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Existence of three solutions for Kirchhoff nonlocal operators of elliptic type

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Abstract. In this paper, we prove the existence of at least three solutions to the following Kirchhoff nonlocal fractional equation:

$$\begin{cases} M\left(\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)dxdy-\int_\Omega|u(x)|^2dx\right)((-\Delta)^su-\lambda u) \\ \qquad\qquad \in \theta(\partial j(x,u(x))+\mu\partial k(x,u(x))), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplace operator, $s \in (0,1)$ is a fix, λ, θ, μ are real parameters and Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary. The approach is fully based on a recent three critical points theorem of Teng [K. Teng, Two nontrivial solutions for hemivariational inequalities driven by nonlocal elliptic operators, Nonlinear Anal. Real World Appl. **14**(2013), 867–874].

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1. Introduction

The aim of this paper is to establish the existence of at least three solutions for the following Kirchhoff nonlocal hemivariational inequalities with the Dirichlet boundary condition:

$$\begin{cases} -M\left(\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)dxdy-\int_\Omega|u(x)|^2dx\right) \\ \qquad\qquad \times(\mathcal{L}_Ku+\lambda u)\in\theta(\partial j(x,u(x))+\mu\partial k(x,u(x))), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{1}$$

where $s \in (0,1)$ is a fix, λ, θ, μ are real parameters, Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, $M : [0,+\infty) \to \mathbb{R}$ is a continuous function, $j, k : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable functions such that for all $x \in \Omega$, $j(x,\cdot), k(x,\cdot)$ are

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locally Lipschitz and $\partial j(x,\cdot), \partial k(x,\cdot)$ denote the generalized subdifferential in the sense of Clarke [5] and

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)dy, \quad x \in \mathbb{R}^n, \tag{2}$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying properties that

(K1) $mK \in L^1(\mathbb{R}^n)$, where $m(x) = \min\{|x|^2, 1\}$;

(K2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$;

(K3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

The homogeneous Dirichlet datum in (1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial\Omega$, consistent with the nonlocal character of the kernel operator $\mathcal{L}_K$.

A typical model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$ where $s \in (0,1)$ $(n > 2s)$ is fixed, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n. \tag{3}$$

Problem (1) in the model case $\mathcal{L}_K = -(-\Delta)^s$ becomes

$$\begin{cases} M\Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)dxdy - \lambda \int_\Omega |u(x)|^2 dx\Big) \\ \qquad\qquad \times((-\Delta)^s u - \lambda u) \in \theta(\partial j(x,u(x)) + \mu \partial k(x,u(x))), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4}$$

Before proving the main results, some preliminary material on function spaces and norms is needed. In what follows we briefly recall the definition of the functional space X_0, firstly introduced in [14], and we give some notations. We denote $\mathrm{Q} = \mathbb{R}^{2n} \setminus \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^n \setminus \Omega \times \mathbb{R}^n \setminus \Omega$. We denote the set X by

$$X = \Big\{u : \mathbb{R}^n \to \mathbb{R} : \ u|_\Omega \in L^2(\Omega),\ (u(x) - u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2n} \setminus \mathcal{O})\Big\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we denote by X_0 the following linear subspace of X

$$X_0 = \{g \in X : \ g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

In this paper, we will prove the existence of nontrivial weak solutions to problem (1). The technical tool is the three critical points theorem of Teng [18] for non-differentiable functionals. By weak solutions of (1) we mean a solution of the following problem

$$\begin{cases} M\ (\ \int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)dxdy - \lambda \int_\Omega |u(x)|^2 dx\ ) \\ \quad \times\Big[\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\eta(x)-\eta(y))K(x-y)dxdy - \lambda \int_\Omega u(x)\eta(x)dx\Big] \\ \quad +\theta\left[-\int_\Omega (u^*,\eta) - \mu(v^*,\eta)\right] = 0, \quad \forall \eta \in X_0, \\ u \ \ \in X_0. \end{cases} \tag{5}$$

where $u^* \in \partial j(x,u)$, $v^* \in \partial k(x,u)$.

We know that X and X_0 are nonempty, since $C_0^2(\Omega) \subseteq X_0$ by Lemma 11 of [14]. Moreover, the linear space X is endowed with the norm defined as

$$||u||_X := ||u||_{L^2(\Omega)} + \left(\int_Q |u(x) - u(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}}. \tag{6}$$

It is easy to see that $||\cdot||_X$ is a norm on X (see, for instance, [15] for a proof). By Lemmas 6 and 7 of [15], in the sequel we can take the function

$$X_0 \ni v \mapsto ||v||_{X_0} = \left(\int_Q |v(x) - v(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}} \tag{7}$$

as a norm on X_0. Also $(X_0, ||\cdot||_{X_0})$ is a Hilbert space, with a scalar product

$$\langle u, v\rangle_{X_0} := \int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy. \tag{8}$$

Note that in (7) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

In what follows, we denote by λ_1 the first eigenvalue of the operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the first eigenvalue of the problem

$$\begin{cases} \mathcal{L}_K u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

For the existence and the basic properties of this eigenvalue we refer to Proposition 9 and Appendix A of [16], where a spectral theory for general integro-differential nonlocal operators was developed.

