NONLINEAR DYNAMIC DEFORMATION SIMULATION FOR HELICAL ROD LIKE OBJECTS

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ARTICLE INFO

***Article history*:**
Received: 11. 09. 2013.
Received in revised form: 11. 10. 2013.
Accepted: 15. 10. 2013.
Keywords:
Helical rod
Kirchhoff dynamic analogy
Cylindrical coordinates
Deformation simulation

Abstract:

In this paper, dynamic deformation simulation of an elastic helical rod with circular cross-section under axial tension is discussed on the basis of the Kirchhoff dynamic analogy. Firstly, equilibrium equations of an elastic rod described by Euler angles are established in the Frenet coordinates of the centerline. To get solutions of the equations, through a cylindrical coordinate system founded by end constraint, mathematical analytical formulations were used to describe elastic rod configuration are gained on the basis of Saint-Venant Principle of Elasticity, in the form of Elliptic functions. Then, based on the conclusions of static analysis, the relationship between geometric parameters and end constraint of helical rods is qualitatively analyzed. Finally, nonlinear dynamic deformation simulation with constraint force change is realized in a virtual environment to verify the effectiveness of the above algorithm.

1 Introduction

With the maturity of virtual simulation technology for rigid objects, simulation of one-dimension flexible objects like elastic curved rods is gradually favored by researchers [1]. Helical state is a prevalent equilibrium form of elastic rods in the nature, such as coiling cables, curly fibers, and stems of climbing plants, spiral bacillus, DNA and spring. Helical rods are equilibrium solutions that have great application backgrounds to DNA molecular biology, textile industry, cable design and pipeline laying procedure. In all the research into elastic rod equilibrium, dynamic analogy is widely applied as an analysis method [2]. Based on Newton's law, the Kirchhoff rod model provides a theoretical frame describing the static and dynamic behaviors of elastic rods and/since Kirchhoff rods can undergo large changes of shape [3]. The nature of the Kirchhoff model is a set of ordinary differential equations, so the configuration of elastic rod is decided by solutions of the equations under special boundary or initial conditions. The problem solving method mainly uses numerical and symbolic methods.

On one hand, the idea of a numerical method is to convert the Kirchhoff equations into nonlinear equations which could be solved with numerical iteration. However, since the Jacobian matrix of mechanical equations has the characteristics of large-scale and large stiffness, algorithm requirement is very high in order to get better convergence. Pai [4] presented a two-phase integration method to model the behaviors of the strand of surgical suture. The full static Kirchhoff equations including distributed external loading and initial curvatures of rods are considered. Sunil [5] used a body chain to calculate the centerline curvature of elastic rods to do hair modeling, animation and rendering. Tao [6] proposed a general elastic rod model using volumetric elastic joints to discrete the Kirchhoff model and thus a long flexible object could be represented by linking elastic joints between rigid edge elements. Liu et al. [7] used quaternion as simulation DOFs to model the material

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twist required for a small simulation time step to solve coupling constraints between the bending moment and the material twisting. On the other hand, the symbolic method is mainly used for elastic rods with circular cross-section, which means that the equations for circular cross-section rods have an analytical solution. So far, the method has obtained some research results. Nizette [8] gave the parametric analytical solution with the form of the Euler angles of the Kirchhoff equations, and made a classification for the shapes of the Kirchhoff filaments based on the geometry of the spinning top solutions. Liu Shu and Andreas Weber [9] presented a symbol-numerical integration method for hair simulation which could be used in different boundary conditions; a unified method was developed to match the parameters and integration constants needed by the explicit solutions and given boundary conditions.

In this paper the focus is on the configuration of circular cross-section elastic rod under known end constraint. In [9], the explicit solutions with Euler angles under boundary conditions are given, but the calculation of integral constants and the convertion from arc coordinates to basic coordinates are very difficult, which are actually not described in details. To solve this problem, in this paper, a cylindrical coordinate system is introduced decided by the given end constrain force and moment on the basis of Saint-Venant Principle, and consequently, cylindrical coordinates reflecting the configuration of elastic rods are clearly expressed. Moreover, mathematical formulations are given for helical rod configuration, which describe the relationship between geometry configuration and the end force and moment. Finally, a dynamic deformation process under changing external forces is verified in a developed virtual platform.

