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BENDING OF THIN-WALLED BEAMS OF SYMMETRICAL OPEN CROSS-SECTIONS WITH INFLUENCE OF SHEAR

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Summary

The theory of bending of thin-walled beams of open cross-sections with the influence of shear, based on the classical Vlasov theory of thin-walled beams of open cross-sections, is developed for cross-sections with one and two axes of symmetry. It is proved that the beam is subjected to bending with the influence of shear caused by forces in the plane of symmetry and in addition to tension/compression due to shear in the case of cross-sections with one axis of symmetry, and to bending with the influence of shear caused by forces in the plane of symmetry in the case of two axes of symmetry. A new shear factor with respect to bending and tension/compression is given. The principal cross-section axes are defined in accordance with the classical theory of thin-walled beams of open sections. Illustrative examples are given, as well as comparisons with the finite element method.

Key words: *bending of thin-walled beams, influence of shear, open sections, single and double symmetrical sections, analytic, FEM*

1. Introduction

The Euler-Bernoulli beam theory as well as the Vlasov thin-walled beam theory [1] do not take into account shear deformations due to shear forces. The shear effect, as well as Poisson's effect, can be included by methods of theory of elasticity [2,3], but in that case the problem is no longer one-dimensional.

Thus, approximate methods to include the shear effect are developed; particularly in the analyses of displacements [4], by deriving an adequate stiffness matrix [5,6]. The concept of shear factors, first introduced by Timoshenko [7,8], was used as the ratio of the maximum shear stress to the average shear stress over a cross-section. Recent approaches to the problem are based on geometric assumptions [9-17] or shear energy relations [5,6]. Numerical examples comparing results obtained by different approaches can be found in [18-20].

In this paper, approximate analytical solutions for stresses along the beam cross-section contour as well as for stresses and displacements along the beam length are given. Beams with cross-sections with one and two axes of symmetry are considered.

Poisson's effect is ignored. Its influence on both the stresses and the displacements in the case of common open cross-sections is small, even for extremely low ratios of beam length to cross section contour dimensions [5]. The warping effect, defined by the "non-uniform warping bending theory" [21], is also ignored. This effect remains very localised close to the clamped ends, where by the non-uniform warping theories warping due to shear is restricted.

2. Strains and displacements

The displacement of an arbitrary point $S(x,s)$ at the middle line in the case of bending of thin-walled beams of open sections with one axis of symmetry can be expressed as

$$u_S = -\frac{\mathrm{d}w}{\mathrm{d}x} z + u + \int_0^s \gamma_{x\xi}\,\mathrm{d}s\,, \tag{1}$$

where $w = w(x)$ is the displacement in the z-direction, i.e. the displacement of the cross-section middle line as a rigid line in the plane of symmetry, $z = z(s)$ is the rectangular coordinate, $u = u(x)$ is the displacement of the cross-section middle line as a rigid line in the x-direction, $\gamma_{x\xi} = \gamma_{x\xi}(x,s)$ is the shear strain in the middle surface, s is the curvilinear coordinate of the middle line, ξ is the tangential axis on the curvilinear coordinate s; O*xyz* is the orthogonal coordinate system, where the z-axis is the axis of symmetry (Fig. 1).

Eq. (1) may be expressed as

$$u_S = \beta z + u + \int_0^s \gamma_{x\xi}\,\mathrm{d}s\,, \tag{2}$$

where $\beta = -\mathrm{d}w/\mathrm{d}x$ is the angular displacement of the middle line as rigid line with respect to the y-axis, orthogonal to the z-axis.

It is assumed that the middle line rotates with respect to the y-axis as rigid line, expressed by the first member of Eq. (2), as in the case of the ordinary theory of bending., In addition, it is assumed that the middle line is displaced due to shear, expressed by the second and third members of Eq. (2) .

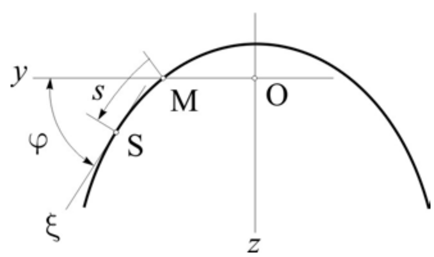

Fig. 1 Cross-section middle-line

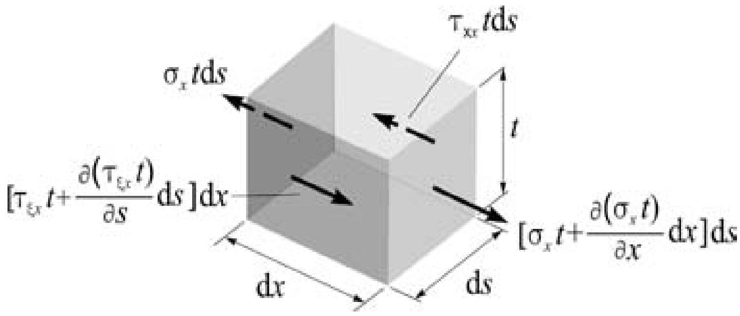

Fig. 2 The equilibrium of the element of the wall

The displacements can be separated as follows:

