The property of Kelley in nonmetric continua

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Abstract. *The main purpose of this paper is to study the property of Kelley in nonmetric continua using inverse systems and limits.*

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1. Introduction

For any topological space X the set of all closed subsets of X is denoted by 2^X. The Vietoris topology on 2^X is the topology with a base

$$\langle U_1,\ U_2,\ ...,\ U_n\rangle\ =\ \left\{F\ :\ F\in 2^X,\ F\subset\bigcup_{i=1}^{n} U_i\ ,F\bigcap U_i\neq\emptyset,\ \ i\ =\ 1,\ ...,\ n\ \right\},$$

where U_1, ..., U_n are open subsets of X [5, p. 162]. If f : X→Y is a continuous mapping, then we define a mapping $2^f : 2^X \to 2^Y$ by $2^f(F) = Cl_Y(f(F))$, $F\in 2^X$. If X is compact, then $2^f(F) = f(F)$, $F\in 2^X$. For a continuum X, C(X) will denote the subspace of 2^X of all subcontinua of X. If f : X→Y is a continuous mapping, then c(f) will denote the restriction $2^f|C(X)$: C(X)→C(Y). Similarly, $C^2(X)$ will denote C(C(X)) and $c^2(f)$ will denote c(c(f)) for a mapping f : X→Y.

We say that $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is *σ-directed* if for each sequence $a_1,a_2,...,a_k,...$ of members of A, there is an a∈A such that $a\geq a_k$, for each $k\in\mathbb{N}$. For each directed set (A,≤) we define

$$A_\sigma = \{\Delta : \emptyset\neq\Delta\subset A,\ card(\Delta)\leq\aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Then A_σ is σ - directed by inclusion [14, Lemma 9.3]. If $\Delta\in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b,\ p_{bb'},\ \Delta\}$ and $X_\Delta = \lim\mathbf{X}^\Delta$. If Δ, $\Gamma\in A_\sigma$ and $\Delta\subseteq\Gamma$, let $p_{\Delta\Gamma}$: $X_\Gamma\to X_\Delta$ denote the map induced by the projections p^Γ_δ: $X_\Gamma\to X_\delta$, $\delta\in\Delta$, of the inverse system X^Γ. Now, we have the following theorem.

Theorem 1. [14, Theorem 9.4] *If* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *is an inverse system, then* $\mathbf{X}_\sigma = \{X_\Delta,\ p_{\Delta\Gamma},\ A_\sigma\}$ *is a σ - directed inverse system and* $\lim\mathbf{X}$ *and* $\lim\mathbf{X}_\sigma$ *are canonically homeomorphic.*

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In the sequel we will need the following theorem.

Theorem 2. *Let X be a compact space. There exists a σ - directed inverse system* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *of compact metric spaces* X_a *and surjective bonding mappings* p_{ab} *such that X is homeomorphic to lim*$\mathbf{X}$.

Proof. Apply [12, pp. 152 , 164] and *Theorem 1.* □

Theorem 3. [12, p. 163, Theorem 2.] *If X is a locally connected compact space, then there exists an inverse system* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *such that each* X_a *is a metric locally connected compact space, each* p_{ab} *is a monotone surjection and X is homeomorphic to lim*$\mathbf{X}$. *Conversely, the inverse limit of such a system is always a locally connected compact space.*

Theorem 4. [7, Corollary 3] *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be a σ - directed inverse system of hereditarily locally connected continua* X_a. *Then* $X = lim\mathbf{X}$ *is hereditarily locally connected.*

2. The property of Kelley

The following definition is well known [13, p. 538].

Definition 1. *A metric continuum X is said to have the property of Kelley provided that given any* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that if* $a,b \in X$, $d(a,b) < \delta$ *and* $a \in A \in C(X)$, *then there exists* $B \in C(X)$ *such that* $b \in B$ *and* $H(A,B) < \epsilon$.

Remark 1. *Each locally connected metric continuum has the property of Kelley [13, Example (6.11), p. 538].*

In nonmetric continua we shall use the following generalization of the property of Kelley [2].