2 Analytical integration of Kirchhoff equations

According to [8]，the centerline of an elastic rod is expressed by the radius vector **r**(s), and spatial configuration of the elastic rod is described by a rotation of any point’s (P) Frenet coordinates *P-NBT* in the centerline relatively to a basic coordinate system $O\text{-}\xi\eta\zeta$. Introducing Euler angles (ϕ, θ, φ), the static Kirchhoff equations on the basis of arc coordinates could be described in equations (1):

$$\begin{cases} A\left[\dfrac{d^2\theta}{\mathrm{ds}^2}-\left(\dfrac{d\phi}{\mathrm{ds}}\right)^2\cos\theta\sin\theta\right]+ \\ C\dfrac{d\phi}{\mathrm{ds}}\sin\theta\left(\dfrac{d\varphi}{\mathrm{ds}}+\dfrac{d\phi}{\mathrm{ds}}\cos\theta\right)-F\sin\theta=0 \\ A\left[\dfrac{d^2\phi}{\mathrm{ds}^2}\sin\theta+2\dfrac{d\phi}{\mathrm{ds}}\dfrac{d\theta}{\mathrm{ds}}\cos\theta\right]- \\ C\dfrac{d\theta}{\mathrm{ds}}\left(\dfrac{d\varphi}{\mathrm{ds}}+\dfrac{d\phi}{\mathrm{ds}}\cos\theta\right)=0 \\ C\dfrac{d}{\mathrm{ds}}\left(\dfrac{d\varphi}{\mathrm{ds}}+\dfrac{d\phi}{\mathrm{ds}}\cos\theta\right)=0 \end{cases}, \quad (1)$$

where A is bending stiffness of rods, C is the torsional stiffness, which are all decided by material constants, s is the arc coordinates, and ***F*** is end force.

For an end constraint elastic rod, the direction of ***F*** is as the ζ axis, and a cylindrical coordinate system *O-XYZ* is established on the basis of Saint-Venant Principle, and the radius vector ***r*** is defined as $\boldsymbol{r}=(\boldsymbol{F}\times\boldsymbol{M})/F^2$, which are all shown in Fig. 1, where ***M*** is the end moment.

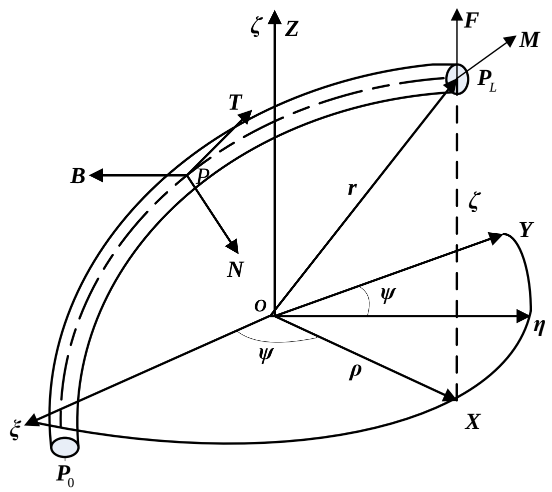

Figure 1. Coordinate systems of an elastic rod

Therefore, cylindrical coordinates (ρ, Ψ, ζ) reflecting configuration of elastic rods shown in equations (2) is expressed, which could be verified that equations (2) are truly the solution to the equations (1) derived by software Mathematica [10]:

$$\begin{cases} \rho(s) = \dfrac{2}{p}\sqrt{a - p\cos\theta(s)} \\ \dfrac{d\psi}{ds} = \dfrac{p}{2}\left(\dfrac{m - l\cos\theta(s)}{a - p\cos\theta(s)}\right) \\ \dfrac{d\zeta}{ds} = \cos\theta(s) \\ \cos\theta(s) = \gamma_1 + (\gamma_2 - \gamma_1) \\ JacobiSN^2\left[\dfrac{p}{4}(\gamma_3 - \gamma_1)s, \sqrt{\dfrac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}}\right] \end{cases}, \quad (2)$$

where a, p, h, m and l are all integral constants, which are decided/defined by the end force and moment; JacobiSN is a first class Elliptic function and γ_i (i=1,2,3) are the three roots of the following equation:

$$\begin{cases} f(\gamma) = (a - p\gamma)(1 - \gamma^2) - (m - l\gamma)^2 \\ \gamma = \cos\theta \end{cases}. \quad (3)$$

3 Centerline of a helical rod

From the perspective of virtual simulation, the torsion of helical rods with a circular cross-section could be neglected and the configuration of helical rods is completely determined by the helix. Geometrically, a helix is decided/defined by the radius R, the helix angle α and the laps N. A helix is shown in Fig. 2. The above basic, Frenet and cylindrical coordinate systems are established in Fig. 2. And another two coordinate systems $P\text{-}x_1y_1z_1$ and $P\text{-}x_2y_2z_2$ are used to describe Euler angles.

