Surjective simplicial inverse systems

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Abstract. *Every topologically complete space embeds as a deformation retract in a topologically complete space which is the limit of a polyhedral inverse system with surjective and simplicial (fixed triangulations) bonding mappings. Moreover, the corresponding homotopy category and its full subcategory are equivalent. The same also holds for several subclasses of the class of all topologically complete spaces: paracompact spaces, Lindelöf spaces, countably compact spaces, strongly paracompact spaces, paracompact (σ-compact) locally compact spaces, compact Hausdorff spaces.*

Key words: *Topologically complete space, (strongly) paracompact (locally compact) space, Lindelöf space, normal covering, nerve, polyhedron, simplicial mapping, canonical mapping, proper mapping, inverse system, limit, resolution.*

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1. Introduction

In 1937 H. Freudenthal [7] proved that every compact metrizable space is the limit of an inverse system (sequence) of compact polyhedra with *surjective* and *simplicial* bonding mappings. On the other side, a well-known example of B. Pasynkov [12] shows that in general, a compact Hausdorff space need *not* be the limit of any surjective polyhedral inverse system. The authors recently showed [3] how to associate, with a topological space X, a polyhedral resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ which is "close" to a surjective and simplicial one. Each projection mapping $p_a : X \to X_a$ in such a resolution is strictly canonical. More precisely, $X_a = |N(\mathcal{U}_a)|$ is the geometric nerve of a normal covering $\mathcal{U}_a \in \mathrm{Cov}(X)$ and the carrier $|p_a(X)| = X_a$. Moreover, all the bonding mappings $p_{aa'} : X_{a'} \to X_a$ are PL, and in some special cases they are simplicial and proper ([3] and [4]).

The problem of surjective and simplicial inverse limits and resolutions of spaces can be also treated in another way. Namely, since a space generally does not admit a surjective and simplicial inverse limit (resolution) development, one may ask if

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there exists a closely related space which does admit such a nice development. In the paper of S.A. Saneblidze [14] one can find a brief sketch of such a construction in the case of a paracompact space. He showed how to associate with a paracompact space X another paracompact space $N(X)$ of the same homotopy type, which is the limit of an inverse system of polyhedra with surjective and simplicial bonding mappings. The basic step in doing this is a construction due to M. D. Alder [1], P. Bacon [2] (Theorem 3.2) and S. Mardešić [9] (Theorem 11).

In this paper we generalize the result of Saneblidze to the class of all topologically complete spaces including a categorical viewpoint. Moreover, we exhibit analogous facts for certain subclasses of topologically complete spaces. The main results can be summarized as follows:
(a) The homotopy category **HCTOP** of topologically complete spaces is equivalent to its full subcategory whose objects are limits of polyhedral inverse systems with surjective and simplicial (with respect to fixed triangulations of the terms) bonding mappings.
(b) Completely analogous results hold for the homotopy categories whose objects are (respectively) all:

(i) paracompact spaces;

(ii) Lindelöf spaces;

(iii) countably compact spaces;

(iv) strongly paracompact spaces;

(v) paracompact locally compact spaces;

(vi) σ-compact locally compact spaces;

(vii) compact Hausdorff spaces.

Moreover, in the cases (iv)-(vii) all bonding mappings in these inverse systems and all projection mappings of the corresponding limits are proper, while in the case (vii) the limit is also a resolution.

Let us recall some indispensable notions, notations and facts. By a space we mean a topological space, and by a mapping a continuous function. The corresponding category is denoted by **TOP.** In this paper we restrict our attention to the class (and the category) **CTOP** of all topologically complete spaces. These are spaces which admit a complete uniform structure. Every paracompact space is topologically complete and every topologically complete space is completely regular (Tychonoff). By $\mathrm{Cov}(X)$ we denote the set of all normal, i.e. numerable coverings of a space X. These are open coverings which admit a subordinate partition of unity. Every open covering of a paracompact space is normal.

POL denotes the class of all polyhedra, i.e. triangulable spaces (CW-topology). If $P = |K|$, where K is a simplicial complex, and $x \in P$, then $\mathrm{st}(x, K) \subseteq P$ denotes the corresponding open star, i.e. the union of all open simplexes of K such that the corresponding closed simplexes contain x. A mapping $f : P \to Q$ of polyhedra is *simplicial* (PL) provided there exist triangulations K and L of P and Q respectively, such that $f : |K| \to |L|$ maps every closed simplex of K linearly onto (into) a closed simplex of L (see [3]).

If $\mathbf{C} \subseteq \mathbf{TOP}$ is a subcategory, then $\mathbf{HC}$ denotes the corresponding homotopy category. Finally, some basic definitions and facts on inverse systems, limits and resolutions can be found in [10].

