 = |\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\,\omega_\nu\,+1|^2,$$

the third part $\Re\left(mg_\nu'(z)/\overline{(z+1)}g_\nu(z)\right)$ gives us

$$\begin{aligned}
\Re\left(\frac{m}{\overline{(\mathrm{e}^{-\mathrm{i}\theta}+1)}}\frac{g_\nu'(\mathrm{e}^{-\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{-\mathrm{i}\theta})}\right) &= m\frac{Q_1(\cos(-\theta);\omega_\nu)+\sin(-\theta)\ S_1(\cos(-\theta);\omega_\nu)}{|\mathrm{e}^{-\mathrm{i}\theta}+1|^2\,|\mathrm{e}^{-2\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\,\omega_\nu\,+1|^2} \\
&= m\frac{Q_1(\cos\theta;\omega_\nu)-\sin\theta\ S_1(\cos\theta;\omega_\nu)}{|\mathrm{e}^{\mathrm{i}\theta}+1|^2\,|\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\,\omega_\nu\,+1|^2} \\
&\le m\frac{Q_1(\cos\theta;\omega_\nu)+\sin\theta\ S_1(\cos\theta;\omega_\nu)}{|\mathrm{e}^{\mathrm{i}\theta}+1|^2\,|\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\,\omega_\nu\,+1|^2} \\
&= \Re\left(\frac{m}{\overline{(\mathrm{e}^{\mathrm{i}\theta}+1)}}\frac{g_\nu'(\mathrm{e}^{\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{\mathrm{i}\theta})}\right), \qquad 0 \le \theta \le \pi.
\end{aligned} \tag{41}$$

Let us turn to the fourth part. As in the third part, let $z = x + \mathrm{i}y$ be a point on the unit circle. Then for any μ and ν, it can be verified that

$$\frac{g'_\nu(z)}{g_\nu(z)}\overline{\frac{g'_\mu(z)}{g_\mu(z)}} = \frac{2z-\omega_\nu}{z^2-z\,\omega_\nu+1}\,\overline{\frac{2z-\omega_\mu}{z^2-z\,\omega_\mu+1}} = \frac{4-2z\overline{\omega}_\mu-2\overline{z}\omega_\nu+\omega_\nu\overline{\omega}_\mu}{4x^2-2x(\overline{\omega}_\mu+\omega_\nu)+\omega_\nu\overline{\omega}_\mu} \tag{42}$$
$$= \frac{(Q_3(x;\omega_\nu,\omega_\mu)+y\ S_3(x;\omega_\nu,\omega_\mu))+\mathrm{i}\,(Q_4(x;\omega_\nu,\omega_\mu)+y\ S_4(x;\omega_\nu,\omega_\mu))}{|4x^2-2x(\omega_\nu+\overline{\omega}_\mu)+\omega_\nu\overline{\omega}_\mu|^2},$$

where

$$\begin{aligned} Q_3(x;\omega_\nu,\omega_\mu) &= 16x^2-16x\Re(\overline{\omega}_\nu+\omega_\mu)+8xy^2\Re(\overline{\omega}_\mu+\omega_\nu)+8\Re(\overline{\omega}_\nu\omega_\mu)\\ &\quad -4y^2\Re(\overline{\omega}_\mu\omega_\nu)+4x^2|\overline{\omega}_\mu+\omega_\mu|^2+|\omega_\mu\omega_\nu|^2-4x\Re(\overline{\omega}_\mu|\omega_\nu|^2+\omega_\nu|\omega_\mu|^2);\\ S_3(x;\omega_\nu,\omega_\mu) &= -8x^2\Im(\omega_\nu-\overline{\omega}_\mu)+8x\Im(\omega_\nu\omega_\mu)+2\Im\overline{\omega}_\nu|\omega_\mu|^2-2\Im\omega_\mu|\omega_\nu|^2;\\ Q_4(x;\omega_\nu,\omega_\mu) &= 8xy^2\Im(\overline{\omega}_\mu+\omega_\nu)-4y^2\Im(\overline{\omega}_\mu\omega_\nu);\\ S_4(x;\omega_\nu,\omega_\mu) &= 8x^2\Re(\omega_\nu-\overline{\omega}_\mu)-4x(|\omega_\nu|^2-|\omega_\mu|^2)+2|\omega_\nu|^2\Re\overline{\omega}_\mu-2|\omega_\mu|^2\Re\overline{\omega}_\nu. \end{aligned} \tag{43}$$