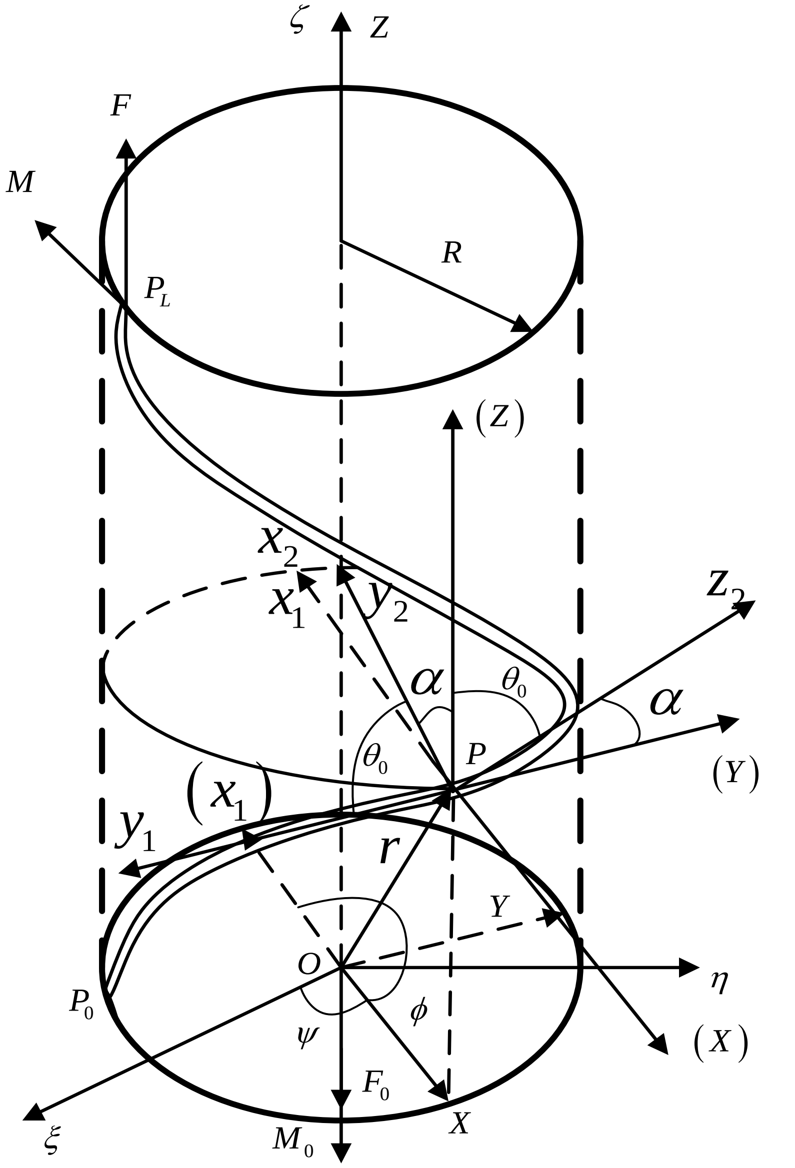

Figure 2. Helical rod

For a helix, the polar radius ρ and the nutation angle θ should be constants. Therefore, the rotation angle between P-$x_1y_1z_1$ and O-XYZ is π, directions of axis x_1 and X are the same. After a rotation angle along θ the axis, P-$x_1y_1z_1$ becomes P-$x_2y_2z_2$ and coincides with the Frenet coordinates.

Because of $d\rho/ds=0$, after considering the relationship between the cylindrical coordinates and the Cartesian coordinates, the following equations (4) could be gained/derived:

$$\begin{cases} \psi = \phi + \pi \\ \rho = R \\ \theta = \theta_0 = \pi/2 - \alpha \\ \dfrac{d\phi}{ds} = \omega_0 = \dfrac{\sin\theta_0}{R} \end{cases} \quad (4)$$

Equations (2) and (4) considered simultaneously, t could be derived reflecting the relationship geometry configuration and the end force and moment as follows:

$$\begin{cases} \rho = R = \dfrac{2}{p}\sqrt{a - p\cos\theta_0} \\ \psi = \psi_0 + \left[1/2 - \dfrac{al - mp}{2(a - p\gamma_0)} \right] s, \\ \zeta = s(\cos\theta_0) \end{cases} \quad (5)$$

where ψ_0 and ζ_0 are cylindrical coordinates at the beginning of the helix.

From equations (5), it is known that if we wanted to do simulation of deformation of helical rods, we should know how to compute the four integral constants decided/determined by the external fore and moment. However, we could only get two nonlinear equations from the equations (5), which is obviously not enough to solve four variables. So, we should give more conditions.