Definition 2. *A continuum X has the property of Kelley if for each a in X, each A in C(X) such that* $a \in A$, *and each open V in C(X) containing A there exists an open set W such that* $a \in W$ *and if* $b \in W$, *then there exists a* $B \in C(X)$ *such that* $b \in B$ *and* $B \in V$.

A continuum X has the *property of Kelley hereditarily* provided that each one of its subcontinua has the property of Kelley.

3. The mapping α_X

Definition 3. *Let X be a continuum. For each* $a \in X$, *let* $\alpha_X(a) = \{A \in C(X),\ a \in A\}$.

If X is a metric continuum, then α_X(a) is a continuum of C(X) [19, p. 292]. This implies that we have a mapping $\alpha_X : X \to C^2(X)$.

Lemma 1. *If X is a nonmetric continuum, then* $\alpha_X(x)$ *is a continuum (in C(X)) for each* $x \in X$, *and there exists a mapping* $\alpha_X : X \to C^2(X)$.

Proof. By virtue of *Theorems 1.* and *3.* there exists a σ - directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric continua X_a such that X is homeomorphic to lim$\mathbf{X}$. Each $\alpha_{X_a}(p_a(x))$ is a continuum of $C(X_a)$ [19, p. 292]. It is clear that

$$c(p_{ab})[\alpha_{X_b}(p_b(x))] \subseteq \alpha_{X_a}(p_a(x)) \tag{1}$$

since for each continuum K of X_b containing $p_b(x)$ the image $p_{ab}(K)$ is a subcontinuum of X_a which contains $p_a(x)$. Consider the inverse system $C(\mathbf{X}) = \{C(X_a), c(p_{ab}), A\}$. From [5, p. 465, 6.3.22.(f)] it follows that C(X) is homeomorphic to lim$C(\mathbf{X})$. Now, $\mathbf{Y} = \{\alpha_{X_a}(p_a(x)),\ c(p_{ab})|\alpha_{X_b}(p_b(x)),\ A\}$ is an inverse system of continua. Moreover, C = lim$\mathbf{Y}$ is a subcontinuum of lim$C(\mathbf{X})$ [5, Theorem 6.1.18.], i.e. C is a subcontinuum of C(X). It remains to prove that $C = \alpha_X(x)$. Let us prove

$$C \subseteq \alpha_X(x). \tag{2}$$

Each point c of C is a thread (C_a) in $\mathbf{Y}$, where C_a is a subcontinuum of X_a such that $p_{ab}(C_b) = C_a$, a≤b. We have again an inverse system of continua whose limit K is a subcontinuum of X containing the point x. This means that $K \in \alpha_X(x)$. Hence, (2) is proved. Arguing similarly one can prove the reverse inclusion

$$C \supseteq \alpha_X(x). \tag{3}$$

We infer that $\alpha_X(x) \in C^2(X)$. Thus, we have the mapping $\alpha_X : X \to C^2(X)$. □

A mapping F assigning to each point y of a topological space Y a closed subset F(y) of a topological space X is *lower (upper) semicontinuous* if for every open set U⊆X the set {y: F(y)⋂U≠ ∅} (the set {y: F(y)⊆U}) is open in Y ([5, p. 89, 1.7.17.], [9, p. 181]).

We infer that a mapping $f : X \to 2^X$ is upper semicontinuous at a point x∈X if for every open set V (in 2^X) containing f(x) there is an open set U containing x such that f(y)⊂V, for all y∈U.