When $\lambda < \lambda_1$, as a norm on X_0 we can take the function

$$X_0 \ni v \mapsto ||v||_{X_0,\lambda} = \left(\int_Q |v(x) - v(y)|^2 K(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \right)^{\frac{1}{2}}, \tag{9}$$

since for any $v \in X_0$ it holds true (for this, see Lemma 10 of [16])

$$m_\lambda ||v||_{X_0} \le ||v||_{X_0,\lambda} \le M_\lambda ||v||_{X_0}, \tag{10}$$

where

$$m_\lambda := \min\left\{ \sqrt{\frac{\lambda_1 - \lambda}{\lambda_1}}, 1 \right\}, \quad M_\lambda := \max\left\{ \sqrt{\frac{\lambda_1 - \lambda}{\lambda_1}}, 1 \right\}.$$

Let $H^s(\mathbb{R}^n)$ be the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$||u||_{H^s(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}. \tag{11}$$

Also, we recall the embedding properties of X_0 into the usual Lebesgue spaces (see Lemma 8 of [15]). The embedding $j : X_0 \hookrightarrow L^v(\mathbb{R}^n)$ is continuous for any $v \in [1, 2^*]$ $(2^* = \frac{2n}{n-2s})$, while it is compact whenever $v \in [1, 2^*)$. Hence, for any $v \in [1, 2^*]$ there exists a positive constant c_v such that

$$||v||_{L^v(\mathbb{R}^n)} \leq c_v ||v||_{X_0} \leq c_v m_\lambda^{-1} ||v||_{X_0,\lambda}, \tag{12}$$

for any $v \in X_0$.

Recently, several studies have been performed for non-local fractional Laplacian equations substituted by superlinear and subcritical or critical nonlinearities; we refer interested readers to [2, 3, 4, 6, 7, 10, 11, 12, 13, 15, 16, 17, 18, 19] and references therein.

Inspired by the above articles, in this paper, we would like to investigate the existence of three solutions to problem (4). The technical tool is critical point theory for non-differentiable functionals.

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our results on the existence of three solutions.

2. Preliminaries

In this section, we present some preliminaries and lemmas that are useful for the proof of the main results. For the convenience of the reader, we also present here the necessary definitions.

Let $(X, ||\cdot||_X)$ be a Banach space, $(X^*, ||\cdot||_{X^*})$ its topological dual, and $\varphi : X \to \mathbb{R}$ a functional. We recall that φ is locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L_U > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L_U ||x - y||_X, \quad \forall x, y \in U.$$

If f is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; h) = \limsup_{w \to u, t \downarrow 0^+} \frac{\varphi(w + th) - \varphi(w)}{t}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi : X \to 2^{X^*}$ is a multifunction. The function $(u, v) \mapsto \varphi^\circ(u; v)$ is upper semicontinuous and

$$\varphi^\circ(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial\varphi(u)\} \text{ for all } v \in X.$$

We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

We say that $u \in X$ is a critical point of locally Lipschitz functional φ if $0 \in \partial\varphi(u)$.

In the proof of our main results, we shall use nonsmooth critical point theory. For this, we first present an important definition.

Definition 1. *An operator $A : X \to X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in X, $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$.*

Definition 2. *A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the nonsmooth Palais-Smale condition (nonsmooth PS-condition for short) if any sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{J(u_n)\}_{n\geq 1}$ is bounded and*

$$\rho(u_n) := \min\{||u^*||_{X^*} : u^* \in \partial\varphi(u_n)\} \to 0 \quad \text{as } n \to +\infty,$$

has a strongly convergent subsequence.

If this is true for every $c \in \mathbb{R}$, we say that J satisfies the nonsmooth (PS)-condition.

Lemma 1 ([9], Proposition 1.1)**.** *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle, \quad \forall u, v \in X,$$
$$\partial\varphi(u) = \{\varphi'(u)\}, \quad \forall u \in X.$$

Lemma 2 ([5], Proposition 2.2.4)**.** *Let $f : X \to \mathbb{R}$ be Lipschitz near u, and let f be continuously differentiable at u. Then $\partial f(u) = \{\nabla f(u)\}$, where $\nabla f(u)$ denotes the Gâteaux derivative of f at u.*

Lemma 3 ([8], Lemma 6)**.** *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with a compact gradient. Then φ is sequentially weakly continuous.*

In the proof of our main results, we shall use Theorem 1. For this, we first present an important definition.

Definition 3. *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semi continuous functional whose restriction to the set dom$(\Psi) = \{u \in X : \Psi(u) < \infty\}$ is continuous. Then, $\Phi + \Psi$ is a Motreanu-Panagiotopoulos functional.*

Definition 4. *Let $\Phi + \Psi$ be a Motreanu-Panagiotopoulos functional, $u \in X$. Then u is a critical point of $\Phi + \Psi$ if for every $v \in X$, $\Phi^0(u; v - u) + \Psi(v) - \Psi(u) \geq 0$.*

The following lemma introduces some basic properties of the generalized gradients:

Lemma 4 (see [5])**.** *Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for every $u, v \in X$, the following conditions hold:*

(i) *$\partial\varphi_1(u)$ is convex and weakly* compact;*

(ii) *the set-value mapping $\partial\varphi_1 : X \to 2^{X^*}$ is weakly* upper semicontinuous;*

(iii) *$\varphi_1^\circ(u; v) = \max_{u^*\in\partial\varphi}\langle u^*, v\rangle \leq L_U||v||$, with L_U as in definition of locally Lipschitz functionals;*

(iv) *$\partial(\lambda\varphi_1)(u) = \lambda\partial\varphi_1(u)$ for every $\lambda \in \mathbb{R}$;*

(v) $\partial(\varphi_1+\varphi_2)(u) \subseteq \partial\varphi_1(u)+\partial\varphi_2(u)$ *for every* $\lambda \in \mathbb{R}$*;*

The goal of this work is to establish some new criteria for system (1) to have at least three weak solutions in X, by means of a very recent abstract critical points result of Teng [18]. First, we recall the following result of ([18, Theorem 3.1]), with easy manipulations, that we are going to use in the sequel.