$$w = w_b + w_a\,, \qquad u = u_a\,, \tag{3}$$

where $w_b = w_b(x)$ is the displacement of the cross-sections as plane sections in the z-direction, as in the case of the ordinary theory of bending, $w_a = w_a(x)$ is additional displacement due to shear in the z-direction, $u_a = u_a(x)$ is the additional displacement due to shear in the x-direction.

Then,

$$\beta = \beta_b + \beta_a, \quad \beta_b = -\mathrm{d}w_b/\mathrm{d}x, \quad \beta_a = -\mathrm{d}w_a/\mathrm{d}x. \tag{4}$$

The strain in the beam longitudinal direction may then be expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} z - \frac{\mathrm{d}u}{\mathrm{d}x} + \int_0^s \frac{\partial \gamma_{x\xi}}{\partial x} \mathrm{d}s. \tag{5}$$

3. Stresses and displacement

Hooke's law may be simplified as

$$\sigma_x = E\varepsilon_x \qquad \tau_{x\xi} = G\gamma_{x\xi}, \tag{6}$$

where E is the modulus of elasticity and G is the shear modulus.

Thus,

$$\sigma_x = -E\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} z + E\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{E}{G}\int_0^s \frac{\partial \tau_{x\xi}}{\partial x} \mathrm{d}s. \tag{7}$$

From the equilibrium of a differential portion of the beam wall (Fig. 2), it may be written

$$\tau_{x\xi} = \frac{1}{t}\left[-\int_0^s \frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}s + f(x)\right], \qquad f = t(\mathrm{M})\cdot\tau_{\xi x}(x,\mathrm{M}) = T_\mathrm{M}(x), \tag{8}$$

where $t = t(s)$ is the wall thickness and M is the starting point of the curvilinear coordinate s.

If $\partial\tau_{x\xi}/\partial x = const.$, referring to (7), one has

$$\tau_{x\xi} = \frac{1}{t}\left[T_M + E\left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} S_y(s) - \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} A(s)\right)\right], \; S_z(s) = \int_0^s y\,\mathrm{d}A, \; A(s) = \int_0^s \mathrm{d}A, \; \mathrm{d}A = t\mathrm{d}s. \tag{9}$$

Eq. (9) may be rewritten as

$$\tau_{x\xi} = \frac{E}{t}\left(-\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} S_y^* + \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} A^*\right), \; S_y^* = \int_{s^*} z\,\mathrm{d}A^*, \; A^* = \int_{s^*} \mathrm{d}A^*, \; \mathrm{d}A^* = t\mathrm{d}s^*, \; \mathrm{d}s^* = -\mathrm{d}s; \tag{10}$$

where $S_y^* = S_y^*(s)$ is the moment of the cut-off portion of area with respect to the y-axis, $A^* = A^*(s)$ is the cut-off portion of the beam wall area with respect to the y-axis, s^* is the curvilinear coordinate of the cut-off portion of the beam wall area, from the free edge, i.e. where $\tau_{x\xi} = 0$.

It is assumed that the normal stress given by Eq. (7) and the shear stress given by Eqs. (9) and (10) are constant across the wall thickness.

4. Equilibrium equations

It is assumed that the beam loads are reduced to loads $q_z = q_z(x)$ in the beam plane of symmetry

$$q_z = \int_L p_z \mathrm{d}s, \tag{11}$$

where $p_z = p_z(x,s)$ are the surface loads with respect to the z-axis and L is the cross-section middle line length.