Lemma 2. *The mapping $\alpha_X : X \to C^2(X)$ is upper semicontinuous at each x∈X.*

Proof. Suppose that α_X is not upper semicontinuous at some point a∈X. This means that there exists a neighborhood V of α_X(a) (in C(X)) such that for each neighborhood U_μ, $\mu \in M$, of a there exists a point $b_\mu \in U_\mu$ such that $\alpha_X(b_\mu) \not\subseteq V$. Hence, for each $\mu \in M$ there exists a point $B_\mu \in \alpha_X(b_\mu) \subseteq C(X)$ such that $B_\mu \in C(X)\setminus V$. By compactness of C(X), the net $\{B_\mu : \mu \in M\}$ has a subnet $\{B_{\mu_\nu} : \mu_\nu \in M\}$ which converge to some point $B \in C(X)\setminus V$ [8, p. 71, Theorem 6; p. 136, Theorem 2]. Let us recall that B_μ is a subcontinuum of X which contains b_μ. Since $b_{\mu_\nu} \in B_{\mu_\nu}$ for each ν, and $\{b_\mu : \mu \in M\}$ converges to a, we have a∈B. This means that $B \in \alpha_X(a)$. This is impossible since $B \in C(X)\setminus V$ and $\alpha_X(a) \subseteq V$. □

The proof of the next theorem is the same as the proof of Theorem 2.2 in [19].

Theorem 5. *The mapping $\alpha_X : X \to C^2(X)$ is continuous if and only if X has the property of Kelley.*

4. Confluent mappings

A mapping $f : X \to Y$ is *confluent* provided every component of the inverse image $f^{-1}(C)$ of a continuum C⊆Y is mapped onto C. Each monotone mapping is confluent.

In the sequel we shall frequently consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow \alpha_X & & \Big\downarrow \alpha_Y \\ C^2(X) & \xrightarrow{c^2(f)} & C^2(Y) \end{array} \tag{4}$$

The diagram (4) commutes if and only if

$$f(\alpha_X(x)) = \alpha_Y(f(x)).$$

From the continuity of f there follows the following relation.

$$\{c(f)(A) : A \in \alpha_X(x)\} \subseteq \{B : B \in \alpha_Y(f(x))\}.$$

Lemma 3. *The diagram (4) commutes if and only if given* $B \in \alpha_Y(f(x))$ *there exists* $A \in \alpha_X(x)$ *such that* $c(f)(A) = B$.

Lemma 4. *The diagram (4) commutes if and only if* $f : X \rightarrow Y$ *is a confluent mapping.*

Proof. See the proof of [19, Theorem 4.2]. □

Theorem 6. [19, Theorem 4.3] *Let* f*:* $X \rightarrow Y$ *be a confluent mapping. If* X *has the property of Kelley, then* Y *has the property of Kelley.*

Corollary 1. *Let* f*:* $X \rightarrow Y$ *be a monotone mapping. If* X *has the property of Kelley, then* Y *has the property of Kelley.*

Corollary 2. *Let* f*:* $X \rightarrow Y$ *be an open mapping. If* X *has the property of Kelley, then* Y *has the property of Kelley.*

Corollary 3. *If a product space has the property of Kelley, then each factor space has the property of Kelley.*

Now we consider inverse systems of continua with the property of Kelley.

Lemma 5. [3, Corollary 4] *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continua and confluent bonding mappings. Then the projections* p_a*,* $a \in A$*, are confluent.*

Theorem 7. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continua and confluent bonding mappings. If each* X_a *has the property of Kelley, then* $X = lim\mathbf{X}$ *has the property of Kelley.*

Proof. By virtue of [5, Theorem 6.1.18] X = lim**X** is a continuum. Let x be any point of X and K any subcontinuum of X such that x∈K. Let $V = < V_1, V_2, ..., V_n >$ be any basis neighborhood of K in C(X). It remains to prove that there exists a neighborhood W of x such that for each point y∈W there exists a continuum L such that y∈L and L∈V in C(X). The proof is broken into several steps.

Step 1. *For every neighborhood U of x there exists an a∈A and an open set U_a containing $p_a(x) = x_a$ such that $x \in p_a^{-1}(U_a) \subset U$.* This follows from the definition of the basis of X.

