On fuzzy BCC-ideals over a t-norm

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Abstract. *Using a t-norm T, the notion of T-fuzzy BCC-ideals of BCC-algebras is introduced, and some of their properties are investigated. Connections between different types of fuzzy BCC-ideals induced by t-norms are described.*

Key words: *T-fuzzy BCC-ideal*

AMS subject classifications: 06F35, 03G25, 94D05

Received October 3, 2000 Accepted November 27, 2000

1. Introduction

As it is well known many classes of algebras (for example: BCK-algebras, Hilbert algebras, Hertz algebras, Heyting algebras, MV-algebras) may be isomorphicaly or anti-isomorphicaly embedded into the class of BCC-algebras. Hence the class of BCC-algebras is important. A special role in the theory of BCC-algebras play ideals of different types and their connections with congruences (cf. [6]) and fuzzy sets (cf. [4]).

Y. B. Jun and K. H. Kim introduced in [7] the notion of fuzzy ideals of BCK-algebras with respect to a given t-norm, and obtained some of their properties. In this paper, we generalize these results to the case of BCC-ideals of BCC-algebras and investigate some of their new properties.

2. Preliminaries

In the present paper a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

A non-empty set G with a constant 0 and a binary operation denoted by juxtaposition is called a *BCC-algebra* if for all $x, y, z \in G$ the following axioms hold:

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(i) $(xy \cdot zy) \cdot xz = 0$,
(ii) $xx = 0$,
(iii) $0x = 0$,
(iv) $x0 = x$,
(v) $xy = 0$ and $yx = 0$ imply $x = y$.

A BCC-algebra satisfying the identity:

(vi) $xy \cdot z = xz \cdot y$

is a *BCK-algebra* (cf. [3]).

Note by the way, that a proper BCC-algebra (i.e., a BCC-algebra which is not a BCK-algebra) has at least four elements (cf. [3]). Moreover, there are proper BCC-algebras in which all proper subalgebras are BCK-algebras (cf. [2]).

A non-empty subset I of a BCC-algebra G is called a *BCC-ideal* of G if (i) $0 \in I$ and (ii) $xy \cdot z, y \in I$ implies $xz \in I$. If (ii) holds only in the case when $z = 0$, i.e., if (iii) $xy, y \in I$ implies $x \in I$, then I is called a *BCK-ideal*.

In BCK-algebras BCK-ideals coincides with BCC-ideals, but in BCC-algebras there are BCK-ideals which are not BCC-ideals. Moreover, in BCC-algebras any BCC-ideal is determined by some congruence (cf. [6]).

Now we review some fuzzy concepts. A *fuzzy set* in a set G is a function $\mu : G \to [0,1]$. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) := \{x \in G \,|\, \mu(x) \geq \alpha\}$ is called an *upper level set* of μ.

Definition 1. *By a t-norm, we mean a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions (cf. [1]):*
(T_1) $T(\alpha,1) = \alpha$,
(T_2) $T(\alpha,\beta) \leq T(\alpha,\gamma)$ *whenever* $\beta \leq \gamma$,
(T_3) $T(\alpha,\beta) = T(\beta,\alpha)$,
(T_4) $T(\alpha, T(\beta,\gamma)) = T(T(\alpha,\beta),\gamma)$

for all $\alpha,\beta,\gamma \in [0,1]$.

A simple example of such defined t-norm is a function $T(\alpha,\beta) = \min\{\alpha,\beta\}$. In the general case $T(\alpha,\beta) \leq \min\{\alpha,\beta\}$ and $T(\alpha,0) = 0$ for all $\alpha,\beta \in [0,1]$. Moreover, $([0,1];T)$ may be considered as a commutative semigroup with 0 as the neutral element. In particular

$$T(T(\alpha,\beta),\, T(\gamma,\delta)) = T(T(\alpha,\gamma),\, T(\beta,\delta))$$

holds for all $\alpha,\beta,\gamma,\delta \in [0,1]$.

The set of all idempotents with respect to T, i.e. the set

$$E_T := \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}$$

is a subsemigroup of a semigroup $([0,1],T)$. If $Im(\mu) \subseteq E_T$, then a fuzzy set μ is called an *idempotent with respect to a t-norm T*. (briefly: *an idempotent T-fuzzy set*). In this case $T(\alpha,\beta) = \min\{\alpha,\beta\}$ for all $\alpha,\beta \in Im(\mu)$ since $\alpha \leq \beta$ implies

$$\alpha = T(\alpha,\alpha) \leq T(\alpha,\beta) \leq \min\{\alpha,\beta\} = \alpha.$$

T-fuzzy BCC-ideals

In what follows, let G denote a BCC-algebra unless otherwise specified.