Fortunately, for helical rods, $f(\gamma)$ and $f'(\gamma)$ should be zero if we want the variable θ be a constant. In summary, four equations for the four variables a, l, m, p are derived; and the conclusion is that as long as the external force and moment are given, the geometry configurations of helical rods are uniquely determined.

4 Simulation results

To verify correctness of the configuration equations of helical rods, a simulation platform is established on the basis of virtual software named Virtools [11] in C++ environments, and the program flow chart is shown in Fig. 3.

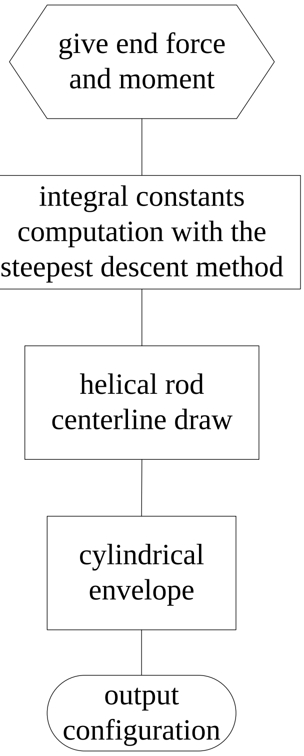

Figure 3. Flow chart of simulation program

Firstly, the cable material constants and end constraints are given, which are shown in Table 1.

Table.1 Deformable simulation parameters

Parameter name	Value
End external force, N	47.5
End external torque, N. m^2	9.5×10^{-3}
Young's module, N/m^2	3.89×10^{9}
Passion ratio	0.25
Diameter, cm	0.5
Length, mm	40
Initial helical angle	Arccos0.2

The whole dynamic deformation process of a helical rod is described as follows. First, geometry parameters of initial configuration of the helical rod are given. Then, with the external force change, the program real-time computes the four integral constants with the steepest descent method so as to get the formulation of centerline of the helical rod.

Finally, functions applied by Virtools are used to draw the centerline and to do cylindrical envelope, gaining the real-time configuration of the helical rod. The results are shown in Fig. 4. It can be seen that the helical radius becomes smaller while the helical angle becomes larger with an increase in external force, and the limiting case is a straight rod.

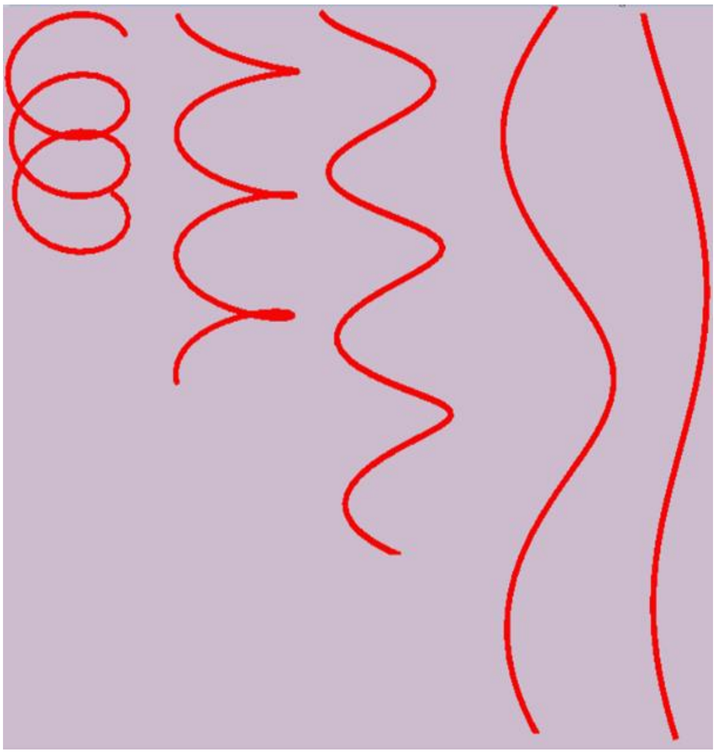

Figure 4. Helical rod deformable process

5 Conclusion

In this paper, Euler angles are used to describe static force equilibrium equations. Mathematical formulations with the form of Elliptic functions expressing two-end constraint elastic rod configuration are developed in a cylindrical coordinate system, which is established on the basis of Saint-Venant Principle of static equivalent forces. For helical rods, configuration expressions are gained by qualitatively analyzing the relationship between geometric parameters and end constraint. A simulation platform is developed in a virtual environment to verify the deformation process under changing external force.

6 Acknowledgements

The work in this paper is supported by the National Natural Science Foundation of China (51175053) and the Fundamental Research Funds for the Central Universities of China (2011QN048, 2012QN049 and 2012TD011) and the Fundamental Research Funds for the Central Universities (3132013311).

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