Step 2. *For each open set $W \subset X$ and each a∈A there exists an open set $W_a \subset X_a$ such that $W = \cup\{p_a^{-1}(W_a) : a \in A\}$.* For each a∈A we define $W_a = \cup\{O_a : O_a$ to be open in X_a and $p_a^{-1}(O_a) \subset W\}$. Then W_a is open in X_a. It is clear that $W \supset \cup\{p_a^{-1}(W_a) : a \in A\}$. On the other hand, if x∈W, then by Step 1., there exists an a∈A and an open set W_a^* containing $p_a(x) = x_a$ such that $x \in p_a^{-1}(W_a^*) \subset W$. It is clear that $W_a^* \subset W_a$. Hence $x \in p_a^{-1}(W_a)$. This means that $W \subset \cup\{W_a : a \in A\}$. From this relation and $W \supset \cup\{p_a^{-1}(W_a) : a \in A\}$ we infer that $W = \cup\{p_a^{-1}(W_a) : a \in A\}$.

Step 3. *For the neighborhood $V = < V_1, V_2, ..., V_n >$ of K (in C(X)) there exists a b∈A such that for each c∈A, c≥b, there exist open subsets $V_1(c), ..., V_n(c)$ of X_c such that $< V_1(c), ..., V_n(c) >$ is an open set in $C(X_c)$ containing $p_c(K)$.* By virtue of Step 2. for each set V_i, i = 1, ..., n, and each a∈A there exists an open set $V_i(a)$, i = 1, ..., n, such that $V_i = \cup\{p_a^{-1}(V_i(a)) : a \in A\}$, i = 1, ..., n. The family $\{p_a^{-1}(V_i(a)) \cap K : a \in A,\ i = 1, ..., n\}$ is an open cover of K. By virtue of the compactness of K there exists a finite subcover $\{p_{a_1}^{-1}(V_i(a_1)), ..., p_{a_k}^{-1}(V_i(a_k))\}$ of $\{p_a^{-1}(V_i(a)) \cap K : a \in A,\ i = 1, ..., n\}$. There exists $b \geq a_i$, i = 1, ..., n, since A is directed. Now, for each c≥b we have a finite family of sets $V_i(c)$, i = 1, ..., n, such that $K \subset \cup\{p_c^{-1}(V_i(c)) : i = 1, ..., n\}$. It is clear that $<V_1(c), ..., V_n(c)>$ is an open set in $C(X_c)$ containing $p_c(K)$. The proof of this Step is completed.

Step 4. *The space X = limX has the property of Kelley.* By virtue of Steps 1 - 3 one can obtain a d∈A and open sets $V_i(d)$, i = 1, ..., n, such that $K \subset \cup\{p_d^{-1}(V_i(d)) : i = 1, ..., n\} = V$. For the continuum $K_d = p_d(K)$ which contains $p_d(x) = x_d$ and for $<V_1(d), ..., V_n(d)>$ there exists a neighborhood W_d of x_d such that if $y_d \in W_d$, then there exists a continuum L_d containing y_d and such that $L_d \in <V_1(c), ..., V_n(c)>$ (since X_d has the property of Kelley). Let $W = p_d^{-1}(W_d)$. It is clear that W is a neighborhood of x. If y is any point of W, then $p_d(y) = y_d$ is in W_d. This means that there exists a continuum L_d containing y_d and such that $L_d \in <V_1(c), ..., V_n(c)>$. It is clear that $y \in p_d^{-1}(L_d)$. Let Q be a component of $p_d^{-1}(L_d)$ which contains y. By *Theorem 5.* it follows $p_d(Q) = L_d$. Let us prove that Q is in V (in C(X)). Suppose that $Q \notin V$ (in C(X)). Then there exists $p_d^{-1}(V_i(d))$ in $\{p_d^{-1}(V_i(d)) : i = 1, ..., n\}$ such that $p_d^{-1}(V_i(d)) \bigcap Q = \emptyset$. Hence $V_i(d) \bigcap p_d(Q) = \emptyset$. This is impossible since $p_d(Q) = L_d$, $L_d \subset \cup\{V_i(d) : i = 1, ..., n\}$ and $L \bigcap V_i(d) \neq \emptyset$, i = 1, ..., n. Finally, the required neighborhood W of x is defined. □

Remark 2. *For another proof of* Theorem 7. *for inverse sequence of metric continua see [1].*

We say that a mapping f : X→Y is *hereditarily confluent (monotone)* [11, p. 17] if f|K is confluent (monotone) for every subcontinuum K of X.