2. Limit of geometric nerves and the functor N

Following Mardešić [9] and Saneblidze [14], let us show how to associate with each mapping $f : X \to Y$ of topological spaces a mapping $N(f) : N(X) \to N(Y)$ of topologically complete spaces, where $N(X)$ and $N(Y)$ are limits of polyhedral inverse system $\boldsymbol{N}(X) = (X_a, p_{aa'}, A)$ and $\boldsymbol{N}(Y) = (Y_b, q_{bb'}, B)$ respectively with surjective and simplicial (with respect to fixed triangulations) bonding mappings $p_{aa'}$ and $q_{bb'}$. Moreover, the construction $f \mapsto N(f)$ will yield a functor $N : \mathbf{TOP} \to \mathbf{CTOP}$. For the sake of completeness and in order to introduce a notation needed in the sequel, we repeat the first part of the proof of [9], Theorem 11.

Let $f : X \to Y$ be a mapping. For every normal covering $\mathcal{V} \in \mathrm{Cov}(Y)$ choose a locally finite partition of unity $(\psi_V, V \in \mathcal{V})$ subordinated to $\mathcal{V}$. Let $|N(\mathcal{V})|$ be the geometric nerve of $\mathcal{V}$, and let $t_\psi : Y \to |N(\mathcal{V})|$ be the canonical mapping of the partition $(\psi_V, V \in \mathcal{V})$, i.e. if $y \in Y$, then the point $t_\psi(y) \in |N(\mathcal{V})|$ has $\psi_V(y)$ as the barycentric coordinate with respect to $V \in \mathcal{V}$. For every $\mathcal{V} \in \mathrm{Cov}(Y)$ consider the corresponding normal covering $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}, f^{-1}(V) \neq \emptyset\} \in \mathrm{Cov}(X)$ and the locally finite partition of unity $(\varphi_V, V \in \mathcal{V})$ defined by $\varphi_V = \psi_V f$. It determines a canonical mapping $s_{\psi f} : X \to |N(f^{-1}(\mathcal{V}))|$. Then there exists a simplicial mapping (embedding) $f_\psi : |N(f^{-1}(\mathcal{V}))| \to |N(\mathcal{V})|$ determined by its values on vertices, i.e. if $V \in \mathcal{V}$ and thus $V \in N(\mathcal{V})^0$, then $f^{-1}(V) \in N(f^{-1}(\mathcal{V}))^0$ and $f_\psi(f^{-1}(V)) = V$. Note that $f_\psi s_{\psi f} = t_\psi f$. For every normal covering $\mathcal{U} \in \mathrm{Cov}(X) \setminus f^{-1}(\mathrm{Cov}(Y))$ choose a locally finite partition of unity $(\varphi_U, U \in \mathcal{U})$ on X. It determines a canonical mapping $s_\varphi : X \to |N(\mathcal{U})|$. Let us denote $M = \mathrm{Cov}(Y)$, $\Lambda_0 = \mathrm{Cov}(X) \setminus f^{-1}(\mathrm{Cov}(Y))$ and $\Lambda = \Lambda_0 \cup M$. Let $M \hookrightarrow \Lambda$ be the inclusion function. The construction proceeds by considering the set $(B, \leq)$ of all finite subsets of M ordered by inclusion. If $b = \{\mathcal{V}_1, \cdots, \mathcal{V}_n\} \in B$, let N_b denote the nerve $N(\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_n)$ and let $Y_b = |N_b|$ be its geometric realization, where the covering $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_n$ consists of all nonempty intersections $V_1 \cap \cdots \cap V_n$, $V_i \in \mathcal{V}_i$, $i \in \{1, \cdots, n\}$. Let $t_b : Y \to Y_b$ be the canonical mapping determined by the partition of unity $(\psi_{(V_1,\cdots,V_n)}, (V_1, \cdots, V_n) \in \mathcal{V}_1 \times \cdots \times \mathcal{V}_n)$, where $\psi_{(V_1,\cdots,V_n)} = \psi^1_{V_1} \cdot \ldots \cdot \psi^n_{V_n}$, and $(\psi^i_{V_i}, V_i \in \mathcal{V}_i)$ are the already chosen partitions of unity. For every related pair $b = \{\mathcal{V}_1, \cdots, \mathcal{V}_n\} \leq \{\mathcal{V}_1, \cdots, \mathcal{V}_n, \cdots, \mathcal{V}_{n'}\} = b'$, let $q_{aa'} : Y_{b'} \to Y_b$ be the simplicial mapping determined by its values on vertices, i.e. $q_{bb'}$ sends the vertex $(V_1, \cdots, V_n, \cdots, V_{n'})$ of $N_{b'}$ to the vertex $(V_1, \cdots, V_n)$ of N_b. Obviously, $q_{bb'}$ is a surjection and one readily sees that $q_{bb'} q_{b'b''} = q_{bb''}$ and $q_{bb'} t_{b'} = t_b$ whenever $b \leq b' \leq b''$. Analogously, we define the ordered set $(A, \leq)$ over Λ, the nerves $N_a = N(\mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n)$ and polyhedra $X_a = |N_a|$, $a = \{\mathcal{U}_1, \cdots, \mathcal{U}_n\} \in A$, the simplicial mappings $p_{aa'} : X_{a'} \to X_a$, $a \leq a'$, and the canonical mappings $s_a : X \to X_a$, $a \in A$. (Of course, if an index $a \in A$ contains some $f^{-1}(\mathcal{V})$, then the mappings of the corresponding partition of unity have $\psi_V f$, $V \in \mathcal{V}$, as some factors.) Further, we extend $M \hookrightarrow \Lambda$ to the inclusion function $j : B \hookrightarrow A$. Since