For a portion of the beam wall, the following equilibrium equations can be written

$$\sum F_x = \int_L \frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}x\,\mathrm{d}s = 0\,,\quad \sum F_z = \int_L \frac{\partial(\tau_{x\xi} t)}{\partial x}\sin\varphi\,\mathrm{d}x\,\mathrm{d}s + q_z\mathrm{d}x = 0\,. \tag{12}$$

Eqs. (12) can be rewritten as

$$\int_L \frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}x\,\mathrm{d}s = 0\,,\qquad \int_L \frac{\partial(\tau_{x\xi} t)}{\partial x}\,\mathrm{d}z + q_z = 0\,,\qquad \sin\varphi = \frac{\mathrm{d}z}{\mathrm{d}s}\,. \tag{13}$$

By integrating by parts one has

$$\int_L \frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}x\,\mathrm{d}s = 0\,,\qquad \left.\frac{\partial(\tau_{x\xi} t)}{\partial x}z\right|_{e_1}^{e_2} - \int_L z\frac{\partial}{\partial s}\left[\frac{\partial(\tau_{x\xi} t)}{\partial x}\right]\mathrm{d}s + q_z = 0 \tag{14}$$

where e_1 and e_2 are the boundaries, where $\tau_{x\xi} = 0$.

Thus,

$$\int_L \frac{\partial \sigma_x}{\partial x}\mathrm{d}A = 0\,,\qquad \int_L z\frac{\partial}{\partial x}\left[\frac{\partial(\tau_{x\xi} t)}{\partial s}\right]\mathrm{d}s - q_z = 0\,, \tag{15}$$

By substituting Eqs. (7) and (9) one has

$$-ES_y\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0\,,\qquad EI_y\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - ES_y\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = q_z\,, \tag{16}$$

where

$$A = \int_A \mathrm{d}A\,,\qquad S_y = \int_A z\,\mathrm{d}A\,,\qquad I_y = \int_A z^2\mathrm{d}A\,, \tag{17}$$

If y is the centroid coordinate, when $S_y = 0$, Eqs. (16) take the following simple form:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0\,,\qquad EI_y\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q_z\,. \tag{18}$$

5. Internal forces and stresses

Integration of the shear stress components $\tau_{x\xi}$ over the cross-sections gives

$$\int_A \tau_{x\xi}\sin\varphi\,\mathrm{d}A = Q_z\,, \tag{19}$$

where $Q_z = Q_z(x)$ is the shear force with respect to the z-axis.

Substitution of Eq. (10) into Eq. (18) gives

$$\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = 0\,,\qquad Q_z = -EI_y\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\,, \tag{20}$$

where

$$\sin\varphi\,\mathrm{d}A = \sin\varphi\, t\mathrm{d}s = t\mathrm{d}z\,,\ \int_L A^*\,\mathrm{d}z = \int_A z\,\mathrm{d}A = S_y = 0\,,\int_L S_y^*\,\mathrm{d}z = \int_A z^2\,\mathrm{d}A = I_y\,.$$

Referring to Eqs. (18) and (20), one has

$$\frac{\mathrm{d}Q_z}{\mathrm{d}x} = -q_z\,. \tag{21}$$

Thus, by substituting Eqs.(20) into (10), the shear stress component $\tau_{x\xi}$ is

$$\tau_{x\xi} = \frac{Q_z S_y^*}{I_y t}\,. \tag{22}$$

Integration of the normal stress over the cross-section gives

$$M_y = \int_A \sigma_x z\,\mathrm{d}A\,, \qquad \int_A \sigma_x\,\mathrm{d}A = 0\,, \tag{23}$$

where $M_y = M_y(x)$ is the bending moment with respect to the *y*-axis.

By substituting Eq.(7) into Eqs. (23), the following equation can be obtained:

$$M_y = -EI_y\frac{\mathrm{d}^2 w}{\mathrm{d}^2 x} - M_y^z\,, \qquad EA\frac{\mathrm{d}u}{\mathrm{d}x} - N^z = 0\,, \tag{24}$$

where

$$M_y^z = -\frac{E}{G}\int_A z\,\mathrm{d}A\int_0^s \frac{\partial\tau_{x\xi}}{\partial x}\mathrm{d}s\,, \qquad N^z = -\frac{E}{G}\int_A \mathrm{d}A\int_0^s \frac{\partial\tau_{x\xi}}{\partial x}\mathrm{d}s\,, \tag{25}$$

i.e. referring to Eq. (22)

$$M_y^z = \frac{Eq_z}{GI_y}\int_A z\,\mathrm{d}A\int_0^s \frac{S_y^*}{t}\mathrm{d}s\,, \qquad N^z = \frac{Eq_z}{GI_y}\int_A \mathrm{d}A\int_0^s \frac{S_y^*}{t}\mathrm{d}s\,, \tag{26}$$

i.e.

$$M_y^z = \frac{Eq_z}{GI_y}\int_A\left(\frac{S_y^*}{t}\right)^2\mathrm{d}A\,, \qquad N^z = \frac{Eq_z}{GI_y}\int_L \frac{A^* S_y^*}{t}\mathrm{d}s \tag{27}$$

