A generalization of the butterfly theorem

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Abstract. *In this paper a new generalization of the well-known butterfly theorem is given using the complex coordinates.*

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Let us prove the following theorem.

Theorem 1. *Let A, B, C, D be four points on a circle $\mathcal{K}$ with the centre O and let M be the orthogonal projection of the point O onto the given straight line $\mathcal{M}$. If M is the midpoint of two points $E = \mathcal{M} \cap AB$ and $F = \mathcal{M} \cap CD$, then M is the midpoint of the points $G = \mathcal{M} \cap AC$ and $H = \mathcal{M} \cap BD$ and the midpoint of the points $K = \mathcal{M} \cap AD$ and $L = \mathcal{M} \cap BC$.*

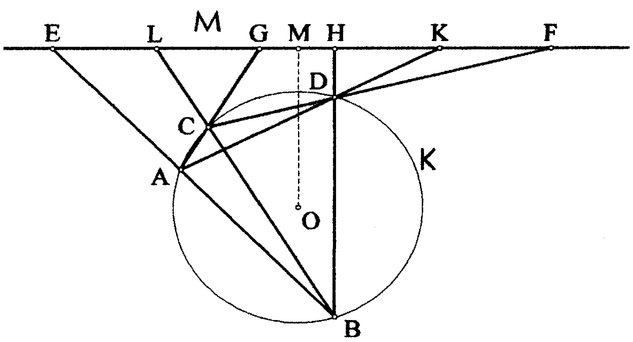

Figure 1

If $G = H = M$ and $\mathcal{M}$ is a chord of $\mathcal{K}$, then we obtain a well-known butterfly theorem (cf. [1] and [2]).

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If $\mathcal{M}$ is a chord of $\mathcal{K}$, then we have Klamkin's generalization of the butterfly theorem (cf. [3]).

If $G = H = M$, then we obtain a Sledge's generalization of the butterfly theorem (cf. [4]).

We shall prove the Theorem using the complex coordinates of the points in a Gauss plane of complex numbers. If a point Z has a complex coordinate z, then we write $Z = (z)$. Let $\bar{z}$ be the conjugated complex number of z. We shall need a lemma.

Lemma 1. *Any straight line $\mathcal{M}$ has an equation of the form*

$$z + t\bar{z} = s, \tag{1}$$

where $Z = (z)$ is any point of this line, $S = (s)$ is the point symmetric to the origin O with respect to the line $\mathcal{M}$, and t is a unimodular number, i.e. $|t| = 1$ or $t\bar{t} = 1$ holds.

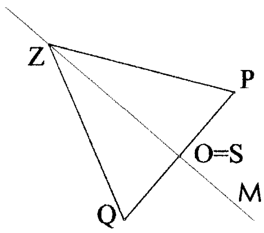

Figure 2

Proof. If $O \notin \mathcal{M}$ (*Figure 2*), then we have the equality $|z - s| = |z|$, i.e. the number $\frac{z-s}{z} = \tau$ is unimodular. Therefore, we have $\frac{\bar{z}-\bar{s}}{\bar{z}} = \frac{1}{\tau}$. Multiplying these two equalities we obtain $(z - s)(\bar{z} - \bar{s}) = z\bar{z}$, i.e. $\bar{s}z + s\bar{z} = s\bar{s}$. But, this is equation (1) if we put

$$t = \frac{s}{\bar{s}}. \tag{2}$$

Obviously $t\bar{t} = 1$. If $O \in \mathcal{M}$, then let $P = (p)$ and $Q = (-p)$ be two points symmetrical with respect to the origin O and with respect to the straight line M (*Figure 3*). Then we have the equality $|z - p| = |z + p|$, i.e. the number $\frac{z-p}{z+p} = \tau$ is unimodular. Therefore, $\frac{\bar{z}-\bar{p}}{\bar{z}+\bar{p}} = \frac{1}{\tau}$. Multiplying these two equalities we obtain $(z + p)(\bar{z} + \bar{p}) = (z - p)(\bar{z} - \bar{p})$, i.e. $\bar{p}z + p\bar{z} = 0$. If we put $t = \frac{p}{\bar{p}}$, then we have equation (1) again, but now $s = 0$ holds.

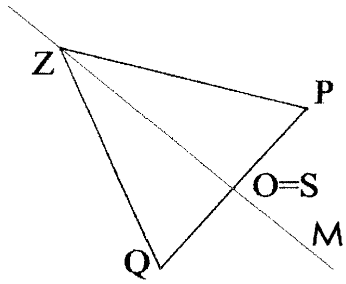

Figure 3

Proof of Theorem. Let $\mathcal{K}$ be the unit circle with the centre in the origin O and let $a, b, c, d, e, f, g, h, k, l$ be the complex coordinates of the points A, B, C, D, E, F, G, H, K, L. The equation

$$z + ab\bar{z} = a + b \tag{3}$$

is the equation of a straight line $\mathcal{L}$ because $|ab| = |a| \cdot |b| = 1$. According to $a\bar{a} = 1$ and $b\bar{b} = 1$, we obtain $a + ab\bar{a} = a + b$ and $b + ab\bar{b} = a + b$, i.e. $A, B \in \mathcal{L}$. Therefore, (3) is the equation of the straight line AB. Now, let $\mathcal{M}$ have equation (1). Substracting equations (1) and (2) we obtain $(t - ab)\bar{z} = s - a - b$. Therefore, for the point $E = \mathcal{M} \cap AB$ the first of two equalities

$$\bar{e} = \frac{s - a - b}{t - ab}, \quad \bar{f} = \frac{s - c - d}{t - cd} \tag{4}$$

holds, and analogously the second equality (4) holds for the point $F = \mathcal{M} \cap CD$. The point $S = (s)$ is symmetrical to the point $O = (0)$ with respect to the line $\mathcal{M}$. Therefore, the points E and F have the midpoint M if and only if $e + f = s$, i.e. $\bar{e} + \bar{f} = \bar{s}$. This condition can be written in the form

$$(t - cd)(s - a - b) + (t - ab)(s - c - d) = \bar{s}(t - ab)(t - cd) \tag{5}$$

because of (4). According to (2), i.e. the equality $\bar{s}t = s$ (which is satisfied in the case $s = 0$, too), equality (5) can be transformed in the form

$$st - (a + b + c + d)t + abc + abd + acd + bcd - abcd\bar{s} = 0,$$

which is symmetrical with respect to the coordinates a, b, c, d. Therefore, we obtain the same condition for $\bar{g} + \bar{h} = \bar{s}$ and for $\bar{k} + \bar{l} = \bar{s}$. Q.E.D.

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