$f^{-1}(\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_n) = f^{-1}(\mathcal{V}_1) \wedge \cdots \wedge f^{-1}(\mathcal{V}_n)$, we can define a simplicial mapping $f_b : X_{j(b)} \to Y_b$, $b = \{\mathcal{V}_1, \cdots, \mathcal{V}_n\} \in B$, by sending the vertex $(V_1, \cdots, V_n)$ of $N_{j(b)} = N(f^{-1}(\mathcal{V}_1) \wedge \cdots \wedge f^{-1}(\mathcal{V}_n))$ to the vertex $(V_1, \cdots, V_n)$ of $N_b = N(\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_n)$. Clearly, if $b \leq b'$, then $f_b p_{j(b),j(b')} = q_{bb'} f_{b'}$.

Thus we have obtained two polyhedral inverse systems $\boldsymbol{N}(X) = (X_a, p_{aa'}, A)$ and $\boldsymbol{N}(Y) = (Y_b, q_{bb'}, B)$ with surjective and simplicial (fixed triangulations on the terms) bonding mappings, and three maps of systems $\boldsymbol{s} \equiv \boldsymbol{s}_{\varphi\psi f} = (s_a) : X \to \boldsymbol{N}(X)$, $\boldsymbol{t} = \boldsymbol{t}_\psi = (t_b) : Y \to \boldsymbol{N}(Y)$ and $\boldsymbol{f} = (j, f_b) : \boldsymbol{N}(X) \to \boldsymbol{N}(Y)$, where all s_a and t_b are canonical mappings, while all f_b are simplicial embeddings. Moreover, $\boldsymbol{f}\boldsymbol{s} = \boldsymbol{t}f$.

Let $N(f) : N(X) \to N(Y)$ be the limit mapping $\lim \boldsymbol{f} : \lim \boldsymbol{N}(X) \to \lim \boldsymbol{N}(Y)$ and let $\boldsymbol{p} = (p_a) : N(X) \to \boldsymbol{N}(X)$ and $\boldsymbol{q} = (q_b) : N(Y) \to \boldsymbol{N}(Y)$ be the limits. Then, for every $b \in B$, $f_b p_{j(b)} = q_b N(f)$ holds. Note that $\boldsymbol{s} : X \to \boldsymbol{N}(X)$ determines a mapping $s \equiv s_{\varphi\psi f} : X \to N(X)$ by means of $s(x)_a = s_a(x)$, $x \in X$, $a \in A$, satisfying $p_a s = s_a$. Similarly, there exists a mapping $t \equiv t_\psi : Y \to N(Y)$ determined by $\boldsymbol{t} : Y \to \boldsymbol{N}(Y)$. Observe that $N(f)s = tf$ holds. Indeed, for every $b \in B$, $q_b N(f)s = f_b p_{j(b)} s = f_b s_{j(b)} = t_b f = q_b t f$.
Finally, recall that polyhedra are paracompact, hence, topologically complete. Since the class of all topologically complete spaces is closed under direct products and closed subsets, the spaces $N(X)$ and $N(Y)$ are topologically complete.