Theorem 1. *Let* X *be a reflexive real Banach space,* Ψ *a convex, proper, lower semicontinuous functional and* $\Phi : X \to \mathbb{R}$ *a locally Lipschitz functional with compact gradient* $\partial\Phi$ *and* Φ *is nonconstant. Suppose that*

(A1) $\Theta : X \to \mathbb{R}$ *is a locally Lipschitz functional with compact gradient* $\partial\Theta$*;*

(A2) There exists an interval $\Lambda \subset \mathbb{R}$ *and a number* $\eta > 0$*, such that for every* $\theta \in \Lambda$ *and every* $\mu \in [-\eta, \eta]$ *the functional* $J_{\theta,\mu} = \Psi + \theta(\Phi + \mu\Theta)$ *is coercive in* X*;*

(A3) The functional $J_{\theta,\mu}$ *satisfies the Palais-Smale condition for every* $\theta \in \Lambda$ *and every* $\mu \in [-\eta, \eta]$*;*

(A4) There exists $r \in (\inf_{u\in X}\Phi(u), \sup_{u\in X}\Phi(u))$ *such that the following two numbers*

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}(I_r)} \frac{\inf_{v\in\Phi^{-1}(r)}\Psi(v) - \Psi(u)}{\Phi(u) - r},$$

$$\varphi_2(r) = \sup_{u\in\Phi^{-1}(I^r)} \frac{\inf_{v\in\Phi^{-1}(r)}\Psi(v) - \Psi(u)}{\Phi(u) - r}$$

satisfy $\varphi_1(r) < \varphi_2(r)$*, where* $I_r = (-\infty, r)$ *and* $I^r = (r, +\infty)$*.*

If $(\varphi_1(r), \varphi_2(r)) \cap \Lambda \neq \emptyset$*, then for every compact interval* $[a, b] \subset (\varphi_1(r), \varphi_2(r)) \cap \Lambda$*, there exists* $\delta \in (0, \eta)$ *such that if* $|\mu| < \delta$*, the functional* $J_{\theta,\mu}$ *admits at least three critical points for every* $\theta \in [a, b]$*.*

We recall a convergence property for bounded sequences in X_0 (see [15], for this we need a Lipschitz boundary):

Lemma 5. *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions (K1)-(K3) and let* $\{u_n\}$ *be a bounded sequence in* X_0*. Then, there exists* $u \in L^p(\mathbb{R}^n)$ *such that, up to a subsequence,* $u_n \to u$ *in* $L^p(\mathbb{R}^n)$*, as* $n \to \infty$*, for any* $p \in [1, 2^*)$*.*

The functional $J_{\theta,\mu} : X_0 \to \mathbb{R}$ corresponding to problem (1) is defined by

$$\begin{aligned} J_{\theta,\mu}(u) &= \frac{1}{2}\overline{M}\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)dxdy - \lambda\int_\Omega |u(x)|^2 dx\right) \\ &\quad -\theta\left[\int_\Omega j(x,u(x))dx + \mu\int_\Omega k(x,u(x))dx\right] \\ &= \frac{1}{2}\overline{M}(\|u\|^2_{X_0,\lambda}) - \theta\left[\int_\Omega j(x,u(x))dx + \mu\int_\Omega k(x,u(x))dx\right], \end{aligned} \tag{13}$$

where $\overline{M}(s) = \int_0^s M(t)dt$.

In order to study problem (1), we will use the functionals $\Phi, \Psi : X_0 \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{2}\overline{M}\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)dxdy - \lambda\int_\Omega |u(x)|^2dx\right),$$
$$\Phi(u) = -\int_\Omega j(x,u(x))dx, \quad \Theta(u) = -\int_\Omega k(x,u(x))dx. \tag{14}$$

Hence, by (9), for any $\lambda < \lambda_1$ and $u \in X_0$ one can get

$$\Psi(u) = \frac{1}{2}\overline{M}(||u||^2_{X_0,\lambda}). \tag{15}$$

Now, we will establish the variational principle for problem (1). For this purpose our hypotheses on the nonsmooth potential $j(x,u)$ and $M(t)$ are the following:

(H1) For all $s \in \mathbb{R}$, the function $x \to j(x,s)$ is measurable;

(H2) For all $x \in \Omega$, the function $s \to j(x,s)$ is locally Lipschitz and $j(x,0) = 0$;

(H3) There exist $a, b \in L^\infty_+(\Omega)$ and $1 \le r < 2$ such that $|s^*| \le a(x) + b(x)|s|^{r-1}$ for all $x \in \Omega$, $x \in \mathbb{R}$ and $s^* \in \partial j(x,s)$;

(M1) there exists $m_0 > 0$ such that $M(t) \ge m_0$, $\forall t \in [0,+\infty)$;

(M2) $M(t)$ is nondecreasing in $t \in [0,+\infty)$.