Thus from Lemma 5 and (42), the fourth part $\sum_{\mu=\nu+1}^{l}\Re\left(g'_\nu(z)\overline{g'_\mu(z)}/g_\nu(z)\overline{g_\mu(z)}\right)$ gives us

$$\begin{aligned} &\sum_{\mu=\nu+1}^{l}\Re\left(\frac{g'_\nu(\mathrm{e}^{-\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{-\mathrm{i}\theta})}\overline{\frac{g'_\mu(\mathrm{e}^{-\mathrm{i}\theta})}{g_\mu(\mathrm{e}^{-\mathrm{i}\theta})}}\right)\\ &\quad=\sum_{\mu=\nu+1}^{l}\left(\frac{Q_3(\cos(-\theta);\omega_\nu)+\sin(-\theta)\ S_3(\cos(-\theta);\omega_\nu)}{|4\cos(-\theta)^2-2(\omega_\nu+\bar{\omega}_\mu)\cos(-\theta)+\omega_\nu\bar{\omega}_\mu|^2}\right)\\ &\quad=\sum_{\mu=\nu+1}^{l}\left(\frac{Q_3(\cos\theta;\omega_\nu)-\sin\theta\ S_3(\cos\theta;\omega_\nu)}{|4\cos\theta^2-2(\omega_\nu+\bar{\omega}_\mu)\cos\theta+\omega_\nu\bar{\omega}_\mu|^2}\right)\\ &\quad\le\sum_{\mu=\nu+1}^{l}\left(\frac{Q_3(\cos\theta;\omega_\nu)+\sin\theta\ S_3(\cos\theta;\omega_\nu)}{|4\cos\theta^2-2(\omega_\nu+\bar{\omega}_\mu)\cos\theta+\omega_\nu\bar{\omega}_\mu|^2}\right)\\ &\quad=\sum_{\mu=\nu+1}^{l}\Re\left(\frac{g'_\nu(\mathrm{e}^{\mathrm{i}\theta})}{g_\nu(\mathrm{e}^{\mathrm{i}\theta})}\overline{\frac{g'_\mu(\mathrm{e}^{\mathrm{i}\theta})}{g_\mu(\mathrm{e}^{\mathrm{i}\theta})}}\right),\qquad 0\le\theta\le\pi. \end{aligned} \tag{44}$$

Using (37), (38), (41) and (44) in (36), we conclude that

$$\left|\frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f(\mathrm{e}^{-\mathrm{i}\theta})}\right|^2 \le \left|\frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f(\mathrm{e}^{\mathrm{i}\theta})}\right|^2,\qquad 0\le\theta\le\pi. \tag{45}$$

Since for any θ, $|f(\mathrm{e}^{-\mathrm{i}\theta})|=|f(\mathrm{e}^{\mathrm{i}\theta})|$, we get from (45)

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})|\le|f'(\mathrm{e}^{\mathrm{i}\theta})|,\qquad 0\le\theta\le\pi, f(\mathrm{e}^{\mathrm{i}\theta})\ne 0.$$

By continuity, the same must hold for those θ for which $f(\mathrm{e}^{\mathrm{i}\theta})=0$. This completes the proof of Theorem 1.

Proof of Corollary 1. For polynomials f in $\mathcal{P}_n^{\vee}$, we have

$$z^{n-1}\, f'(\frac{1}{z}) + z\, f'(z) = n\, f(z).$$

From the interpolation formula (31) of Aziz and Mohammad given in Lemma 3, with $a = \mathrm{e}^{\mathrm{i}\alpha}$, where $\alpha \in \mathbb{R}$ and $z = \mathrm{e}^{\mathrm{i}\theta}$ is a complex number on the unit circle, we get

$$\mathrm{e}^{\mathrm{i}(\theta+\alpha)}\, f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\, f'(\mathrm{e}^{-\mathrm{i}\theta}) = \frac{(1+\mathrm{e}^{\mathrm{i}\alpha})^2}{n\ \mathrm{e}^{\mathrm{i}\alpha}} \sum_{\nu=1}^{n} c_\nu(a)\ f(\mathrm{e}^{\mathrm{i}\theta}\xi_\nu),$$

which can be written as

$$\mathrm{e}^{\mathrm{i}(\theta+\alpha)}\, f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\, f'(\mathrm{e}^{-\mathrm{i}\theta}) = n \sum_{\nu=1}^{n} d_\nu(\mathrm{e}^{\mathrm{i}\alpha})\, f(\mathrm{e}^{\mathrm{i}\theta}\xi_\nu),$$

where

$$\sum_{\nu=0}^{n} |d_\nu(\mathrm{e}^{\mathrm{i}\alpha})| = \sum_{\nu=0}^{n} \left| \frac{c_\nu(\mathrm{e}^{\mathrm{i}\alpha})}{n^2\ \mathrm{e}^{\mathrm{i}\alpha}/(1+\mathrm{e}^{\mathrm{i}\alpha})^2} \right| = 1.$$

For $p \geq 1$, we have

$$\left|\mathrm{e}^{\mathrm{i}(\theta+\alpha)}\, f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\, f'(\mathrm{e}^{-\mathrm{i}\theta})\right|^p \leq n^p \sum_{\nu=1}^{n} d_\nu(\mathrm{e}^{\mathrm{i}\alpha})\ \left|f(\mathrm{e}^{\mathrm{i}\theta}\xi_\nu)\right|^p.$$