Definition 2. *A fuzzy set μ in G is called a* fuzzy BCC-ideal of G with respect to a t-norm T *(briefly, a T-*fuzzy BCC-ideal *) if*

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(xz) \geq T(\mu(xy \cdot z), \mu(y))$

for all $x, y, z \in G$.

A fuzzy set μ satisfying (F_1) and

(F_3) $\mu(x) \geq T(\mu(xy), \mu(y))$

*is called a T-*fuzzy BCK-ideal *of G.*

This means that any T-fuzzy BCC-ideal is a T-fuzzy BCK-ideal, but not conversely as shows the example given below.

Example 1. *The function T_m defined by $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha, \beta \in [0, 1]$ is a t-norm (cf. [10]).*

Let $G = \{0, a, b, c, d\}$ be a BCC-algebra with the following multiplication:

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

By routine calculations, we known that a fuzzy set μ in G defined by $\mu(0) = \mu(a) = 0.9$ and $\mu(b) = \mu(c) = \mu(d) = 0.3$ is a T_m-fuzzy BCK-ideal of G, which is not a T_m-fuzzy BCC-ideal because $\mu(db) < T_m(\mu(da \cdot b), \mu(a))$.

Proposition 1. *In a BCK-algebra every T-fuzzy BCK-ideal is a T-fuzzy BCC-ideal.*

Proof. Let μ be a T-fuzzy BCK-ideal of a BCK-algebra G, where T is a t-norm. Since (vi) holds in G for all $x, y, z \in G$, we have

$$\mu(xz) \geq T(\mu(xz \cdot y), \mu(y)) = T(\mu(xy \cdot z), \mu(y))$$

and so μ is a T-fuzzy BCC-ideal of G. □

Remark that in the case $T(\alpha, \beta) = \min\{\alpha, \beta\}$ our T-fuzzy BCC-ideals (BCK-ideals) are fuzzy BCC-ideals (BCK-ideals) described in [4] and [5]. On the other hand, any fuzzy BCC-ideal (BCK-ideal) is a T-fuzzy BCC-ideal (BCK-ideal) for every t-norm T because for $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ for every t-norm T. But there are T-fuzzy BCC-ideals which are not fuzzy BCC-ideals.

Example 2. *Let a BCC-algebra G and a t-norm T_m be as in the above example. Then a fuzzy set ρ in G defined by $\rho(0) = 0.9$, $\rho(a) = \rho(b) = 0.6$, $\rho(c) = \rho(d) = 0.5$ is – as is not difficult to see – a T_m-fuzzy BCC-ideal, which is not a fuzzy BCC-ideal since $\rho(c0) < \min\{\rho(cb \cdot 0), \rho(b)\}$.*

This example proves also that a T-fuzzy BCC-ideal is not a fuzzy BCK-ideal in general.

Proposition 2. *If μ is a T-fuzzy BCC-ideal of G, then $U(\mu;1)$ is either empty or a BCC-ideal of G.*

Proof. Assume that $U(\mu;1) \neq \emptyset$. Then there exists $x \in U(\mu;1)$, and so $\mu(0) \geq \mu(x) = 1$, i.e., $0 \in U(\mu;1)$.

Let $x, y, z \in G$ be such that $xy \cdot z \in U(\mu;1)$ and $y \in U(\mu;1)$. Then

$$\mu(xz) \geq T(\mu(xy \cdot z),\, \mu(y)) \geq T(1,1) = 1$$

so that $xz \in U(\mu;1)$. Hence $U(\mu;1)$ is a BCC-ideal of G. □

Note that in the above example $U(\rho;1)$ is empty but $U(\rho;0.6) = \{0, a, b\}$ is not a BCC-ideal (BCK-ideal also). If a T-fuzzy BCC-ideal (BCK-ideal) μ is idempotent, then $T(\alpha,\beta) = \min\{\alpha,\beta\}$ for all $\alpha,\beta \in Im(\mu)$, and in the consequence (cf. [4]), its each non-empty upper level set $U(\mu;\alpha)$ is a BCC-ideal (BCK-ideal) of the corresponding BCC-algebra. Moreover, from [5] follows that an idempotent T-fuzzy BCC-ideal is order reversing.

Let f be a mapping defined on G. If υ is a fuzzy set in $f(G)$, then the fuzzy set $\mu = \upsilon \circ f$ in G (i.e., the fuzzy set defined by $\mu(x) = \upsilon(f(x))$ for all $x \in G$) is called the *preimage of υ under f*.