For example, in what follows, it holds that conditions (M1) and (M2) hold:

$$M(t) = pt^{p-1} + 1, \quad p \ge 1,\ \forall t \in [0,+\infty).$$

Then

$$\overline{M}(t) = t^p + x, \quad \forall t \in [0,+\infty).$$

Now, by the Formulas of $M(t)$ it is obvious that (M1) and (M2) hold true and that $\overline{M}(t)$ is convex.

First of all, note that X_0 is a Hilbert space and the functionals Ψ, Φ and Θ are Frechét differentiable in X_0. Also, note that the map $u \mapsto ||u||^2_{X_0,\lambda}$ is lower semicontinuous in the weak topology of X_0 and M is a continuous function, so that the functional Ψ is lower semicontinuous in the weak topology of X_0. Also, by (M2), the functional Ψ is a convex functional.

Therefore, we have the following remark.

Remark 1. *By Definition 3, the functional $J_{\theta,\mu}$ is of a Motreanu-Panagiotopoulos functional on X.*

Proposition 1. *Assume that $j(x,u)$ and $k(x,u)$ satisfy hypotheses (H1)-(H3), the functional $J_{\theta,\mu} : X_0 \to \mathbb{R}$ is well defined and locally Lipschitz on X_0. Moreover, every critical point $u \in X_0$ of $J_{\theta,\mu}$ is a solution of problem (1).*

According to Proposition 1, we know that in order to find solutions of problem (1), it suffices to obtain the critical points of the functional $J_{\theta,\mu}$.

3. Main results

In this section we present our main results. Now, we will apply Theorem 1 to obtain some existence and multiplicity results to problem (1).

Before our main result, we need the following lemmas.

Lemma 6. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3) and $\lambda < \lambda_1$. Assume that $j(x, u)$ and $k(x, u)$ satisfy hypotheses (H1)-(H3) and M satisfies conditions (M1) and (M2), the functional $J_{\theta,\mu} : X_0 \to \mathbb{R}$ is coercive for every $\theta, \mu \in \mathbb{R}$.*

Proof. By (H2), (H3) and the Lebourg's mean value theorem, we have

$$\begin{aligned} |j(x, u)| &= |j(x, u) - j(x, 0)| = |(u^*, u)| \leq a(x)|u| + b(x)|u|^r, \\ |k(x, u)| &= |k(x, u) - k(x, 0)| = |(u^*, u)| \leq a(x)|u| + b(x)|u|^r, \end{aligned} \tag{16}$$

for all $u \in \mathbb{R}$ and $x \in \Omega$. Thus, by (M1), (12) and (16), one can get

$$\begin{aligned} J_{\theta,\mu}(u) &= \frac{1}{2}\overline{M}(||u||^2_{X_0,\lambda}) - \theta \left[\int_\Omega j(x, u(x))dx + \mu \int_\Omega k(x, u(x))dx\right] \\ &\geq \frac{m_0}{2}||u||^2_{X_0,\lambda} - |\lambda|(1 + |\mu|)\Big[||a||_\infty||u||_{L^1(\Omega)} + ||b||_\infty||u||^r_{L^r(\Omega)}\Big] \\ &\geq \frac{m_0}{2}||u||^2_{X_0,\lambda} - |\lambda|(1 + |\mu|)\Big[||a||_\infty\frac{c_1}{m_\lambda}||u||_{X_0,\lambda} + ||b||_\infty\frac{c_r^r}{m_\lambda^r}||u||^r_{X_0,\lambda}\Big]. \end{aligned}$$

Since $1 < r < 2$, then $J_{\theta,\mu}$ is coercive for every $\theta, \mu \in \mathbb{R}$. □

Lemma 7. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3) and $\lambda < \lambda_1$. Assume that $j(x, u)$ satisfies hypotheses (H1)-(H3). Then, the functional $\Phi : X_0 \to \mathbb{R}$ is a locally Lipschitz functional with a compact gradient.*

Proof. Clearly, Φ is locally Lipschitz on X_0. Now we shall show that the set-valued function $\partial\Phi : X_0 \to 2^{X_0}$ is compact. To this end, let us fix a bounded sequence $\{u_n\} \subset X_0$ and $u_n^* \in \partial\Phi(u_n)$ for all $n \in \mathbb{N}$ such that $\langle u_n^*, v\rangle = \int_\Omega(u_n^*(x), v(x))dx$ for every $v \in X_0$. Let $L > 0$ be a Lipschitz constant for Φ, restricted to a bounded set where the sequence $\{u_n\}$ lies, then $||u_n^*||_{X_0^*} \leq L$ for all $n \in \mathbb{N}$. Up to a subsequence, $\{u_n^*\}$ weakly converges to some u^* in $(X_0)^*$. We shall show that the convergence is strong. Assume to the contrary, that is, we assume there exists $\epsilon > 0$ such that $||u_n^* - u^*||_{(X_0)^*} > \epsilon$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$, there exists $v_n \in B^N(0, 1)(B^N(0, 1) = \{u \in X_0 : \ ||u||_{X_0,\lambda} \leq 1\})$ such that