Let us show that our "N-construction" is functorial. First observe that the systems $\boldsymbol{N}(X)$ and $\boldsymbol{N}(Y)$ do not depend on the choice of locally finite partitions of unity of normal coverings of the spaces X and Y. (The canonical mappings $t_b : Y \to Y_b$, $b \in B$, and $s_a : X \to X_a$, $a \in A$, do depend on the choice of these partitions, and some s_a depend also on f; however, for any two choices , the corresponding mappings are contiguous.) Moreover, $\boldsymbol{N}(X)$ and $\boldsymbol{N}(Y)$ do not depend on the mapping $f : X \to Y$ appearing in the construction. (The mapping f determines a partition of the set $\mathrm{Cov}(X)$, yielding the corresponding inclusion function $j : B \to A$ of $\boldsymbol{f}$ on the indexing sets.) Furthermore, the identity mapping 1_X induces the identity map $\mathbf{1}_{\boldsymbol{N}(X)}$ of the system $\boldsymbol{N}(X)$, while $\boldsymbol{gf} = \boldsymbol{g}\boldsymbol{f} : \boldsymbol{N}(X) \to \boldsymbol{N}(Z)$ whenever $f : X \to Y$ and $g : Y \to Z$, because of $(gf)^{-1}(\mathcal{W}) = g^{-1}(f^{-1}(\mathcal{W}))$, $\mathcal{W} \in \mathrm{Cov}(Z)$. Therefore, the correspondence $X \mapsto \boldsymbol{N}(X)$, $f \mapsto \boldsymbol{f}$, determines a functor $\boldsymbol{N} : \mathbf{TOP} \to pro\text{-}\mathbf{POL}$. Since the inverse limit $\lim : pro\text{-}\mathbf{POL} \to \mathbf{CTOP}$ is a functor (the restriction functor of $\lim : pro\text{-}\mathbf{TOP} \to \mathbf{TOP}$), the composition $\lim \boldsymbol{N} = N : \mathbf{TOP} \to \mathbf{CTOP}$ is also a functor.

We intend to prove that in the case of a topologically complete space X, the space $N(X)$ is closely related to X. More precisely, in that case X embeds as a deformation retract in $N(X)$. First, we prove the main lemma:

Lemma 1. *Let X be a topologically complete space and let $s : X \to N(X)$ be the mapping obtained by the N-construction using an arbitrary mapping $f : X \to Y$ (or $g : Y \to X$). Then there exists a surjective and closed mapping $r \equiv r_s : N(X) \to X$ having compact fibers and satisfying $rs = 1_X$.*

Proof. Let $y = (y_a) \equiv (p_a(y)) \in N(X)$. For each $a = \{\mathcal{U}_1, \cdots, \mathcal{U}_n\} \in A$, choose a point $x_a \in s_a^{-1}(y_a) \subseteq X$ and consider the minimal (closed) simplex σ_{y_a} of N_a containing y_a. Then $\sigma_{y_a} = [W_a^0, \cdots, W_a^m]$ and $\bigcap_{i=0}^{m} W_a^i \equiv G_a \neq \emptyset$, where

$W_a^i \in \mathcal{W}_a \equiv \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n$ and $N_a = N(\mathcal{W}_a)$. Note that

$$s_a^{-1}(y_a) \subseteq s_a^{-1}(\mathrm{Int}(\sigma_{y_a})) = s_a^{-1}(\bigcap_{i=0}^{m} \mathrm{st}(W_a^i)) = \bigcap_{i=0}^{m} s_a^{-1}(\mathrm{st}(W_a^i)) \subseteq \bigcap_{i=0}^{m} W_a^i \equiv G_a.$$

Namely, since $s_a : X \to |N_a|$ is canonical, $s_a^{-1}(\mathrm{st}(W)) \subseteq W$ holds for every $W \in \mathcal{W}$. Let us show that $(x_a, a \in A)$ is a Cauchy net in X with respect to the uniformity generated by $\mathrm{Cov}(X)$. By the construction of $\boldsymbol{N}(X)$, if $a \leq a' = \{\mathcal{U}_1, \cdots, \mathcal{U}_n, \cdots, \mathcal{U}_{n'}\}$, then $s_{a'}^{-1}(y_{a'}) \subseteq s_a^{-1}(y_a)$, $\sigma_{y_{a'}} = [W_{a'}^0, \cdots, W_{a'}^m, \cdots, W_{a'}^{m'(y_{a'})}]$, $W_{a'}^j \in \mathcal{W}_{a'} \equiv \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n \wedge \cdots \wedge \mathcal{U}_{n'}$, $\bigcap_{j=0}^{m'} W_{a'}^j \equiv G_{a'} \subseteq G_a$ and $p_{aa'}(\sigma_{y_{a'}}) = \sigma_{y_a}$. To conclude that $(x_a, a \in A)$ is a Cauchy net in X, it suffices to prove the following fact:

($*$) $(\forall \mathcal{U} \in \mathrm{Cov}(X))(\exists a \in A)(\exists U \in \mathcal{U})(\forall a' \geq a) G_{a'} \subseteq U$.

Indeed, first choose an $a \in A$ such that $\mathcal{W}_a \leq \mathcal{U}$. Then consider any $W \in \mathcal{W}_a$ corresponding to a vertex of the simplex $\sigma_{y_a} = [W_a^0, \cdots, W_a^m]$. Finally, take a $U \in \mathcal{U}$ such that $W \subseteq U$. Consequently, if $a' \geq a$, then $G_{a'} \subseteq G_a \subseteq W \subseteq U$.
Since X is topologically complete, it is complete with respect to the (finest) uniform structure given by the set of all normal coverings of X. Therefore, the net $(x_a, a \in A)$ converges to a unique point $x \in X$. Moreover, by ($*$), the point x is uniquely determined by the point y, i.e. it does not depend on the particular choice of $x_a \in s_a^{-1}(y_a)$, $a \in A$. (In other words, $\{s_a^{-1}(y_a) \mid a \in A\}$ is a Cauchy family in X with respect to the uniformity, and $\{x\} = \cap\{s_a^{-1}(y_a) \mid a \in A\}$.)