Referring to Eqs. (20) and (24), the following equations can be written:

$$-EI_y\frac{\mathrm{d}^3 w}{\mathrm{d}^3 x} = \frac{\mathrm{d}M_y}{\mathrm{d}x} + \frac{\mathrm{d}M_y^z}{\mathrm{d}x} = Q_z + 0\,, \qquad EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mathrm{d}N^z}{\mathrm{d}x} = 0\,, \tag{28}$$

and according to Eq. (18)

$$-EI_y\frac{\mathrm{d}^4 w}{\mathrm{d}^4 x} = \frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} = \frac{\mathrm{d}Q_z}{\mathrm{d}x} = -q_z\,, \qquad \frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = 0\,, \tag{29}$$

It is assumed that $q_z = const$; if $q_z \neq const$, Eqs. (29) give only an approximate solution to the problem.

The normal stress given by Eq. (7), referring to Eqs. (22) and (24), can be expressed as

$$\sigma_x = \frac{M_y}{I_y} z + \frac{M_y^z}{I_y} z + \frac{N^z}{A} - \frac{E}{G} \cdot \frac{q_z}{I_y} \int_0^s \frac{S_y^*}{t} \mathrm{d}s \,. \tag{30}$$

The internal forces given by Eq. (27) can also be written as

$$M_y^z = \frac{EI_y}{GA} \kappa_{zz} q_z \,, \qquad N^z = \frac{E}{G} L_s \kappa_{xz} q_z \,, \tag{31}$$

where

$$\kappa_{zz} = \frac{A}{I_y^2} \int_A \left(\frac{S_y^*}{t} \right)^2 \mathrm{d}A \,, \qquad \kappa_{xz} = \frac{1}{I_y L_s} \int_A \frac{A^* S_y^*}{t^2} \mathrm{d}A \tag{32}$$

are the shear factors with respect to the w-displacements and to the u-displacements during the w-displacements, respectively; L_s is an arbitrary length of the middle line.

Hence, the normal stress given by (30) can also be written as

$$\sigma_x = \frac{M_y}{I_y} z + \frac{E\kappa_{zz}}{GA} q_z z + \frac{E\kappa_{xz}}{GA} L_s q_z - \frac{E}{GI_y} q_z \int_0^s \frac{S_y^*}{t} \mathrm{d}s \,. \tag{33}$$

6. Differential equations with separated displacements

Eqs. (24), according to Eqs. (27) and (32), can be expressed as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M_y}{EI_y} - \frac{\kappa_{zz}}{GA} q_z \,, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{L_s \kappa_{xz}}{GA} q_z \,. \tag{34}$$

Eqs. (34), referring to Eqs. (3), can be separated as

$$\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2} = -\frac{M_y}{EI_y} \,, \tag{35}$$

and

$$\frac{\mathrm{d}^2 w_a}{\mathrm{d}x^2} = -\frac{\kappa_{zz}}{GA} q_z \,, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u_a}{\mathrm{d}x} = \frac{L_s \kappa_{xz}}{GA} q_z \,. \tag{36}$$

Integration of Eqs. (36), taking into account Eq. (21), gives

$$\frac{\mathrm{d}w_a}{\mathrm{d}x} = -\beta_a = \frac{Q_z \kappa_{zz}}{GA} \,, \qquad u_a = -\frac{Q_z L_s \kappa_{xz}}{GA} \tag{37}$$

where the integration constants are ignored; it is assumed that the angular displacement β_a and additional linear displacement u_a do not depend on the boundary conditions.

Eq. (35) is the well known equation of the classical theory of bending of thin-walled beams, where

$$EI_y \frac{\mathrm{d}^3 w_b}{\mathrm{d}x^3} = -\frac{\mathrm{d}M_y}{\mathrm{d}x} = -Q_z \,, \; EI_y \frac{\mathrm{d}^4 w_b}{\mathrm{d}x^4} = -\frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} = -\frac{\mathrm{d}Q_z}{\mathrm{d}x} = q_z \,, \; \frac{\mathrm{d}w_b}{\mathrm{d}x} = -\beta_b \,. \tag{38}$$

Eqs. (36) and (37) take into account the displacement due to shear. The integration of the first equation of Eqs. (36) gives

$$w_a = \frac{\kappa_{zz}}{GA} M_y + C_w, \tag{39}$$

where C_w is the integration constant with respect to the w-displacements.

Eqs. (39) can also be written as

$$w_a = \frac{M_y}{GA_s} + C_w, \tag{40}$$

where $A_s = A/\kappa_{zz}$ is the shear area with respect to the w-displacements.