Integrating both sides with respect to θ from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} \left|\ \mathrm{e}^{\mathrm{i}(\theta+\alpha)}\ f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\ f'(\mathrm{e}^{-\mathrm{i}\theta})\right|^p\ d\theta \leq n^p \int_{-\pi}^{\pi} \left|f(\mathrm{e}^{\mathrm{i}\theta})\right|^p\ d\theta.$$

Since the above inequality is true for every α in $[0, 2\pi]$, integrating both sides with respect to α and changing the order of integration, we get

$$\int_{-\pi}^{\pi} \int_{0}^{2\pi} \left|\ \mathrm{e}^{\mathrm{i}(\theta+\alpha)}\ f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\ f'(\mathrm{e}^{-\mathrm{i}\theta})\right|^p\ d\alpha\ d\theta \leq 2\pi n^p \int_{-\pi}^{\pi} \left|f(\mathrm{e}^{\mathrm{i}\theta})\right|^p\ d\theta. \tag{46}$$

The left-hand side of the inequality (46) is

$$\begin{aligned}
&\int_{-\pi}^{\pi} \int_{0}^{2\pi} \left|\ \mathrm{e}^{\mathrm{i}(\theta+\alpha)}\ f'(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}(n-1)\theta}\ f'(\mathrm{e}^{-\mathrm{i}\theta})\right|^p\ d\alpha\ d\theta \\
&\quad = \int_{-\pi}^{0} \int_{0}^{2\pi} |\ f'(\mathrm{e}^{\mathrm{i}\theta})|^p\ \left|1 - \mathrm{e}^{\mathrm{i}(n-2)\theta-\mathrm{i}\alpha}\ \frac{f'(\mathrm{e}^{-\mathrm{i}\theta})}{f'(\mathrm{e}^{\mathrm{i}\theta})}\right|^p\ d\alpha\ d\theta \\
&\qquad + \int_{0}^{\pi} \int_{0}^{2\pi} |\ f'(\mathrm{e}^{-\mathrm{i}\theta})|^p\ \left|1 - \mathrm{e}^{\mathrm{i}(2-n)\theta+\mathrm{i}\alpha}\ \frac{f'(\mathrm{e}^{\mathrm{i}\theta})}{f'(\mathrm{e}^{-\mathrm{i}\theta})}\right|^p\ d\alpha\ d\theta \\
&\quad \geq 2 \int_{-\pi}^{0} \left|f'(\mathrm{e}^{\mathrm{i}\theta})\right|^p\ d\theta \int_{0}^{2\pi} \left|1+\mathrm{e}^{\mathrm{i}\alpha}\right|^p\ d\alpha.
\end{aligned} \tag{47}$$

Inequality (47) follows from the fact that

$$\left|f'(\mathrm{e}^{-\mathrm{i}\theta})/f'(\mathrm{e}^{\mathrm{i}\theta})\right| \geq 1 \text{ for } -\pi \leq \theta \leq 0,$$
$$\left|f'(\mathrm{e}^{\mathrm{i}\theta})/f'(\mathrm{e}^{-\mathrm{i}\theta})\right| \geq 1 \text{ for } 0 \leq \theta \leq \pi,$$

and

$$\int_0^{2\pi} |1 + r\mathrm{e}^{\mathrm{i}\gamma}|^p d\gamma \geq \int_0^{2\pi} |1 + \mathrm{e}^{\mathrm{i}\gamma}|^p d\gamma \text{ for every } |r| \geq 1 \text{ and } p \geq 1.$$

Also, for $f \in \mathcal{P}_n^{\vee}$, $|f(\mathrm{e}^{-\mathrm{i}\theta})| = |f(\mathrm{e}^{\mathrm{i}\theta})|$ for $0 \leq \theta \leq \pi$. From (46) and (47) we conclude that

$$\int_{-\pi}^{0} |f'(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta \leq n^p \; C_p \int_{-\pi}^{0} |f(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta,$$

where C_p is as given in (11). □

Proof of Corollary 2. Let f be a polynomial in $\mathcal{P}_n^{\vee}$ such that $|f(\mathrm{e}^{-\mathrm{i}\theta})| \leq M$ for $0 \leq \theta \leq \pi$. Since $|f(\mathrm{e}^{-\mathrm{i}\theta})| = |f(\mathrm{e}^{\mathrm{i}\theta})|$ for every f in $\mathcal{P}_n^{\vee}$, it implies that $|f(\mathrm{e}^{\mathrm{i}\theta})| \leq M$ for $-\pi \leq \theta \leq \pi$. We also observe that $g(z) \equiv z^n \overline{f(1/\bar{z})} = \overline{f(\bar{z})}$. Then from inequality (32) in Lemma 4, for $z = \mathrm{e}^{\mathrm{i}\theta}$

$$|f'(\mathrm{e}^{\mathrm{i}\theta})| + |g'(\mathrm{e}^{\mathrm{i}\theta})| = |f'(\mathrm{e}^{-\mathrm{i}\theta})| + |f'(\mathrm{e}^{\mathrm{i}\theta})| \leq nM, \qquad -\pi \leq \theta \leq \pi. \tag{48}$$