Theorem 8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continua and hereditarily confluent bonding mappings. If each X_a has the property of Kelley hereditarily, then X = lim***X** *has the property of Kelley hereditarily.*

Proof. Let K be any subcontinuum of X. For each a∈A we have the continuum $K_a = p_a(K)$. Each mapping $q_{ab} = p_{ab}|K_b$ is confluent. Now we have the inverse system $\mathbf{K} = \{K_a, q_{ab}, A\}$ of continua and confluent bonding mappings. Using *Theorem 7.* we complete the proof. □

Lemma 6. *If X is a continuum which has the property of Kelley hereditarily and if $f: X \to Y$ is confluent, then Y has the property of Kelley hereditarily.*

Proof. Let K be any subcontinuum of Y. Then $L = f^{-1}(K)$ is a subcontinuum of X which has the property of Kelley since X is hereditarily of Kelley. The restriction $g = f|L$ is confluent $g: L \to K$. Applying *Theorem 6.* we infer that K has the property of Kelley. The proof is completed.

Theorem 9. *Each locally connected continuum has the property of Kelley.*

Proof. If X is a metric locally connected continuum, then it has the property of Kelley [19]. Suppose that X is a nonmetric locally connected continuum. By virtue of *Theorems 1.* and *3.* there exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric spaces X_a such that X is homeomorphic to lim$\mathbf{X}$ and the bonding mappings p_{ab} are monotone surjection. Each X_a has the property of Kelley since each X_a is a locally connected metric continuum (as a continuous image of the locally connected continuum X). By virtue of *Theorem 7.* X has the property of Kelley. □

Any metric continuum has the property of Kelley at each point of a dense G_δ-set. It is natural to ask the following question.

QUESTION. Is it true that each nonmetric continuum has the property of Kelley at some point?

A continuum X is said to be *hereditarily locally connected* if each subcontinuum of X is locally connected.

Theorem 10. (see [6]) *A metric continuum X is hereditarily locally connected if and only if X has the property of Kelly hereditarily and is hereditarily arcwise connected.*

Now we shall prove the following generalization of this Theorem.

Theorem 11. *A locally connected nonmetric continuum X is hereditarily locally connected if and only if X has the property of Kelley hereditarily and is hereditarily arcwise connected.*

Proof. a) Let X be a locally connected continuum X which is hereditarily Kelley and hereditarily arcwise connected, i.e. for every subcontinuum K of X and for every pair x,y of points of K there exists an arc L (possibly nonmetric) such that x and y are end-points of L and $L \subseteq K$. Let us prove that X is hereditarily locally connected. By virtue of *Theorems 1.* and *3.* there exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected continuum, each p_{ab} is monotone and X is homeomorphic to lim$\mathbf{X}$. Let us prove that each X_a is hereditarily locally connected. By virtue of *Theorem 10.* it suffices to prove that X_a is hereditarily of Kelley and hereditarily arcwise connected. Let us prove that X_a is hereditarily of Kelley. It is known that the projections p_a are monotone and, consequently, confluent. From *Lemma 6.* it follows that X_a is hereditarily of Kelley. Let us prove that X_a is hereditarily arcwise connected. Let K_a be a subcontinuum of X_a. Let x_a,y_a be a pair of the points of K_a. Now $p_a^{-1}(K_a) = K$ is a subcontinuum of X since p_a is monotone. There exists a pair x,y of the points in K such that $x_a = p_a(x)$ and $y_a = p_a(y)$. There exists an arc L such that x,y∈L and L⊂K. It is clear that $x_a, y_a \in p_a(L) \subset K_a$. Moreover, $p_a(L)$ is arcwise connected as a continuous image of the arc L [17, Theorem 9.]. Hence X_a is hereditarily locally connected. By virtue of *Theorem 4.*, X is hereditarily locally connected.