Proposition 3. *Let T be a t-norm and let $f : G \to G'$ be an onto homomorphism of BCC-algebras, υ a T-fuzzy BCC-ideal of G' and μ the preimage of υ under f. Then μ is a T-fuzzy BCC-ideal of G. Moreover, if υ is idempotent, then so is μ.*

Proof. For any $x \in G$, we get

$$\mu(x) = \upsilon(f(x)) \leq \upsilon(0') = \upsilon(f(0)) = \mu(0)\,.$$

Let $x, z \in G$. Then

$$\mu(xz) = \upsilon(f(xz)) = \upsilon(f(x)f(z)) \geq T(\upsilon(f(x)y' \cdot f(z)),\, \upsilon(y'))$$

for any $y' \in G'$.

Let y be an arbitrary preimage of y' unless f. Then

$$\begin{aligned} \mu(xz) &\geq T(\upsilon(f(x)y' \cdot f(z)),\, \upsilon(y')) \\ &= T(\upsilon(f(x)f(y) \cdot f(z)),\, \upsilon(f(y))) \\ &= T(\upsilon(f(xy \cdot z)),\, \upsilon(f(y))) \\ &= T(\mu(xy \cdot z),\, \mu(y))\,. \end{aligned}$$

Since y' is arbitrary, the above inequality is true for all $y \in G$, i.e.,

$$\mu(xz) \geq T(\mu(xy \cdot z),\, \mu(y))$$

for all $x, y, z \in G$, which proves that μ is a T-fuzzy BCC-ideal.

Now, if υ is idempotent and $\alpha \in Im(\mu)$, then $\alpha = \mu(x) = \upsilon(f(x)))$ for some $x \in G$. Hence $Im(\mu) \subseteq Im(\upsilon) \subseteq E_T$, and therefore μ is an idempotent T-fuzzy BCC-ideal. □

3. Fuzzy BCC-ideals induced by norms

Now we present some methods of constructions of T-fuzzy BCC-ideals.

Definition 3. *Let T be a t-norm and let μ and ν be two fuzzy sets in G. Then the* T-product of μ and ν, *denoted by $[\mu\cdot\nu]_T$, is defined by*

$$[\mu\cdot\nu]_T(x)=T(\mu(x),\nu(x))$$

for all $x\in G$.

Obviously $[\mu\cdot\nu]_T$ is a fuzzy set in G and $[\mu\cdot\nu]_T=[\nu\cdot\mu]_T$.

Theorem 1. *Let T be a t-norm and let μ and ν be two T-fuzzy BCC-ideals of G. If a t-norm T^* dominates T, i.e., if*

$$T^*(T(\alpha,\gamma),T(\beta,\delta))\geq T(T^*(\alpha,\beta),T^*(\gamma,\delta))$$

for all $\alpha,\beta,\gamma,\delta\in[0,1]$, then T^*-product $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy BCC-ideal of G.

Proof. At first observe that

$$[\mu\cdot\nu]_{T^*}(0)=T^*(\mu(0),\nu(0))\geq T^*(\mu(x),\nu(x))=[\mu\cdot\nu]_{T^*}(x)$$

for all $x\in G$. Similarly, for $x,y,z\in G$ we have

$$\begin{aligned}[\mu\cdot\nu]_{T^*}(xz) &= T^*(\mu(xz),\nu(xz))\\ &\geq T^*(T(\mu(xy\cdot z),\mu(y)),\,T(\nu(xy\cdot z),\nu(y)))\\ &\geq T(T^*(\mu(xy\cdot z),\nu(xy\cdot z)),\,T^*(\mu(y),\nu(y)))\\ &= T([\mu\cdot\nu]_{T^*}(xy\cdot z),[\mu\cdot\nu]_{T^*}(y))\,,\end{aligned}$$

which proves that $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy BCC-ideal of G. □

Corollary 1. *The T-product of two T-fuzzy BCC-ideals of G is a T-fuzzy ideal of the same BCC-algebra G.*

Theorem 2. *Let T and T^* be t-norms in which T^* dominates T. Let $f:G\to G'$ be an onto homomorphism of BCC-algebras. For any T-fuzzy BCC-ideals μ and ν of G', we have*

$$f^{-1}([\mu\cdot\nu]_{T^*})=[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}.$$

Proof. Let $x\in G$. Then

$$\begin{aligned}[f^{-1}([\mu\cdot\nu]_{T^*})](x) &= [\mu\cdot\nu]_{T^*}(f(x))=T^*(\mu(f(x)),\nu(f(x)))\\ &= T^*([f^{-1}(\mu)](x),\,[f^{-1}(\nu)](x))=[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}(x)\,,\end{aligned}$$

completing the proof. □

Corollary 2. *If $f:G\to G'$ is an onto homomorphism of BCC-algebras, then $f^{-1}([\mu\cdot\nu]_T)=[f^{-1}(\mu)\cdot f^{-1}(\nu)]_T$ for any T-fuzzy BCC-ideals μ and ν of G'.*

Theorem 3. *Let T be a t-norm and let $G = G_1 \times G_2$ be the direct product of BCC-algebras G_1 and G_2. If μ_1 (resp. μ_2) is a T-fuzzy BCC-ideal of G_1 (resp. G_2), then $\mu = \mu_1 \times \mu_2$ is a T-fuzzy BCC-ideal of G defined by*

$$\mu(x) = \mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$$

for all $(x_1, x_2) = x \in G$.