From Theorem 1, $|f'(\mathrm{e}^{-\mathrm{i}\theta})| \leq |f'(\mathrm{e}^{\mathrm{i}\theta})|$ for $0 \leq \theta \leq \pi$. So, from (48) we get

$$2|f'(\mathrm{e}^{-\mathrm{i}\theta})| \leq |f'(\mathrm{e}^{-\mathrm{i}\theta})| + |f'(\mathrm{e}^{\mathrm{i}\theta})| \leq nM, \qquad 0 \leq \theta \leq \pi. \tag{49}$$

The result follows from (49). It is easy to verify that the equality holds for $f(z) = (z^2+1)^{\frac{n}{2}}$, when n is even and $f(z) = (z+1)^n$, when n is odd. □

3.2. Proof of Theorem 6 and its corollaries

Let $\{z_\nu\}, \nu = 1, 2, \dots$ be the zeros of f other than 0 in $\{z \in \mathbb{C} : \Re z \geq 0, \Im z \geq 0\}$. The number of such zeros can be finite or infinite. Besides, to each zero z_ν there corresponds a zero $-z_\nu$. A zero of f at the origin, if there is any, must be of even multiplicity, say $2k$. For these reasons, the Hadamard factorization of f takes the form

$$f(z) = cz^{2k}\mathrm{e}^{\mathrm{i}\tau z/2} \prod_{\nu} \left(1 - \frac{z^2}{z_\nu^2}\right),$$

where c is a constant and k is a non-negative integer. Now, let us write

$$x_\nu = \Re z_\nu \;\text{ and }\; y_\nu = \Im z_\nu$$

so that $x_\nu \geq 0$ and $y_\nu \geq 0$.

Case 1. f has only real zeros

In this case, for any real x different from 0 that is not a zero of f, we have

$$\frac{f'(x)}{f(x)} = \frac{2k}{x} + \sum_{\nu} \left(\frac{1}{x_\nu + x} - \frac{1}{x_\nu - x}\right) + \mathrm{i}\frac{\tau}{2}.$$

The real part of $f'(x)/f(x)$ is clearly an odd function of x and so

$$\frac{f'(-x)}{f(-x)} = -\left(\frac{2k}{x} + \sum_{\nu}\left(\frac{1}{x_\nu + x} - \frac{1}{x_\nu - x}\right)\right) + \mathrm{i}\frac{\tau}{2}.$$

From the definition of the class $\mathcal{F}_\tau^\vee$ it is clear that $|f(-x)| = |f(x)|$ for any real x. Hence $|f'(-x)| = |f'(x)|$. Since it holds for any x such that $f(x) \neq 0$, by continuity it also holds for those values for x for which $f(x) = 0$.

Case 2. The zeros of f are not all real

In this case, for any real x different from 0 that is not a zero of f, we have

$$\frac{f'(x)}{f(x)} = A_f(x) + \mathrm{i}\left(\frac{\tau}{2} + B_f(x)\right),$$

where

$$A_f(x) := \frac{2k}{x} + \sum_{\nu}\left(\frac{x_\nu + x}{(x_\nu + x)^2 + y_\nu^2} - \frac{x_\nu - x}{(x_\nu - x)^2 + y_\nu^2}\right)$$

and

$$B_f(x) := 4x\sum_{\nu}\left(\frac{x_\nu y_\nu}{((x_\nu + x)^2 + y_\nu^2)((x_\nu - x)^2 + y_\nu^2)}\right).$$

Consequently, for any real $x \neq 0$ such that $f(x) \neq 0$ we have

$$\left|\frac{f'(x)}{f(x)}\right| = \sqrt{(A_f(x))^2 + \left(B_f(x) + \frac{\tau}{2}\right)^2}.$$

Now note that $B_f(x)$ is an odd function that is positive for $x > 0$. Hence

$$\left|B_f(-x) + \frac{\tau}{2}\right| < \left|B_f(x) + \frac{\tau}{2}\right|, \qquad x > 0, f(x) \neq 0.$$

Since $|f(-x)| = |f(x)|$, we find that $|f'(-x)| \leq |f'(x)|$ for any positive x if $f(x) \neq 0$. However, by continuity, the same must also hold for those values of x for which $f(x) = 0$. The proof of Theorem 6 is thus complete.