Let us define $r : N(X) \to X$ by putting $r(y) = x = \lim(x_a, a \in A)$.

Let us prove that $rs = 1_X$. If $x \in X$, let $y = s(x)$. Then $(rs)(x) = r(s(x)) = r(y) = x$. Indeed, $x \in s_a^{-1}(s(x)_a)$ for all $a \in A$, because of $p_a s = s_a$, $a \in A$. Hence, a particular choice of the net could be $(x_a = x, a \in A)$. Observe that we have also proved that r is surjective.

In order to prove continuity of r, let $x = r(y)$ and let H be an open neighbourhood of x in X. Since X is completely regular, there exists an open neighbourhood G of x in X such that $G \subseteq \mathrm{Cl}(G) \subseteq H$, and the open covering $\mathcal{U} \equiv \{G, X \setminus \{x\}\} \in \mathrm{Cov}(X)$ (see [10], App. 1.3, Theorem 2). Let an $a \in A$ and a $U \in \mathcal{U}$ be chosen by ($*$) for y and $\mathcal{U}$. Then the corresponding intersection set $G_a \subseteq U$. Since $\mathcal{U}$ consists of the two members, G and $X \setminus \{x\}$, and since $x \in s_a^{-1}(y_a) \subseteq G_a \nsubseteq X \setminus \{x\}$, $U = G$ must hold. Now, by continuity of $p_a : N(X) \to X_a$, there exists an open set $V \subseteq N(X)$, $y \in V$, such that $p_a(V) \subseteq \mathrm{st}(y_a, N_a)$. Let us show that $r(V) \subseteq G_a$. Let $y' \in V$. Then $p_a(y') = y'_a \in \mathrm{st}(y_a, N_a)$ and $r(y') \in s_a^{-1}(y'_a)$. Observe that $y'_a \in \mathrm{st}(W_a^i, N_a)$ for every vertex W_a^i belonging to σ_{y_a}. Since $s_a : X \to |N_a|$ is canonical, $s_a^{-1}(y'_a) \in W_a^i \subseteq X$ must hold for all these W_a^i, hence, $r(y') \in s_a^{-1}(y'_a) \subseteq G_a$. Therefore, $r(V) \subseteq G_a \subseteq G \subseteq H$, and the claim is proved.

Let us now prove that the fibers of r are compact. Let $x \in X$. Then $s(x) \in r^{-1}(x)$. If $r^{-1}(x) = \{s(x)\}$, there is nothing to prove. Let $\{s(x)\} \subsetneqq r^{-1}(x)$. Notice that, for every $y \in r^{-1}(x)$ and every $a \in A$, $\sigma_{y_a} = \sigma_{(s(x))_a}$. Namely, if $\sigma_{(s(x))_a} = [W_a^0, \cdots, W_a^m]$, then $x = r(s(x)) \in G_a^{s(x)}$. Since $r(y) = x$ too, the simplexes $\sigma_{y_a} = [W'^0_a, \cdots, W'^{m'}_a]$ and $\sigma_{(s(x))_a}$ have a common face. Moreover, since they satisfy the minimality condition in the nerve N_a and since $s_a : X \to X_a$ is a canonical mapping with respect to $\mathcal{W}_a$, the simplexes σ_{y_a} and $\sigma_{(s(x))_a}$ must coincide. Recall that $p_{aa'} : X_{a'} \to X_a$, $a \leq a'$, are surjective and simplicial mappings with the fixed triangulations N_a on X_a, $a \in A$. Thus the simplexes $\sigma_{(s(x))_a}$, $a \in A$, "survive" in the limit space $N(X)$, i.e. there is a continuum $K_x \subseteq N(X)$ such that $p_a(K_x) = \sigma_{(s(x))_a}$, $a \in A$. (More precisely, if $a \in A$ and σ is a simplex of $X_a = |N_a|$, then for every $a' \geq a$ there is a simplex σ' of $X_{a'} = |N_{a'}|$, $\dim \sigma' \geq \dim \sigma$, which maps simplicially onto σ, $p_{aa'}(\sigma') = \sigma$. Since simplexes are nonempty continua, there exists the limit "simplex" - a nonempty continuum K in $N(X)$.) Obviously, $r^{-1}(x) \subseteq K_x$ because of $y_a \in \sigma_{s(x)_a} = p_a(K_x)$ whenever $y \in r^{-1}(x)$. Consequently, $r^{-1}(x)$ is compact as a closed subset of K_x.