$$\langle u_n^* - u^*, v_n\rangle > \epsilon. \tag{17}$$

Since $\{v_n\}$ is bounded in X_0, then up to a subsequence, there is a $v \in X_0$ such that $v_n \rightharpoonup v$ in X_0 and $v_n \to v$ in $L^q(\Omega)$ $(1 \leq q \leq 2)$ (see Lemma 5). From (H3), one can get

$$\begin{aligned} \langle u_n^* - u^*, v_n\rangle &= \langle u_n^*, v_n - v\rangle + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\leq C_1(||v_n - v||_{L^1} + ||v_n - v||_{L^q}) + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \to 0, \end{aligned}$$

as $n \to +\infty$, which contradicts (17). □

Lemma 8. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3) and $\lambda < \lambda_1$. Assume that $j(x, u)$ and $k(x, u)$ satisfy hypotheses (H1)-(H3) and M satisfies conditions (M1) and (M2). Then, the functional $J_{\theta,\mu}$ satisfies the (PS)-condition for every $\theta, \mu \in \mathbb{R}$.*

Proof. By Definition 2, suppose $\{u_n\} \subset X_0$ satisfies

$$|J_{\theta,\mu}(u_n)| \le C \quad \text{and} \quad \rho(u_n) = \min\{||u^*||_{X^*} : u^* \in \partial J_{\theta,\mu}(u_n)\} \to 0. \tag{18}$$

Since $\partial J_{\theta,\mu}(u_n) \subset (X_0)^*$ is a weak* compact set and the norm function in a Banach space is weakly semi-continuous, by Weierstrass theorem, we can find $u_n^* \in \partial J_{\theta,\mu}(u_n)$ such that

$$\rho(u_n) = ||u_n^*||_{(X_0)^*} \quad \text{and} \quad u_n^* = Au_n - \theta(v_n + \mu w_n), \quad \text{for every } n \ge 1 \tag{19}$$

with $v_n \in L^{r'}(\Omega)$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $v_n \in \partial j(x, u_n(x))$, $w_n \in \partial k(x, u_n(x))$ for all $x \in \Omega$. Here $A : X_0 \to (X_0)^*$ is an operator defined by

$$\begin{aligned}\langle Au, v\rangle = {} & M\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x) - u(y)|^2 K(x-y)dxdy - \lambda\int_\Omega |u(x)|^2 dx\right)\\ & \times\Big[\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy - \lambda\int_\Omega u(x)v(x)dx\Big],\end{aligned}$$

for all $v \in X_0$.

Since, $J_{\theta,\mu}$ is coercive, then the sequence $\{u_n\}$ in X_0 is bounded and so by passing to a subsequence if necessary, by Lemma 5, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X_0,\\ u_n \to u, & \text{strongly in } L^p(\mathbb{R}^n)\ (1 \le p < 2^*),\\ u_n \to u, & \text{a.e. in } \mathbb{R}^n. \end{cases} \tag{20}$$

We note that the nonlinear operator $A : X_0 \to (X_0)^*$ is strongly monotone, that is

$$\langle Au - Av, u - v\rangle \ge c||u - v||^2_{X_0,\lambda}, \quad \text{for all } u, v \in E^\alpha.$$

In fact, by (M1), then for all $u, v \in X_0$ we have

$$\begin{aligned}&\langle Au - Av, u - v\rangle\\ &\quad = M\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |(u-v)(x) - (u-v)(y)|^2 K(x-y)dxdy - \lambda\int_\Omega |(u-v)(x)|^2 dx\right)\\ &\qquad \times\Big[\int_{\mathbb{R}^n\times\mathbb{R}^n} |(u-v)(x) - (u-v)(y)|^2 K(x-y)dxdy - \lambda\int_\Omega |(u-v)(x)|^2 dx\Big],\\ &\quad \ge m_0||u - v||^2_{X_0,\lambda}.\end{aligned}$$

Clearly, the strongly monotonicity property implies that A satisfies $(S)_+$.

Consequently, it suffices to prove the following fact

$$\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0. \tag{21}$$

Indeed, from definition 2 and (19), we have

$$\begin{aligned}\epsilon_n||u_n-u||^2_{X_0,\lambda} \geq \langle u_n^*, u_n-u\rangle \\ = \langle Au_n, u_n-u\rangle - \theta\Bigg[\int_\Omega v_n(x)(u_n(x)-u(x))dx \\ +\mu\int_\Omega w_n(x)(u_n(x)-u(x))dx\Bigg]\end{aligned}$$

with $\epsilon_n \downarrow 0$. By (20) and Hölder inequality, we can get

$$\int_\Omega v_n(x)(u_n(x)-u(x))dx + \mu\int_\Omega w_n(x)(u_n(x)-u(x))dx \to 0$$

as $n \to +\infty$. So, $\limsup_{n\to+\infty}\langle Au_n, u_n-u\rangle \leq 0$. Thus (1) holds. Since A is of type $(S)_+$, so we obtain $u_n \to u$ in X_0. Thus, the functional $J_{\theta,\mu}$ satisfies the (PS)-condition for every $\theta, \mu \in \mathbb{R}$. □

Our first result is as follows.