Proof of Corollary 3. Let $p \geq 1$ be any real number. From the interpolation formula (29) given in Lemma 1, we get

$$\left|\frac{\mathrm{e}^{\mathrm{i}\gamma} f'(x) + \mathrm{e}^{\mathrm{i}\tau x} f'(-x)}{\tau}\right|^p \leq \sum_{n=-\infty}^{\infty} \frac{c_n}{\tau}\left|f\left(x - s + \frac{n\pi}{\tau}\right)\right|^p.$$

If we integrate both sides of the above inequality with respect to x on the real line, we have

$$\int_{-\infty}^{\infty} |\mathrm{e}^{\mathrm{i}\gamma} f'(x) + \mathrm{e}^{\mathrm{i}\tau x} f'(-x)|^p dx \leq \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx.$$

The above integral is true for any $0 \leq \gamma \leq 2\pi$, therefore by integrating both sides with respect to γ on the interval $[0, 2\pi]$ we get

$$\int_0^{2\pi}\int_{-\infty}^{\infty} |\mathrm{e}^{\mathrm{i}\gamma} f'(x) + \mathrm{e}^{\mathrm{i}\tau x} f'(-x)|^p dx d\gamma \leq 2\pi\ \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx. \tag{50}$$

The integral on the left-hand side of (50) may be written as

$$\int_0^{2\pi}\int_{-\infty}^{0}|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx\ d\gamma+\int_0^{2\pi}\int_0^{\infty}|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx\ d\gamma. \tag{51}$$

The first integral $\int_0^{2\pi}\int_{-\infty}^{0}|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx\ d\gamma$ in (51), after the change of order of integration can be written as

$$\begin{aligned}\int_{-\infty}^{0}\int_0^{2\pi}&|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx d\gamma\\ &=\int_{-\infty}^{0}|f'(x)|^p dx\int_0^{2\pi}\left|1+e^{i\tau x-i\gamma}\frac{f'(-x)}{f'(x)}\right|^p d\gamma\\ &\geq\int_{-\infty}^{0}|f'(x)|^p dx\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma.\end{aligned} \tag{52}$$

Inequality (52) follows because for $x\leq 0$, $|f'(-x)/f'(x)|\geq 1$ from Theorem 6 and $\int_0^{2\pi}|1+re^{i\gamma}|^p d\gamma\geq\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma$ for every $|r|\geq 1$ and $p\geq 1$.
Similar reasoning applied to the second integral $\int_0^{2\pi}\int_0^{\infty}|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx d\gamma$ in (51) gives

$$\int_0^{2\pi}\int_0^{\infty}|e^{i\gamma}f'(x)+e^{i\tau x}f'(-x)|^p dx d\gamma\geq\int_0^{\infty}|f'(-x)|^p dx\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma, \tag{53}$$

as once again from Theorem 6 we have $|f'(x)/f'(-x)|\geq 1$ when $x\geq 0$.
Thus from (50), (52) and (53) we get

$$\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma\left(\int_{-\infty}^{0}|f'(x)|^p\ dx+\int_0^{\infty}|f'(-x)|^p\ dx\right)\leq 2\pi\tau^p\int_{-\infty}^{\infty}|f(x)|^p dx. \tag{54}$$

Note that

$$\int_{-\infty}^{0}|f'(x)|^p\ dx+\int_0^{\infty}|f'(-x)|^p\ dx=2\int_{-\infty}^{0}|f'(x)|^p\ dx. \tag{55}$$

Also, for $f\in\mathcal{F}_\tau^\vee$, we have $|f(x)|=|f(-x)|$, and so

$$\int_{-\infty}^{\infty}|f(x)|^p dx=2\int_{-\infty}^{0}|f(x)|^p dx. \tag{56}$$

From (54), (55), and (56) we get

$$\int_{-\infty}^{0}|f'(x)|^p dx\leq\tau^p\ C_p\int_{-\infty}^{0}|f(x)|^p dx,$$

where C_p is as given in (11). □

Proof of Corollary 4. Let $f \in \mathcal{F}_\tau^\vee$ such that $|f(x)| \le M$ for $x \le 0$. Since $f \in \mathcal{F}_\tau^\vee$, we have $|f(x)| = |f(-x)|$ for $x \in \mathbb{R}$ and hence $|f(x)| \le M$ for $-\infty < x < \infty$. So from inequality (30) in Lemma 2 we have

$$|f'(x)| + |f'(-x)| \le M\tau, \qquad x \in \mathbb{R}. \tag{57}$$

Also, from Theorem 6, $|f'(-x)| \ge |f'(x)|$ for $x \le 0$, and (57) then gives us

$$|f'(x)| \le \frac{M\tau}{2}, \qquad x \le 0.$$

It is easy to verify that the equality holds in (22) for $f(x) = M(1 + \mathrm{e}^{\mathrm{i}\tau z})/2$. □