The normal stress may then be written as

$$\sigma_x = \frac{M_y}{I_y} y + \frac{E}{GA_s} q_z z + \frac{EL_s \kappa_{xz}}{GA} q_z - \frac{E}{GI_y} q_z \int_0^s \frac{S_y^*}{t} \mathrm{d}s \,. \tag{41}$$

7. Shear strain energy

According to Hooke's law, taking into account Eq. (6) and the first equation of Eqs. (37), the average shear stresses with respect to the displacements can be expressed as

$$\tau_{x\xi,av} = G\gamma_{x\xi,av} = G\frac{\mathrm{dw}_a}{\mathrm{d}x} = G\frac{\kappa_{zz}}{A} Q_z, \tag{42}$$

where $\gamma_{x\xi,av}$ is the average shear strain with respect to the displacement w_a.

The average shear stresses can also be expressed as

$$\tau_{x\xi,av} = Q_z / A_s \,. \tag{43}$$

The shear energy of the beam element may be expressed as

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G} \int_A \tau_{x\xi}^2 \mathrm{d}A, \tag{44}$$

i.e.

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G} \frac{Q_z^2}{I_y^2} \int_A \left(\frac{S_y^*}{t} \right)^2 \mathrm{d}A \,. \tag{45}$$

The shear energy can also be written by average shear deformations as

$$\mathrm{d}U = \frac{\mathrm{d}x}{2} (-\beta_a) Q_z = \frac{\mathrm{d}x}{2} \gamma_{x\xi,av} Q_z \,. \tag{46}$$

i.e. taking into account Eq. (42)

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G} \frac{\kappa_{zz}}{A} Q_z^2 \,. \tag{47}$$

The shear factor κ_{zz} can be obtained by equating (47) and (45). The result is equal to the obtained shear factor.

8. Boundary conditions

Boundary conditions can be defined as follows, at the starting section A,

$$w_a = 0\,, \tag{48}$$

hence, referring to Eq. (40),

$$C_w = -\frac{M_{yA}}{GA_s}\,, \tag{49}$$

where M_{yA} is the bending moment at $x = x_A$.

The total displacements then are:

$$w = w_b + \frac{M_y - M_{yA}}{GA_s}, \qquad u_a = -\frac{Q_z L_s \kappa_{xz}}{GA}\,. \tag{50}$$

For the hinged sections it may be written:

$$w\big|_{x=x_A} = w_b\big|_{x=x_A} = 0\,, \quad \left.\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}\right|_{x=x_A} = 0 \quad \left(M_{yA} = 0\right);$$

$$w\big|_{x=x_B} = w_b\big|_{x=x_B} = 0\,, \quad \left.\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}\right|_{x=x_B} = 0 \quad \left(M_{yB} = 0\right). \tag{51}$$

For the clamped sections:

$$w\big|_{x=x_A} = w_b\big|_{x=x_A} = 0\,, \quad \left.\frac{\mathrm{d}w_b}{\mathrm{d}x}\right|_{x=x_A} = 0 \quad \left(\beta_A^b = 0\right);$$

$$w\big|_{x=x_B} = w_b\big|_{x=x_B} + \frac{1}{GA_s}\left(-EI_y \left.\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}\right|_{x=x_B} + EI_y \left.\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}\right|_{x=x_A}\right) = 0\,, \ \left.\frac{\mathrm{d}w_b}{\mathrm{d}x}\right|_{x=x_B} = 0 \quad \left(\beta_B^b = 0\right). \tag{52}$$

For the free section:

$$\left.\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}\right|_{x=x_A} = 0 \quad \left(M_{yA} = 0\right), \quad \left.\frac{\mathrm{d}^3 w_b}{\mathrm{d}x^3}\right|_{x=x_A} = 0 \quad \left(Q_{zA} = 0\right). \tag{53}$$

9. Double symmetrical cross-section

For double symmetrical cross-sections the normal stress given by Eq. (41) becomes

$$\sigma_x = \frac{M_y}{I_y} z + \frac{E}{GA_s} q_z - \frac{E}{GI_y} q_z \int_0^s \frac{S_y^*}{t}\,\mathrm{d}s\,, \tag{54}$$

where due to symmetry

$$\kappa_{xz} = 0\,, \tag{55}$$

i.e. referring to the second equation of Eqs. (32)

$$\int_A \frac{A^* S_y^*}{t^2}\,\mathrm{d}A = 0\,. \tag{56}$$

The total displacements become

$$w = w_b + \frac{M_y - M_{yA}}{GA_s}, \qquad u_a = 0\,. \tag{57}$$

10. Illustrative examples

The I-section with two axes of symmetry (Fig. 3.a) and the symmetrical U-section (Fig. 3.b) are considered.