It remains to prove that r is a closed mapping. Let $F \subseteq N(X)$ be a closed subset and let $x \in \mathrm{Cl}(r(F))$. We have to prove that $x \in r(F)$. If $s(x) \in F$, then $x = rs(x) \in r(F)$. Let $s(x) \notin F = \mathrm{Cl}(F)$. Since $N(X)$ is regular, there exists a local base of neighbourhoods $(U_\mu, \mu \in M)$ at the point $s(x)$ in $N(X)$ such that $U_\mu \cap F = \varnothing$, $\mu \in M$. (M is ordered by inclusion.) If M is finite, then $s(x)$ is an isolated point, hence $x \in r(F)$. Let M be an infinite set. Observe that $s^{-1}(U_\mu) \cap r(F) \neq \varnothing$, $\mu \in M$. Choose any $x^\mu \in s^{-1}(U_\mu) \cap r(F)$, $\mu \in M$. Then (x^μ) is a net in $r(F) \subseteq X$ converging to x. Namely, s embeds X onto $s(X) \subseteq N(X)$ as a closed subspace. Thus $(V_\mu, \mu \in M)$, $V_\mu = s^{-1}(U_\mu) = s^{-1}(U_\mu \cap s(X)) \approx U_\mu \cap s(X)$, is a local base of neighbourhoods at x in X. Notice that $r^{-1}(x^\mu) \cap F \neq \varnothing$, $\mu \in M$. Choose any net (y^μ), $y^\mu \in r^{-1}(x^\mu) \cap F$. Then $y^\mu \neq y^{\mu'}$ whenever $\mu \neq \mu'$, and $r(y^\mu) = x^\mu$, $\mu \in M$. Let us show that (y^μ) has an accumulation point in F. Consider, for each $a \in A$, the points $y^\mu_a = p_a(y^\mu)$ and $s(x^\mu)_a = p_a s(x^\mu) = s_a(x^\mu)$, $\mu \in M$, as well as $s(x)_a = p_a s(x) = s_a(x)$ in X_a. Then $(s(x^\mu)_a) \to s(x)_a$ because of $(x^\mu) \to x$, and thus $s(x^\mu)_a \in \mathrm{st}(\sigma_{s(x)_a}, N_a)$, whenever $\mu \geq \mu_0$ for some $\mu_0 \in M$. Since $s_a : X \to X_a = |N_a|$ is a canonical mapping, almost all of $s_a(x^\mu) = s(x_\mu)_a$ belong to some finite subcomplex, i.e. some compact subpolyhedron $P \subseteq \mathrm{Cl}(\mathrm{st}(\sigma_{s(x)_a}, N_a))$. Therefore, there exists a closed simplex $\sigma_a \subseteq P$ containing a subnet $(s(x^\mu)_a)$, $\mu \in M' \subseteq M$, of $(s(x^\mu)_a)$ such that $\sigma_{s(x)_a} \subseteq \sigma_a$. Furthermore, by definition of the mapping r, $y^\mu_a \in \sigma_a$ for every $\mu \in M'$. Recall that $p_{aa'}$ is surjective and simplicial with respect to fixed triangulations. Therefore, the simplexes σ_a, $a \in A$, "survive" in the limit space $N(X)$, i.e. there is a continuum $K_x \subseteq N(X)$ such that $p_a(K_x) = \sigma_a$, $a \in A$. By our construction, for every $\mu \in M'$, $r^{-1}(x^\mu) \subseteq K_x$ holds. Hence, (y^μ), $\mu \in M'$, is a net in the compact set $K_x \cap F$. Consequently, there exists a subnet (y^μ), $\mu \in M'' \subseteq M'$, converging to a point $y \in K_x \cap F$. Clearly, $r(y) \in r(F)$. On the other hand, $x = \lim\limits_{\mu \in M''}(x^\mu) = \lim\limits_{\mu \in M''}(r(y^\mu)) = r(\lim\limits_{\mu \in M''}(y^\mu)) = r(y) \in F$. This completes the proof of the lemma. □

Remark 1. *Comparing our definition of $r : N(X) \to X$ with that of Saneblidze [14], one can see the difference. Namely, in [14],* r is defined by means of the family *$\{|\sigma_{y_a}| \mid a \in A\}$ in X, where $|\sigma_{y_a}| = \bigcap\limits_{i=0}^{m} \mathrm{Cl}(W^i_a)$, referring also to V. I. Ponomarev [13].*

Lemma 2. *Under previous assumptions, $s(X) \approx X$ is a deformation retract of $N(X)$. More precisely, $rs = 1_X$ and $sr \simeq 1_{N(X)}$.*