Proof of Corollary 5. Let f satisfy the conditions given in Corollary 5. Then according to Corollary 4, for $x \le 0, |f'(x)| \le M\tau/2$. From Rahman and Tariq ([18, Lemma 3]), $h_f(\pi/2) \le 0$. Thus we have $h_{f'}(\pi/2) \le h_f(\pi/2) \le 0$ as well. Consider the function $g(z) = \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\bar{z})}$. Then $g(z)$ is an entire function of exponential type τ and $g(z) = f(-z)$. From Corollary 4, $|g'(x)| \le M\tau/2$ for $x \ge 0$. Also, $h_{g'}(\pi/2) = h_{f'}(-\pi/2) = \tau$. Then according to Theorem 6.2.3 ([4], page 82), for $x \ge 0, y \ge 0$,

$$|g'(x + \mathrm{i}y)| \le \frac{M\tau}{2}\,\mathrm{e}^{\tau y}.$$

Since $g(z) = f(-z)$, we have for $x \le 0, y \le 0$,

$$|f'(x + \mathrm{i}y)| \le \frac{M\tau}{2}\,\mathrm{e}^{-\tau y}.$$

It is easy to see that the equality holds for the function $M(1 + \mathrm{e}^{\mathrm{i}\tau z})/2$. □

3.3. Proof of Theorem 10

Let f, whose zeros lie in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Let $\varepsilon > 0$ be an arbitrary real number. Define the function g_ε as follows

$$g_\varepsilon(z) = \mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}z}\frac{\sin\frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z}f(z). \tag{58}$$

It is obvious that $g_\varepsilon(z)$ is an entire function of exponential type $\tau + \varepsilon$. Also,

$$\mathrm{e}^{\mathrm{i}(\tau+\varepsilon)z}g_\varepsilon(-z) = \mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}z}\frac{\sin\frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z}\mathrm{e}^{\mathrm{i}\tau z}f(-z) = \mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}z}\frac{\sin\frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z}f(z) = g_\varepsilon(z).$$

Thus, $g_\varepsilon(z)$ belongs to $\mathcal{F}_{\tau+\varepsilon}^\vee$.
Note that the zeros of $g_\varepsilon(z)$ are the zeros of $\sin\frac{\varepsilon}{2}z$ or the zeros of $f(z)$. Since the zeros of $\sin z$ are all real, the zeros of $g_\varepsilon(z)$ also lie in the first and third quadrants. Hence, according to Theorem 6,

$$|g_\varepsilon'(-x)| \le |g_\varepsilon'(x)|, \qquad x \ge 0. \tag{59}$$

Next, we will show that g_ε is bounded on the real line. The assumption that $M^p(f) < \infty$ gives us ([12, Theorem 1]), $f(x) = O(|x|^{\frac{1}{p}})$ as $|x| \to \infty$. It means there exist a positive real number $x_0 \in \mathbb{R}$ and a real number $N_1 \in \mathbb{R}$ such that $|f(x)| \le N_1 |x|^{\frac{1}{p}}$ for $|x| \ge x_0$. Thus for $|x| \ge x_0$,

$$|g_\varepsilon(x)| = \left| \mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}x} \frac{\sin\frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x} f(x) \right| \le N_1 \left| \frac{\sin\frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x} \right| |x|^{\frac{1}{p}} \le N_1 \frac{2}{\varepsilon|x|^{1-\frac{1}{p}}} \le N_1 \frac{2}{\varepsilon\,|x_0|^{1-\frac{1}{p}}}.$$

On the interval $[-x_0, x_0]$, g_ε is continuous and hence bounded. So there exists a real number N_2 such that $|g_\varepsilon(x)| \le N_2$ for $x \in [-x_0, x_0]$. Let $K = \max(2N_1/\varepsilon|x_0|^{1-\frac{1}{p}}, N_2)$. Then $|g_\varepsilon(x)| \le K$ for $x \in \mathbb{R}$. Thus g_ε is bounded on the real line and belongs to $\mathcal{F}^{\vee}_{\tau+\varepsilon}$. Hence Lemma 1 (with τ replaced by $\tau+\varepsilon$), when applied to the function $g_\varepsilon(z)$, gives us for $x \in \mathbb{R}$

$$-\mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\gamma} g'_\varepsilon(x) + \mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x} g'_\varepsilon(-x)\right\} = \sum_{n=-\infty}^{\infty} c_n g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right),$$

where

$$c_n = \frac{1}{(s(\tau+\varepsilon) - n\pi)^2} \{1 + (-1)^n\} \{1 - (-1)^n \cos\gamma\} (\tau+\varepsilon), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

γ is any real number, $s = -\gamma/(\tau+\varepsilon)$, and $\sum_{n=\infty}^{\infty} |c_n| = \tau+\varepsilon$.
From the above interpolation formula we have

$$\frac{-\mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\gamma} g'_\varepsilon(x) + \mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x} g'_\varepsilon(-x)\right\}}{\tau+\varepsilon} = \sum_{n=-\infty}^{\infty} d_n g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right), \tag{60}$$

where $d_n = c_n/(\tau+\varepsilon)$ and $\sum_{n=-\infty}^{\infty} |d_n| = 1$. Thus right-hand side of (60) is a convex combination of $\{g_\varepsilon\,(x - s + n\pi/\tau+\varepsilon)\}_{n=-\infty}^{\infty}$. So for $p \ge 1$ we get

$$\left| \frac{-\mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\gamma} g'_\varepsilon(x) + \mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x} g'_\varepsilon(-x)\right\}}{\tau+\varepsilon} \right|^p \le \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right) \right|^p,$$

which gives us

$$\left| \mathrm{e}^{\mathrm{i}\gamma} g'_\varepsilon(x) + \mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p \le (\tau+\varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right) \right|^p. \tag{61}$$