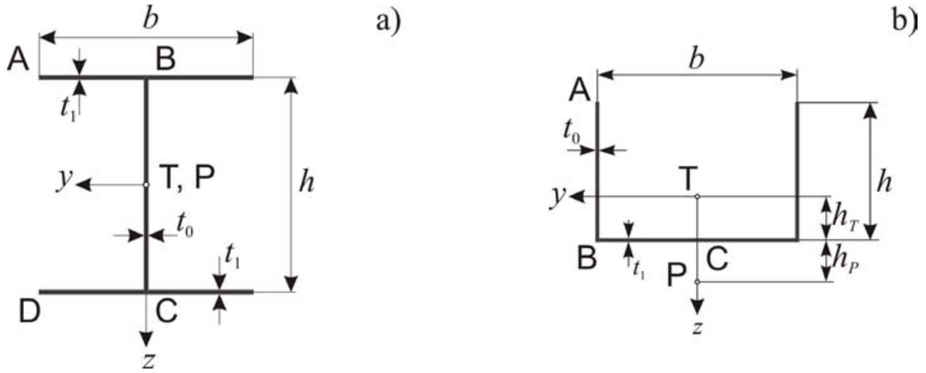

Fig. 3 Analysed cross-sections: a) double symmetrical I-section; b) symmetrical U-section

The shear factors for the double symmetrical I-section, according to Eqs. (32) and (55) are:

$$\kappa_{zz} = \frac{6(2+\psi)^3\left(30+10\psi+\psi^2+5\psi\rho^2\right)}{5\psi\left[12+\psi(8+\psi)\right]^2}, \qquad \kappa_{xz} = 0\,, \tag{58}$$

where: $A_1 = bt_1, \quad A_0 = bt_0, \quad \psi = A_0/A_1, \quad \rho = b/h, \quad L_s = h, \quad h_0 = h/2\,.$

The shear factors for the symmetrical U-section, according to Eqs. (32), are:

$$\kappa_{zz} = \frac{3\left[2\left(8+55\psi+140\psi^2+160\psi^3+80\psi^4+16\psi^5\right)+5\psi\rho^2(1+2\psi)^3\right]}{20\psi(1+2\psi)^2(2+\psi)^2}$$

$$\kappa_{xz} = \frac{1+\rho^2(1+2\psi)}{4(1+2\psi)(2+\psi)}, \tag{59}$$

where: $A_1 = bt_1, A_0 = ht_0, \psi = A_0/A_1, \rho = b/h, L_s = h, \ h_T = h\psi/(1+2\psi), \ h_P = 3h\psi/(1+6\psi)\,.$

Shear factors given by (58) and (59) for the double symmetrical I-section and the symmetrical U-section beam ($b = h = 1000$ mm, $t_1 = t_0$) are compared with those presented in [12] as shown in Table 1.

Table 1 Comparison of the shear factor

	Presented theory	Senjanović [12]
I - section	$\kappa_{zz} = 3{,}380$	$\frac{1}{k_0}(\nu = 0) = \kappa_{zz} = 3{,}380$
U - section	$\kappa_{zz} = 1{,}95$	$\frac{1}{k_0}(\nu = 0) = \kappa_{zz} = 1{,}95$

A comparison of the normalised maximal vertical displacements at the free end of the cantilevered I-, T- and U-section beam ($b = h/2,\quad t_1 = t_0 = h/20$) subjected to the end concentrated force F with material properties $E = 200$ GPa and $\nu = 0$ is given in Table 2.

Table 2 Comparison of the normalised vertical displacements $w_{\max} / w_{b,\max}$

	Presented theory	El Fatmi [21]
I - section	1,353	1,329
T - section	1,181	1,171
U - section	1,146	1,142

A series of examples have been analysed by applying the FEM using Autodesk Algor Simulation Pro in order to compare the results with those obtained analytically with the presented theory (BIS – **B**ending with **I**nfluence of **S**hear). Membrane elements with 2 DOF are used. The mesh was generated with square elements with sides of $h/40$.

Due to symmetry, only one half of the beam is modelled. Fig. 4 shows the boundary conditions that are used: at the simply supported end and at $x = l/2$ (Fig. 4.a), at the clamped end and at $x = l/2$ (Fig. 4.b). The sign ☑ means that certain displacement, translation T or rotation R, is constrained.