Proof. For any $x = (x_1, x_2) \in G$ we have

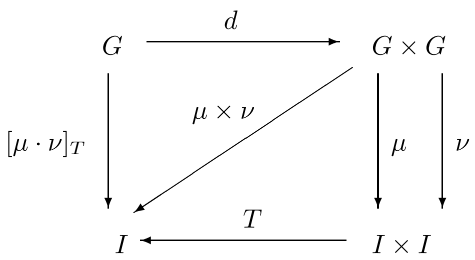
$$\begin{aligned} \mu(x) &= (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1),\, \mu_2(x_2)) \\ &\leq T(\mu_1(0_1),\, \mu_2(0_2)) = (\mu_1 \times \mu_2)(0_1, 0_2) = \mu(0). \end{aligned}$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in G$. Then

$$\begin{aligned} \mu(xz) &= (\mu_1 \times \mu_2)((x_1, x_2)\,(z_1, z_2)) = (\mu_1 \times \mu_2)(x_1z_1,\, x_2z_2) \\ &= T(\mu_1(x_1z_1),\, \mu_2(x_2z_2)) \\ &\geq T(T(\mu_1(x_1y_1 \cdot z_1),\, \mu_1(y_1)),\, T(\mu_2(x_2y_2 \cdot z_2),\, \mu_2(y_2))) \\ &= T(T(\mu_1(x_1y_1 \cdot z_1),\, \mu_2(x_2y_2 \cdot z_2)),\, T(\mu_1(y_1),\, \mu_2(y_2))) \\ &= T((\mu_1 \times \mu_2)(x_1y_1 \cdot z_1,\, x_2y_2 \cdot z_2),\, (\mu_1 \times \mu_2)(y_1, y_2)) \\ &= T((\mu_1 \times \mu_2)((x_1, x_2)\,(y_1, y_2) \cdot (z_1, z_2)),\, (\mu_1 \times \mu_2)(y_1, y_2)) \\ &= T(\mu(xy \cdot z),\, \mu(y))\,. \end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is a T-fuzzy BCC-ideal of G. □

The relationship between T-fuzzy BCC-ideals $\mu \times \nu$ and $[\mu \cdot \nu]_T$ can be viewed via the following diagram



where $I = [0, 1]$ and $d : G \to G \times G$ is defined by $d(x) = (x, x)$.

It is not difficult to see that $[\mu \cdot \nu]_T$ *is the preimage of* $\mu \times \nu$ *under* d.

Note by the way, that our T-product of fuzzy sets is different from the product studied by Liu [8] and Sessa [9].

Now we generalize the product of two T-fuzzy BCC-ideals to the product of $n \geq 2$ T-fuzzy BCC-ideals. We first need to generalize the domain of t-norm T to $\prod_{i=1}^{n} [0, 1]$ as follows:

Definition 4. *The function $T_n : \prod_{i=1}^{n} [0, 1] \to [0, 1]$ is defined by*

$$T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$, $T_2 = T$ and $T_1 = id$ (identity).

Using the induction on n, we have the following lemma.

Lemma 1. *For a t-norm T and every $\alpha_i, \beta_i \in [0,1]$, where $1 \leq i \leq n$ and $n \geq 2$, we have*

$$T_n(T(\alpha_1,\beta_1),T(\alpha_2,\beta_2),\ldots,T(\alpha_n,\beta_n)) = T(T_n(\alpha_1,\alpha_2,\ldots,\alpha_n),T_n(\beta_1,\beta_2,\ldots,\beta_n)) \quad \square.$$

Basing on this *Lemma* and *Theorem 3.* we can prove

Theorem 4. *Let T be a t-norm and let $G = \prod_{i=1}^{n} G_i$ be the direct product of BCC-algebras $\{G_i\}_{i=1}^{n}$. If μ_i is a T-fuzzy BCC-ideal of G_i, where $1 \leq i \leq n$, then $\mu = \prod_{i=1}^{n} \mu_i$ defined by*

$$\mu(x) = (\prod_{i=1}^{n} \mu_i)(x_1,x_2,\ldots,x_n) = T_n(\mu_1(x_1),\mu_2(x_2),\ldots,\mu_n(x_n))$$

for all $x = (x_1,x_2,\ldots,x_n) \in G$, is a T-fuzzy BCC-ideal of G. Moreover, if all μ_i are T-idempotent, then so is μ.

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