Theorem 2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions (K1)-(K3). Assume that $j(x,u)$ and $k(x,u)$ satisfy conditions (H1)-(H3) and M satisfies conditions (M1) and (M2), and suppose $j(x,u)$ satisfies the following conditions:*

(H4) There exists $2 < \alpha_0 < 2^*$ such that $\limsup_{|u|\to 0}\frac{\max\{|u^*|:u^*\in\partial j(x,u)\}}{|u|^{\alpha_0-1}} < \infty$ uniformly for all $x \in \Omega$;

(H5) There exist $0 < \mu_0 < r_0$ where r_0 is a positive constant, $c_0 > 0$ and $M_0 > 0$ such that $c_0 < j(x,u) \leq -\mu_0 j^\circ(x,u;-u)$ for all $u \in \mathbb{R}^N$ with $|u| \geq M_0$ and $x \in \Omega$.

Then, for any non-degenerate closed interval $[a,b]$ with $[a,b] \subset (\varphi_1(0),\infty)$, there exists $\delta > 0$ such that problem (1) admits at least three solutions on X_0 for all $\lambda < \lambda_1$, $\theta \in [a,b]$ and $\mu \in (-\delta,\delta)$.

Proof. Since $\Phi(0) = 0$, we claim that $\Phi(tu) \to -\infty$ as $t \to +\infty$. To this end, let $\mathcal{N}$ be the Lebesgue-null set outside of which hypotheses (H3) and (H5) hold and let $x \in \Omega\backslash\mathcal{N}$, $u \in \mathbb{R}$ with $|u| \geq M_0$. We set $\mathcal{J}(x,\lambda_2) = j(x,\lambda_2 u)$, $\lambda_2 \in \mathbb{R}$. Clearly, $\mathcal{J}(x,\cdot)$ is locally Lipschitz. By Rademarcher's theorem, we see that for every $x \in \Omega$, $\lambda_2 \to \mathcal{J}(x,\lambda_2)$ is differentiable a.e. on $\mathbb{R}$ and at a point of differentiability $\lambda_2 \in \mathbb{R}$, we have $\frac{d}{d\lambda_2}\mathcal{J}(x,\lambda_2) \in \partial\mathcal{J}(x,\lambda_2)$. Moreover, by Chain rule (see [5, Theorem 2.3.10]), we have $\partial\mathcal{J}(x,\lambda_2) \subset \partial_u j(x,\lambda_2 u)u$, hence $\lambda_2\partial\mathcal{J}(x,\lambda_2) \subset \partial_u j(x,\lambda_2 u)\lambda_2 u$. From (H5), one can get

$$\lambda_2\frac{d}{d\lambda_2}\mathcal{J}(x,\lambda_2) \geq -\mathcal{J}^\circ(x,\lambda_2 s;-\lambda_2 s) \geq \frac{1}{\mu_0}\mathcal{J}(x,\lambda_2) \quad \Longrightarrow \quad \frac{\frac{d}{d\lambda_2}\mathcal{J}(x,\lambda_2)}{\mathcal{J}(x,\lambda_2)} \geq \frac{1}{\lambda_2\mu_0}.$$

Moreover, the two inequalities hold true for almost $\lambda_2 \geq 1$.

By integrating from 1 to λ_0 from the above inequality, we get $\ln \frac{\mathcal{J}(x,\lambda_0)}{\mathcal{J}(x,1)} \geq \ln \lambda_0^{\frac{1}{\mu_0}}$. So, we have proved that for $x \in \Omega\backslash\mathcal{N}$, $|u| \geq M_1$ and $\lambda_2 \geq 1$, we have $j(x, \lambda_0 s) \geq \lambda_0^{\frac{1}{\mu_0}} j(x, s)$.

Let $z(x) = \min\{j(x, u) : \ |u| = M_1\}$, clearly $z \in L^2(\Omega, \mathbb{R}^+)$ and $z(x) \geq c_0$ for every $x \in \Omega$. Therefore, for every $x \in \Omega\backslash\mathcal{N}$ and $|u| \geq M_1$, we have

$$j(x,u) = j(x, |u|M_1^{-1}M_1u|u|^{-1}) \geq \Big(\frac{|u|}{M_1}\Big)^{\frac{1}{\mu_0}} j\Big(x, \frac{u}{|u|}M_1\Big) \geq z(x)\Big(\frac{|u|}{M_1}\Big)^{\frac{1}{\mu_0}}. \quad (22)$$

On the other hand, by means of the equivalence between two norms in finite-dimensional space, for any finite-dimensional subspace $U \subset X_0$ and any $u \in U$, there exists a constant $C > 0$ such that

$$||u||_\delta = \left(\int_\Omega |u(x)|^\delta dx\right)^{\frac{1}{\delta}} \geq C||u||_{X_0,\lambda}, \quad \delta \geq 1.$$

Then, by (14) and (22) one can get

$$\begin{aligned}\Phi(u) &= -\int_\Omega j(x, u(x))dx \leq -\int_\Omega z(x)\Big(\frac{|u(x)|}{M_1}\Big)^{\frac{1}{\mu_0}} dx \\ &\leq -c_0\Big(\frac{1}{M_1}\Big)^{\frac{1}{\mu_0}} ||u||_{\frac{1}{\mu_0}}^{\frac{1}{\mu_0}} \leq -c_0C\Big(\frac{1}{M_1}\Big)^{\frac{1}{\mu_0}} ||u||_{X_0,\lambda}^{\frac{1}{\mu_0}},\end{aligned}$$

thus

$$\Phi(tu) \leq -c_0C\Big(\frac{1}{M_1}\Big)^{\frac{1}{\mu_0}} t^{\frac{1}{\mu_0}} ||u||_{X_0,\lambda}^{\frac{1}{\mu_0}},$$