The beams under uniformly distributed load per unit length q_z were analysed, where:

$$h = 400\text{ mm},\quad b = h,\quad t_1 = t_2 = t_0 = h/40,\quad E = 210\text{ GPa},\quad \nu = 0{,}3\,.$$

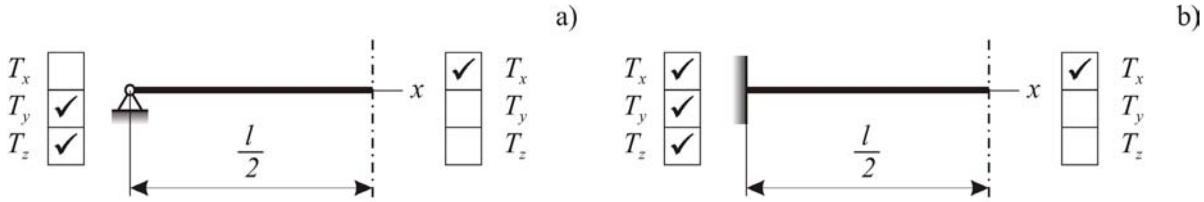

Fig. 4 The boundary conditions: a) a simply supported beam; b) a clamped beam

Some results, compared also with the FEM analysis, are presented in Table 3 and in Figs 5 and 6.

The normal stresses in the x-direction at a selected point of the beam cross-section are normalised as: $\sigma_x / \sigma^b_{x,\max}$, $\sigma^{FEM}_x / \sigma^b_{x,\max}$, where σ_x is the normal stress in the x-direction at the selected point obtained analytically by Eq. (41), or Eq. (54) for the double symmetrical section, σ^{FEM}_x is the maximal normal stress in the x-direction at that point obtained by applying the FEM, and $\sigma^b_{x,\max}$ is the maximal normal stress in the x-direction at a certain point/wall of the beam cross-section obtained by applying the ordinary bending theory (the **E**uler-**B**ernoulli **b**ending **t**heory – EBBT).

Table 3 Normalised maximal normal stresses

L/h	Double symm. I-section (point B, Fig. 3.a)				Symm. U-section (point A, Fig. 3.b)			
	Simply supported		Clamped		Simply supported		Clamped	
	BIS	FEM	BIS	FEM	BIS	FEM	BIS	FEM
3	1,171	1,170	1,512	1,498	1,116	1,113	1,347	1,323
5	1,061	1,061	1,184	1,184	1,042	1,042	1,125	1,124

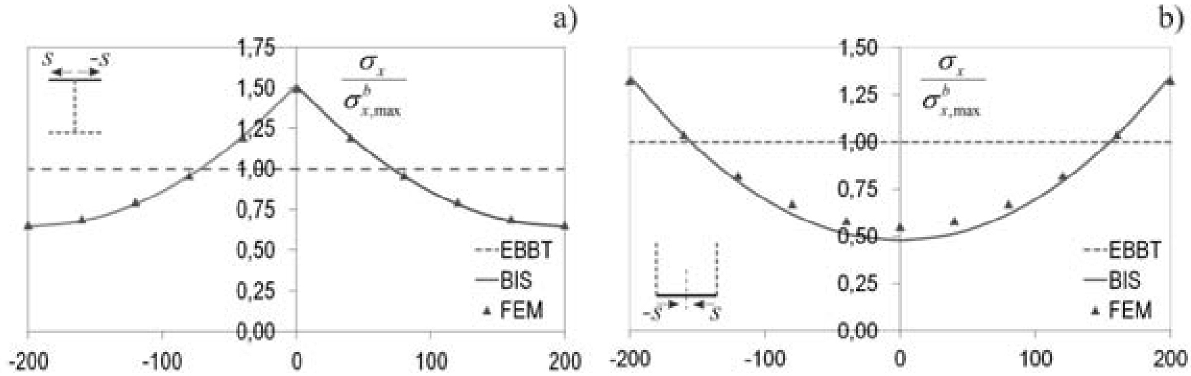

Fig. 5 Normalised normal stresses at the clamped beam midspan (L=3h): a) at the top flange of the I-section, b) at the horizontal wall of the U-section

The normalised vertical displacements are expressed as: $w_P / w^b_{P,\max}$, $w^{FEM}_P / w^b_{P,\max}$, where w_P is the total vertical displacement of the pole P obtained analytically by the first equation of Eqs. (50), or Eq. (57) for the double symmetrical section, w^{FEM}_P is the vertical displacement of the point B obtained by applying the FEM, and $w^b_{P,\max}$ is the vertical displacement of the pole P according to the ordinary theory of bending.