b) Let X be hereditarily locally connected. Now, X is hereditarily arcwise connected since X is a continuous image of an arc (see [15] and [17]). In order to complete the proof it remains to prove that X is hereditarily of Kelley. Let K be any subcontinuum of X. Clearly K is hereditarily locally connected. By *Theorem 9.* we infer that K has the property of Kelley. □

Theorem 12. *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be an inverse system of hereditarily locally connected continua and hereditarily confluent bonding mappings. Then* $X = lim\mathbf{X}$ *is hereditarily locally connected if and only if X is hereditarily arcwise connected.*

Proof. If X is hereditarily locally connected, then X is hereditarily arcwise connected as a continuous image of an arc (see [15] and [17]). Conversely, if X is hereditarily arcwise connected, then X is hereditarily locally connected since it is hereditarily of Kelley (*Theorem 11.*). □

5. The property of Kelley and smoothness of continua

A continuous mapping f : X→Y is said to be *monotone relative to a point p∈X* if for each subcontinuum Q of Y such that f(p)∈Q the inverse image $f^{-1}(Q)$ is connected [1, p. 184].

Theorem 13. [3, Theorem 1.] *Let* $\mathbf{X} = \{X^\lambda,\ f^{\lambda\mu},\ \Lambda\}$ *be an inverse system with the limit X. If there exists a thread* $p = \{p^\lambda\}$ *such that for each* $\lambda \in \Lambda$ *with* $\alpha \leq \lambda$ *the bonding mapping* $f^{\alpha\lambda}$ *is monotone relative to* p^λ*, then the projection* π^α *is monotone relative to p.*

We say that a Hausdorff continuum X is *smooth at the point p∈X* [16](for metric continua [10, p. 81]) if for each convergent net $\{x_n,\ n\in D\}$ of points of X and for each subcontinuum K of X such that p,x∈K, where $x = \lim\{x_n : n \in D\}$, there exists a net $\{K_i, i\in E\}$ of subcontinua of X such that each K_i contains p and some x_n and $\mathrm{Lim}K_n = K$.

Theorem 14. [16, Proposition 1.] *Let p be an arbitrary point of a continuum X. The following are equivalent:*

(i) *X is smooth at p,*

(ii) *for each subcontinuum N of X such that p∈N and for each open set V which contains N there exists an open connected set K such that N⊆K⊆V.*

A continuum X is locally connected at the point p if X is smooth at the point p.

Lemma 7. [10, Corollary 3.4] *A continuum X is locally connected if and only if it is smooth at each of its points.*

The proof of the following theorem is similar to the proof of *Theorem 7.*

Theorem 15. *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be an inverse system of continua and monotone projections* $p_a : lim\mathbf{X} \to X_a$. *If each* X_a *is smooth, then* $X = lim\mathbf{X}$ *is smooth.*

Proof. A straightforward modification of the proof of *Theorem 7.* using (ii) of *Theorem 14.* instead of the definition of the property of Kelley. □

Let p be a fixed point of a continuum X. For each point x∈X consider the family {K : K∈C(X), p,x∈K} of all subcontinua K of X containing both p and x [1, p. 185]. We define

$$F[X,p](x) = \{K : K \in C(X), \quad p,x \in K\}.$$

If X is a metric continuum, then for each x∈X F[X,p](x) is a compact and arcwise connected subset of C(X), i.e. F[X,p](x) is an element of $C^2(X)$ [1, p. 185]. If X is a nonmetric continuum, then (as in the proof of *Theorem 1.*) one can prove that F[X,p](x) is a continuum, i.e. F[X,p](x) is an element of $C^2(X)$ Thus, we have the mapping F[X,p](x) : $X \to C^2(X)$.

Lemma 8. *The mapping F[X,p] is continuous if and only if the continuum X is smooth at the point p.*

Proof. The proof is the same as the proof of Proposition 2. of [1]. □

Lemma 9. *Let a continuous surjection* $f : X \to Y$ *and points* $p \in X$ *and* $q \in Y$ *with* $q = f(p)$ *be given. If* $F_1 = F[X,p]$ *and* $F_2 = F[Y,q]$, *then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow F_1 & & \Big\downarrow F_2 \\ C^2(X) & \xrightarrow{c^2(f)} & C^2(Y) \end{array} \qquad (5)$$

commutes if and only if f is monotone relative to p.