Since $0 < \mu_0 < r_0$ and $c_0C\Big(\frac{1}{M_1}\Big)^{\frac{1}{\mu_0}} > 0$, then for any $u \in U \subset X_0\backslash\{0\}$ we have $\Phi(tu) \to -\infty$ as $t \to +\infty$. Hence the claim is true. Then, for large $t_0 > 0$, we take $u_0 = t_0 u$ with $u \in U \subset X_0\backslash\{0\}$ fixed, then $\Phi(u_0) < 0$, that is, $u_0 \in \Phi^{-1}(-\infty, 0)$, hence that $\mathbb{R}_0^- \subset (\inf \Phi, \sup \Phi)$ follows from the locally Lipschitz continuity of Φ.

If we denote

$$\lambda^* = \varphi_1(0) = \inf_{u\in\Phi^{-1}(I_0)} \frac{-\Psi(u)}{\Phi(u)}, \quad I_0 = (-\infty, 0). \quad (23)$$

By the above argument, we see that λ^* is well defined.

Similarly to the proof of (4.5) in [1], one can get

$$\limsup_{r\to 0^-} \varphi_1(r) \leq \varphi_1(0) = \lambda^*. \quad (24)$$

Also, from (H3) and (H4), we can deduce that $|j(x, u)| \leq C_1|u|^{\alpha_0}$, for every $u \in \mathbb{R}$, where $C_1 > 0$ is a constant. So, for every $u \in X_0$, it is easy to deduce that $|\Phi(u)| \leq C_1 C_{\alpha_0}^{\alpha_0} ||u||^{\alpha_0} = C_2||u||^{\alpha_0}$, where $C_2 > 0$ is a constant. Therefore, given $r < 0$ and $u \in \Phi^{-1}(r)$, by (M1), we have

$$-r = -\Phi(u) \leq C_2||u||_{X_0,\lambda}^{\alpha_0} = C_3\left(m_0\frac{||u||_{X_0,\lambda}^2}{2}\right)^{\frac{\alpha_0}{2}} \leq C_3(\Psi(u))^{\frac{\alpha_0}{2}}, \quad (25)$$

where $C_3 = \left(\frac{2}{m_0}\right)^{\frac{\alpha_0}{2}} C_2$. Since $0 \in \Phi^{-1}((r, +\infty))$, by definition on $\varphi_2(r)$ and (25), we have

$$\varphi_2(r) \geq \frac{1}{|r|} \inf_{v \in \Phi^{-1}(r)} \Psi(v) \geq C_3^{-\frac{2}{\alpha_0}} |r|^{\frac{2}{\alpha_0}-1}.$$

In view of $\alpha_0 > 2$, so that the above inequalities imply that $\lim_{r\to 0^-} \varphi_2(r) = +\infty$. Consequently, we have proved that

$$\lim_{r\to 0^-} \varphi_1(r) = \varphi_1(0) = \lambda^* < \lim_{r\to 0^-} \varphi_2(r) = +\infty.$$

This yields that for all integers $n \geq n^* = 2 + [\lambda^*]$ there exists a number $r_n < 0$ so close to zero such that $\varphi_1(r_n) < \lambda^* + \frac{1}{n} < n < \varphi_2(r_n)$. Hence, since by Lemma 6 we have $\Lambda = \mathbb{R}$, by Theorem 1, for every compact interval

$$[a, b] \subset (\lambda^*, \infty) = \bigcup_{n=n^*}^{\infty} \left[\lambda^* + \frac{1}{n}, n\right] \subset \bigcup_{n=n^*}^{\infty} (\varphi_1(r_n), \varphi_2(r_n)) \bigcap \Lambda,$$

there exists $\delta > 0$ such that problem (1) admits at least three solutions for every $\theta \in [a, b]$ and $\mu \in (-\delta, \delta)$. Therefore, we finish the proof. □

In the following result we replace condition (H5) by conditions (H6).

Theorem 3. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3) and $\lambda < \lambda_1$. Assume that hypotheses (H1)-(H4), (M1) and (M2) hold, suppose $j(x, u)$ satisfies the following condition:*
(H6) $\sup_{u\in\mathbb{R}} j(t, u) > 0$ for all $t \in \Omega$.

Then, for any non-degenerate closed interval $[a, b]$ with $[a, b] \subset (\varphi_1(0), \infty)$, there exists $\delta > 0$ such that problem (1) admits at least three solutions on X_0 for all $\lambda < \lambda_1$, $\theta \in [a, b]$ and $\mu \in (-\delta, \delta)$.

Proof. From the proof of Theorem 2, we only need to prove that $\Phi^{-1}(-\infty, 0) \neq \emptyset$. To this end, we prove that there exists $u_1 \in X$ such that $\Phi(u_1) < 0$. By (H6), for every $x \in \overline{\Omega}$, there is $t_x \in \mathbb{R}$ such that $j(x, t_x) > 0$. For $x \in \mathbb{R}^N$, denoted by N_x a neighborhood of x which is the product of N compact intervals. From (H6) and $j(x, t) \in C(\overline{\Omega} \times \mathbb{R})$, for any $x_0 \in \overline{\Omega}$, there are $N_{x_0} \subset \mathbb{R}^N$, $t_{x_0} \in \mathbb{R}$, and $\delta_0 > 0$, such that $j(x, t_{x_0}) > \delta_0 > 0$ for all $x \in N_{x_0} \bigcap \overline{\Omega}$.