Let $T > 0$ be an arbitrary real number. Then, integrating both sides of (61) with respect to x we get

$$\begin{aligned}
&\frac{1}{2T} \int_{-T}^{T} \left| \mathrm{e}^{\mathrm{i}\gamma} g'_\varepsilon(x) + \mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p dx \\
&\qquad \le (\tau+\varepsilon)^p \frac{1}{2T} \int_{-T}^{T} \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right) \right|^p dx \\
&\qquad = (\tau+\varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \frac{1}{2T} \int_{-T}^{T} \left| g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right) \right|^p dx.
\end{aligned}$$

We can change the order of integration in the last inequality because the series on right-hand side of (61) is absolutely convergent and hence uniformly convergent. Applying Lemma 4 followed by Lemma 1 given in [12] we get

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right|^p dx \\
&\qquad \le (\tau+\varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|g_\varepsilon\left(x - s + \frac{n\pi}{\tau+\varepsilon}\right)\right|^p dx \\
&\qquad = (\tau+\varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g_\varepsilon(x)|^p dx = (\tau+\varepsilon)^p M^p g_\varepsilon(x).
\end{aligned}$$

Thus $M^p\left\{e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right\}$, the mean value of $\{e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\}$, exists for each real number γ and $\varepsilon > 0$. From the definition of limit superior, for every $\delta > 0$ there exists a positive $T_0 \in \mathbb{R}$ such that

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \left|e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right|^p dx &< M^p\{e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\} + \delta \\
&\le (\tau+\varepsilon)^p M^p g_\varepsilon(x) + \delta \qquad (62)
\end{aligned}$$

for all $T \ge T_0 > 0$, $\gamma \in \mathbb{R}$, and $\varepsilon > 0$.

Since (62) is true for each γ, integrating both sides with respect to γ from 0 to 2π and changing the order of integration which is justified by Fubini's Theorem as the function $\left|e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right|^p$ is continuous, we get

$$\frac{1}{2T}\int_{-T}^{T}\int_0^{2\pi} \left|e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right|^p d\gamma dx < 2\pi\{(\tau+\varepsilon)^p M^p g_\varepsilon(x) + \delta\}. \quad (63)$$

By considering the iterated integral on the left-hand side of (63), we get

$$\begin{aligned}
&\int_{-T}^{T}\int_0^{2\pi} \left|e^{i\gamma} g_\varepsilon'(x) + e^{i(\tau+\varepsilon)x} g_\varepsilon'(-x)\right|^p d\gamma dx \\
&\qquad = \int_{-T}^{0}\int_0^{2\pi} |g_\varepsilon'(x)|^p \left|1 + e^{-i\gamma + i(\tau+\varepsilon)x} \frac{g_\varepsilon'(-x)}{g_\varepsilon'(x)}\right|^p d\gamma dx \\
&\qquad\quad + \int_{0}^{T}\int_0^{2\pi} |g_\varepsilon'(-x)|^p \left|1 + e^{i\gamma - i(\tau+\varepsilon)x} \frac{g_\varepsilon'(x)}{g_\varepsilon'(-x)}\right|^p d\gamma dx \\
&\qquad \ge \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^p d\gamma \left(\int_{-T}^{0} |g_\varepsilon'(x)|^p dx + \int_0^{T} |g_\varepsilon'(-x)|^p dx\right) \\
&\qquad = 2\int_0^{2\pi} \left|1 + e^{i\gamma}\right|^p d\gamma \left(\int_{-T}^{0} |g_\varepsilon'(x)|^p dx\right).
\end{aligned}$$

Then multiplying both sides by $1/2T$, from (63) we get

$$\begin{aligned}\frac{2}{2T}\int_0^{2\pi}\left|1+\mathrm{e}^{\mathrm{i}\gamma}\right|^p d\gamma\left(\int_{-T}^{0}|g_\varepsilon'(x)|^p dx\right)\\ \leq \frac{1}{2T}\int_{-T}^{T}\int_0^{2\pi}\left|\mathrm{e}^{\mathrm{i}\gamma}g_\varepsilon'(x)+\mathrm{e}^{\mathrm{i}(\tau+\varepsilon)x}g_\varepsilon'(-x)\right|^p d\gamma dx\\ \leq 2\pi\{(\tau+\varepsilon)^p M^p g_\varepsilon(x)+\delta\}. \qquad (64)\end{aligned}$$

Inequality (64) is true for all $T \geq T_0$, so taking limit superior when $T \to \infty$, we get