Proof. The proof is the same as the proof of Proposition 3. of [1]. □

The following Theorem for inverse sequences of metric continua was proved in the paper [1].

Theorem 16. [1, Theorem 1.] *Let* $\{X^i, f^i\}_{i=1}^{\infty}$ *be an inverse sequence such that, for each i=1,2,... (a) the continuum* X^i *is smooth at a point* p^i*;(b)* $f^i(p^{i+1}) = p^i$*;(c)* f^i *is monotone relative to* p^{i+1}*.Then the inverse limit continuum* $X = lim\{X^i, f^i\}$ *is smooth at the thread* $p = \{p^i\}_{i=1}^{\infty}$.

The proof given there works in the general situation. Hence, we have the following theorem.

Theorem 17. *Let* $\mathbf{X} = \{X_a, f_{ab}, A\}$ *be an inverse system such that, for each* $a \in A$*:*

1. *The continuum* X_a *is smooth at a point* p_a*;*
2. $p = (p_a : a \in A)$ *is a thread;*
3. f_{ab} *is monotone at* p_b.

Then the inverse limit $X = \lim\mathbf{X}$ is smooth at the thread p.

A topological space X is said to be *rim-metrizable* if X admits a basis of open sets whose boundaries are metrizable.

Theorem 18. *Let X be a rim-metrizable continuum. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X_a, $a \in A$, are metric continua, p_{ab} are monotone surjections and X is homeomorphic to $\lim\mathbf{X}$.*

Proof. By *Theorem 2.* there exists an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ of metric spaces Y_a such that X is homeomorphic to $\lim\mathbf{Y}$. Let $q_a : X \to Y_a$ be the natural projection, $a \in A$. By the monotone-light factorization of q_a we obtain a space X_a, a monotone mapping $q'_a : X \to X_a$ and a light mapping $q''_a : Xa \to Y_a$ such that $q_a = q''_a q'_a$. Moreover, by Lemma 8. of [12], for each a,b∈A, there exists a monotone mapping $p_{ab} : X_b \to X_a$. Hence, we have an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$. It is readily seen that X is homeomorphic to $\lim\mathbf{X}$. Moreover, by Theorem 3.2 of [18] it follows that X_a is rim-metrizable. Finally, by Theorem 1.2. of [18] we have $w(X_a) = w(Y_a) = \aleph_0$, i.e. X_a is a metric continuum. □

A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. If a metric dendroid has the property of Kelley, then it is smooth [4, Corollary 5.].

Theorem 19. *If a rim-metrizable dendroid X has the property of Kelley, then X is smooth.*

Proof. By *Theorem 18.* there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a and monotone bonding mappings p_{ab} such that X is homeomorphic to $\lim\mathbf{X}$. Every X_a is hereditarily unicoherent since p_{ab} are monotone surjections. Moreover, every X_a is arcwise connected. Namely, let x_a,y_a be a pair of points of X_a. There exists a pair x,y of points in X such that $x_a = p_a(x)$ and $y_a = p_a(y)$. There exists an arc L such that x,y∈L and L⊂X. It is clear that $x_a, y_a \in p_a(L) \subset X_a$. Moreover, $p_a(L)$ is arcwise connected as a continuous image of the arc L [17, Theorem 9.]. Hence X_a is arcwise connected. By *Theorem 6.* X_a has the property of Kelley. From [4, Corollary 5.] it follows that every X_a is smooth. Using *Theorem 17.* we infer that X is smooth. □

From the proof of *Theorem 19.* the following theorem follows.

Theorem 20. *Every rim-metrizable nonmetric dendroid is homeomorphic to the inverse limit of a σ-directed inverse system of metric dendroids.*

QUESTION 2. Is it true that every dendroid with the property of Kelley is smooth?

QUESTION 3. Is every nonmetric dendroid homeomorphic to the limit of an inverse system of metric dendroids (to the limit of a σ-directed inverse system of metric dendroids)?

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