Since $\Omega \subseteq \mathbb{R}^N$ is bounded, $\overline{\Omega}$ is compact, then we can find $N_{x_1}, N_{x_2}, \ldots, N_{x_n}$ such that $\Omega \subset \bigcup_{i=1}^{n} N_{x_i}$ and $N_{x_i} \bigcap N_{x_j} = \partial N_{x_i} \bigcap \partial N_{x_j}$ $(i \neq j)$ and, also, we can find positive constants $t_{x_1}, t_{x_2}, \ldots, t_{x_n} \in \mathbb{R}$ and n positive number $\delta_1, \delta_2, \ldots, \delta_n$ such that

$$j(x, t_{x_i}) > \delta_i > 0 \quad \text{uniformly for t} \in N_{x_i} \bigcap \overline{\Omega},\ i = 1, 2, \ldots, n. \tag{26}$$

Set $\delta_0 = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, $t_0 = \max\{t_{x_1}, t_{x_2}, \ldots, t_{x_n}\}$ and

$$L = \sup_{|t|<|t_0|, x\in\overline{\Omega}} |j(x, t)|. \tag{27}$$

Thus, we can fix a closed set $\mathcal{A}_{x_i} \subset \text{int}(N_{x_i} \bigcap \Omega)$ such that

$$\text{meas}(\mathcal{A}_{x_i}) > \frac{L\text{meas}(N_{x_i} \bigcap \overline{\Omega})}{\delta_0 + L}, \tag{28}$$

where meas(B) denotes the Lebesgue measure of set B. We consider a function $u_1 \in X_0$ such that $|u_1(x)| \in [0, t_0]$ and $u_1 \equiv t_{x_i}$ for all $x \in \mathcal{A}_{t_i}$. For instance, we can set $u_1 = \sum_{i=1}^n u_1^i$, where $u_1^i \in C_0^\infty(N_{x_i} \bigcap \overline{\Omega})$ and

$$u_1^i = \begin{cases} t_{x_i}, & t \in \mathcal{A}_{x_i}, \\ 0 \leq u_1^i < t_{x_i}, & t \in (N_{x_i} \bigcap \Omega) \setminus \mathcal{A}_{t_i}. \end{cases}$$

Therefore, from (26)-(28) we get

$$\begin{aligned} \Phi(u_1) &= -\int_\Omega j(x, u_1) dx = -\int_{\bigcup_{i=1}^n (N_{x_i} \bigcap \Omega)} j(x, u_1) dx \\ &= -\int_{\bigcup_{i=1}^n \mathcal{A}_{t_i}} j(x, u_1) dx - \int_{(\bigcup_{i=1}^n N_{x_i} \bigcap \Omega) \setminus \bigcup_{i=1}^n \mathcal{A}_{x_i}} F(x, u_1) dx \\ &\leq -\sum_{i=1}^n \delta_i \text{meas}(\mathcal{A}_{t_i}) + \sum_{i=1}^n L\Big[\text{meas}(N_{x_i} \bigcap \Omega) - \text{meas}(\mathcal{A}_{t_i})\Big] \\ &< -\sum_{i=1}^n \Big[(\delta_0 + L)\text{meas}(\mathcal{A}_{x_i}) - L\text{meas}(N_{x_i} \bigcap \overline{\Omega})\Big] \\ &< 0. \end{aligned}$$

Therefore, we complete the proof. □

Theorem 4. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3) and $\lambda < \lambda_1$. Assume that hypotheses (H1)-(H4), (M1) and (M2) hold, suppose $j(x, u)$ satisfies the following condition:*

(H7) There exists $1 < \beta < 2$ such that $\liminf_{|u| \to \infty} \frac{\max\{|u^*| : u^* \in \partial j(x,u)\}}{|u|^{\beta-1}} > 0$ uniformly for all $x \in \Omega$.

Then, for any non-degenerate closed interval $[a, b]$ with $[a, b] \subset (\varphi_1(0), \infty)$, there exists $\delta > 0$ such that problem (1) admits at least three solutions on X_0 for all $\lambda < \lambda_1$, $\theta \in [a, b]$ and $\mu \in (-\delta, \delta)$.

Proof. From the proof of Theorem 2, we only need to prove that $\Phi^{-1}(-\infty, 0) \neq \emptyset$. For our purpose, from (H3) and (H7) we have $j(x, u) \geq C_5 |u|^\beta - C_6$, where C_5 and C_6 are positive constants. Thus, one can get

$$\Phi(u) = -\int_\Omega j(x, u(x)) dx \leq -C_5 \int_\Omega |u(x)|^\beta dx + C_6 |\Omega| = -C_5 ||u||_\beta^\beta + C_6 |\Omega|.$$

So,

$$\lim_{u \in X_0, ||u||_\beta \to \infty} \Phi(u) = -\infty,$$

so that $\mathbb{R}_0^- \subset (\inf \Phi, \sup \Phi)$ follows from the locally Lipschitz continuity of Φ. □

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