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^{2\pi}\left|1+\mathrm{e}^{\mathrm{i}\gamma}\right|^p d\gamma\left(\int_{-T}^{0}|g_\varepsilon'(x)|^p dx\right) \leq 2\pi\{(\tau+\varepsilon)^p M^p g_\varepsilon(x)+\delta\}. \qquad (65)$$

Since, δ is an arbitrary positive real number, letting $\delta \to 0$ we get

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^{2\pi}\left|1+\mathrm{e}^{\mathrm{i}\gamma}\right|^p d\gamma\left(\int_{-T}^{0}|g_\varepsilon'(x)|^p dx\right) \leq 2\pi(\tau+\varepsilon)^p\{M^p g_\varepsilon(x)\}. \qquad (66)$$

Note that from (59) for every $x \in \mathbb{R}$ such that $x \geq 0$, $|g_\varepsilon(-x)| \leq |g_\varepsilon(x)|$, we have

$$M^p g_\varepsilon(x) = \limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g_\varepsilon(x)|^p dx \leq 2\limsup_{T\to\infty}\frac{1}{T}\int_{-T}^{0}|g_\varepsilon(x)|^p. \qquad (67)$$

Then from (66) and (67), we get

$$\limsup_{T\to\infty}\frac{1}{T}\int_{-T}^{0}|g_\varepsilon'(x)|^p dx \leq (\tau+\varepsilon)^p C_p \limsup_{T\to\infty}\frac{1}{T}\int_{-T}^{0}|g_\varepsilon(x)|^p, \qquad (68)$$

where C_p is as given in (11).
For any $x \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0} g_\varepsilon(x) = \lim_{\varepsilon\to 0}\mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}x}\frac{\sin\frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x}f(x) = f(x), \qquad (69)$$

and

$$\lim_{\varepsilon\to 0} g_\varepsilon'(x) = f'(x). \qquad (70)$$

Inequality (68) is true for every $\varepsilon > 0$, therefore by letting $\varepsilon \to 0$, and using (69) and (70), we get (28).

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References

[1] A. Aziz, Q. G. Mohammad, *Simple proof of a theorem of Erdős and Lax*, Proc. Amer. Math. Soc. **80**(1980), 119–122.

[2] S. N. BERNSTEIN, *Sur une propriété des fonctions entières*, Comptes rendus **176**(1923), 1603–1605.

[3] S. N. BERNSTEIN, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Gauthier–Villars, Paris, 1926.

[4] R. P. BOAS, JR, *Entire functions*, Academic Press, New York, 1954.

[5] K. K. DEWAN, N. K. GOVIL, *An inequality for self-inversive polynomials*, J. Math. Anal. Appl. **95**(1982), 490.

[6] C. FRAPPIER, Q. I. RAHMAN, ST. RUSCHEWEYH, *New inequalities for polynomials*, Trans. Amer. Math. Soc. **288**(1985), 69–99.

[7] C. FRAPPIER, Q. I. RAHMAN, ST. RUSCHEWEYH, *Inequalities for polynomials*, J. Approx. Theory **44**(1985), 73–81.

[8] N. K. GOVIL, *L^p inequalities for entire functions of exponential type*, Math. Inequal. Appl. **6**(2003), 445–452.

[9] N. K. GOVIL, V. K. JAIN, G. LABELLE, *Inequalities for polynomials satisfying $p(z) \equiv z^n p(1/z)$*, Proc. Amer. Math. Soc. **57**(1976), 238–242.

[10] N. K. GOVIL, Q. I. RAHMAN, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc. **137**(1969), 501–517.

[11] N. K. GOVIL, D. H. VETTERLINE, *Inequalities for a class of polynomials satisfying $p(z) \equiv z^n p(1/z)$*, Complex variables Theory Appl. **31**(1996), 285–191.

[12] A. R. HARVEY, *The mean of a function of exponential type*, Amer. J. Math. **70**(1948), 181–202.

[13] M. A. MALIK, *On the derivative of a polynomial*, J. London Math. Soc. **1**(1969), 57–60.

[14] Q. I. RAHMAN, G. SCHMEISSER, *L^p inequalities for entire functions of exponential type*, Trans. Amer. Math. Soc. **320**(1990), 91–103.

[15] Q. I. RAHMAN, G. SCHMEISSER, *Analytic theory of polynomials*, Clarendon Press, Oxford, 2002.

[16] Q. I. RAHMAN, Q. M. TARIQ, *An inequality for self-reciprocal polynomials*, East J. Approx. **12**(2006), 43–51.

[17] Q. I. RAHMAN, Q. M. TARIQ, *On Bernstein's inequality for entire functions of exponential type*, Comput. Methods Funct. Theory **7**(2007), 167–184.

[18] Q. I. RAHMAN, Q. M. TARIQ, *On Bernstein's inequality for entire functions of exponential type*, J. Math. Anal. Appl. **359**(2009), 168–180.

[19] A. ZYGMUND, *A remark on conjugate series*, Proc. London Math. Soc. **34**(1932), 292–400.