Proof. We only need to prove the homotopy relation. Let $y \in N(X) \setminus s(X)$, $x = r(y)$ and $\{y^0\} = r^{-1}(x) \cap s(X)$, i.e. $y^0 = sr(y)$. We need to prove that the fiber $r^{-1}(x)$ contracts to the point y^0. Recall that $\sigma_{y'_a} = \sigma_{y^0_a}$, for every $y' \in$

$r^{-1}(x)$ and every $a \in A$. Therefore, for each $a \in A$, there exists a linear homotopy $(p_a sr)\mid_{r^{-1}(x)} \simeq p_a \mid_{r^{-1}(x)}$ within the simplex $\sigma_{y_a^0}$, which contracts $p_a(r^{-1}(x))$ to the point $p_a sr(r^{-1}(x)) = y_a^0$. Of course, the same holds for every fiber of r in $N(X)$. Thus, $p_a sr \simeq p_a$ in every polyhedron $X_a = |N_a|$, $a \in A$. Since $p_{aa'}$ are simplicial with respect to a fixed triangulation N_a on each X_a, these homotopy relations "survive" in the limit space $N(X)$. Consequently, $sr \simeq 1_{N(X)}$, and the lemma is proved. □

Remark 2. *Observe that the mapping $r : N(X) \to X$ is perfect. This implies that* Lemmas 1. *and* 2. *also hold in the case of paracompact (Lindelöf; countable compact; strongly paracompact; paracompact locally compact; σ-compact locally compact; Hausdorff compact) spaces. The needed facts from general topology of perfect mappings can be found in [5] and [6].*

The next theorem summarizes all previous results concerning the construction $X \mapsto N(X)$, $X \in \mathrm{Ob}(\mathbf{CTOP})$, including the appropriate restrictions mentioned in *Remark 2.* In some of the special cases one achieves improvements concerning the mappings using results from [3] and [4].

Theorem 1. *With every topologically complete space X one can associate functorialy a topologically complete space $N(X)$ of the same homotopy type, together with mappings $s_X : X \to N(X)$ and $r_X : N(X) \to X$ such that*

(i) *$N(X)$ is the limit space of a polyhedral inverse system $\boldsymbol{N}(X) = (X_a, p_{aa'}, A)$ with surjective and simplicial bonding mappings (with respect to a fixed triangulation N_a on each X_a), where N_a is the nerve of a normal covering of X;*

(ii) *s_X is the limit mapping of the corresponding canonical mappings $s_a : X \to |N_a| = X_a$, $a \in A$, and it embeds X as a closed subspace $s_X(X) \subseteq N(X)$ of $N(X)$;*

(iii) *$r_X s_X = 1_X$;*

(iv) *the fibers $r_X^{-1}(x) \subseteq N(X)$, $x \in X$, are compact;*

(v) *r_X is a surjective, closed and proper mapping, i.e. it is perfect;*

(vi) *$s_X r_X \simeq 1_{N(X)}$, hence, $s_X(X)$ is a deformation retract of $N(X)$.*

Moreover, in some special cases one achieves:

(a) *If X is paracompact (Lindelöf; countable compact), so is $N(X)$.*

(b) *If X is strongly paracompact (paracompact locally compact; σ-compact locally compact), so is $N(X)$, and all the mappings $p_{aa'}$ and p_a are proper.*

(c) *If X is compact Hausdorff, so is $N(X)$, and the limit $\boldsymbol{p} : N(X) \to \boldsymbol{N}(X)$ is also a resolution.*

(d) *If X is non-metrizable (non-second countable), then $N(X)$ is non-metrizable (non-second countable).*

Remark 3. *To prove (c), it is enough to use in the N-construction only the cofinal* subset $\mathcal{C} \subseteq \mathrm{Cov}(X)$ *of all finite open coverings. See also [10]. The statement (d) is obvious by [5], XI. 5.2 Theorem, by (3) and (4).*

3. Equivalence of the homotopy categories

In this section we consider the functor $\widetilde{N}$ induced by $N : \mathbf{TOP} \to \mathbf{CTOP}$ on the homotopy categories. One trivially verifies that

$$\widetilde{N} : \mathbf{HTOP} \to \mathbf{HCTOP},$$

$\widetilde{N}(X) = N(X)$ and $\widetilde{N}([f]) = [N(f)]$, is indeed a functor. We are especially interested in the restrictions of $\widetilde{N}$ to the categories with the classes of objects considered in the previous section. A few basic facts on categories and functors which we need one can find in [8] .

Lemma 3. *Let $f : X \to Y$ be a mapping of topologically complete spaces, and let $r_X : N(X) \to X$ and $s_Y : Y \to N(Y)$ be mappings considered in* Theorem 1. *Then $N(f) \simeq s_Y f r_X : N(X) \to N(Y)$, i.e. $\widetilde{N}([f]) = [s_Y f r_X]$.*

Proof. Consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \uparrow r_X & & \downarrow s_Y \\ N(X) & \xrightarrow[]{s_Y f r_X} & N(Y) \end{array}$$

Recall that $N(f)s_X = s_Y f$ holds by the N-construction. Apply now *Theorem 1.* (vi) to establish $N(f) \simeq N(f)s_X r_X = s_Y f r_X$, and the claim follows. □

Lemma 4. $\widetilde{N}\,|_{\mathbf{HCTOP}}$: **HCTOP** → **HCTOP** *is fully faithful.*

Proof. Let $f,\, g : X \to Y$ satisfying $\widetilde{N}\,[f] = \widetilde{N}[g]$ be given. Then $s_Y f r_X \simeq s_Y g r_X$ by *Lemma 3.*. Applying *Theorem 1.*, (iii) and (vi), one establishes $f \simeq g$. Thus the functor $\widetilde{N}$ is faithful. Let a mapping $g : \widetilde{N}(X) = N(X) \to N(Y) = \widetilde{N}(Y)$ be given. Take $f = r_Y g s_X : X \to Y$. Then $\widetilde{N}\,[f] = [s_Y f r_X] = [s_Y r_Y g s_X r_X] = [g]$. Hence $\widetilde{N}$ is full, and the lemma is proved. □

Let **S** be the full subcategory of **CTOP** whose objects are limits of polyhedral inverse systems with surjective and simplicial (with respect to a fixed triangulation on each polyhedron in the system) bonding mappings. Clearly, **HS** is a full subcategory of **HCTOP**.

Theorem 2. **HCTOP** *is equivalent to* **HS**. *Moreover,*

$$\mathbf{HCTOP} \;\underset{\hookleftarrow\atop J}{\overset{\widetilde{N}\atop\to}{}}\; \mathbf{HS}$$

is an equivalence pair of functors with natural isomorphism

$$\rho : \widetilde{N}J \rightsquigarrow 1_{\mathbf{HS}} \quad \text{and} \quad \sigma : 1_{\mathbf{HCTOP}} \rightsquigarrow J\widetilde{N}$$

determined by the classes of homotopy classes of mappings $r_Y : N(Y) \to Y$, $Y \in \mathrm{Ob}\mathbf{S}$, *and* $s_X : X \to N(X)$, $X \in \mathrm{Ob}\mathbf{CTOP}$, *respectively, i.e.* $\rho = \{\rho_Y = [r_Y] \mid Y \in \mathrm{Ob}\mathbf{S}\}$ *and* $\sigma = \{\sigma_X = [s_X] \mid X \in \mathrm{Ob}\mathbf{CTOP}\}$.

Quite analogous statements hold for the full homotopy subcategories of **HCTOP** *(and their corresponding subcategories* $\mathbf{HS}'$*) whose objects are all*
- *paracompact spaces;*
- *Lindelöf spaces;*
- *countable compact spaces;*
- *strongly paracompact spaces;*
- *paracompact locally compact spaces;*
- *σ-compact locally compact spaces;*
- *compact Hausdorff spaces.*

Proof. Let $Y \in \mathrm{Ob}\mathbf{S}$. Then $J(Y) = Y$ and $\widetilde{N}J(Y) = N(Y)$. Let $[f] : Y \to Y'$ be a morphism of $\mathrm{Mor}\mathbf{HS}$. Because of $r_{Y'}N(f) = r_{Y'}s_{Y'}fr_Y \simeq fr_Y$, the equality $[r_{Y'}]\,[N(f)] = [f]\,[r_Y]$ holds. Therefore, the following diagram in **HS** commutes:

$$\begin{array}{ccc}
\widetilde{N}J(Y) = N(Y) & \xrightarrow{[r_Y]} & Y = 1_{\mathbf{HS}}(Y) \\
{\scriptstyle \widetilde{N}J[f]=[N(f)]}\downarrow & & \downarrow{\scriptstyle [f]=1_{\mathbf{HS}}[f]} \\
\widetilde{N}J(Y') = N(Y') & \xrightarrow{[r_{Y'}]} & Y' = 1_{\mathbf{HS}}(Y')
\end{array}$$

Hence, $\rho : \widetilde{N}J \rightsquigarrow 1_{\mathbf{HS}}$, determined by the class $\{\rho_Y = [r_Y] \mid Y \in \mathrm{Ob}\mathbf{S}\}$, is a natural transformation of functors. Since each r_Y is a homotopy equivalence, ρ is a natural isomorphism. Quite analogously one can prove that $\sigma : 1_{\mathbf{HCTOP}} \rightsquigarrow J\widetilde{N}$ is a natural isomorphism of functors. All the remaining claims follow now by *Theorem 1.*. □

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