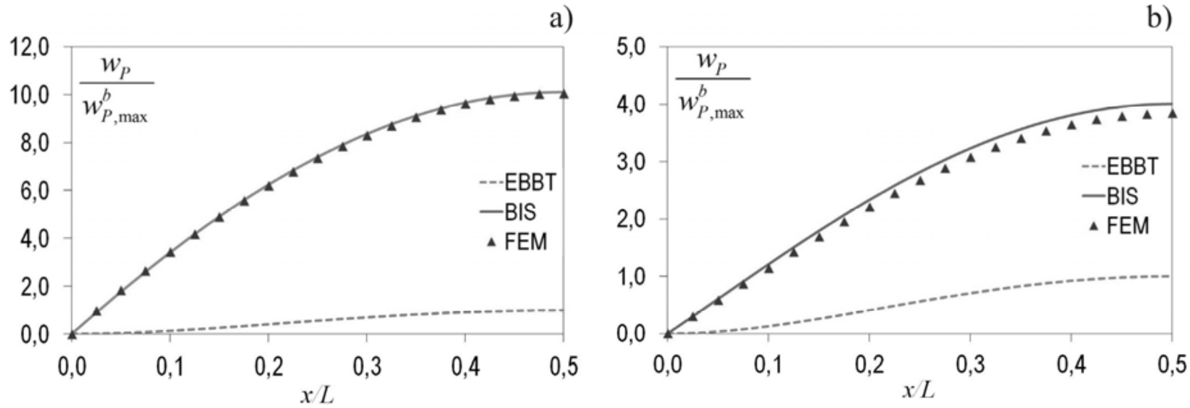

Fig. 6 Normalised vertical displacements at the clamped beam (L=3h): a) I-section, b) U-section

11. Conclusion

A theory of bending of thin-walled beams with the influence of shear for sections with one and two axes of symmetry is developed. The theory is based on the classical Timoshenko bending theory. The shear factor with respect to the bending in the beam plane of symmetry is given in an analytical form. It is proved that the beam with a single symmetrical section, loaded in the plane of symmetry, is subjected also to tension/compression due to shear.

Thus, a new factor of shear is given, with respect to tension/compression due to shear. In the case of a double symmetrical section this factor vanishes: the beam is subjected to bending with the influence of shear only.

For various types of cross-sections with one and two axes of symmetry, the shear factors are given in the parametric forms.

Stresses can be obtained in the analytical form both along the cross-section middle line and the beam length. Various boundary conditions and loadings are considered.

Several examples are analyzed in comparison with the finite element method. Excellent agreements of the results for displacements are obtained, as well as for stresses. Some discrepancies for normal stresses are noticed at beam ends in the case of clamped ends, as a result of different boundary conditions, both in the presented theory and the finite element method. Corresponding cross-section functions are given in the appendix.

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Appendix: Cross-section functions

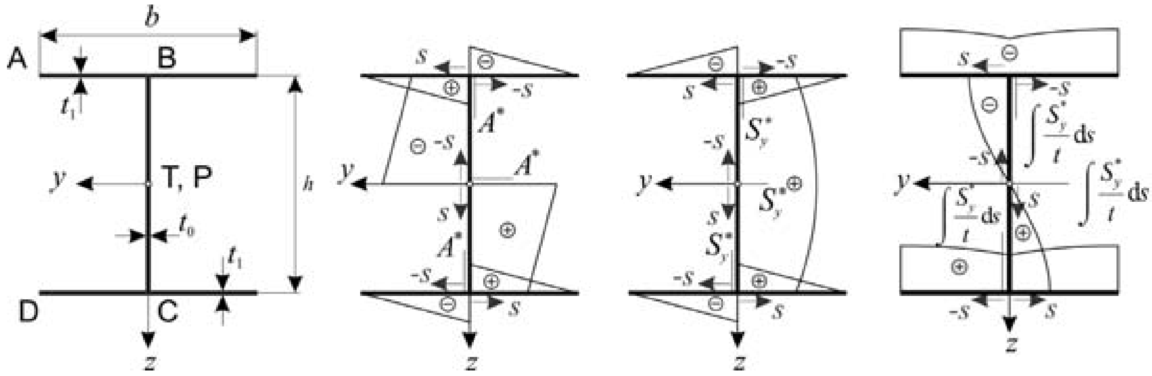

Fig. A1 Double symmetrical I-section: Cross-section functions

Cross-section functions for the double symmetrical I-section (Fig. A1) are:

$$A^* = \left(\frac{b}{2} - s\right)t_1,\quad S_y^* = -\frac{h}{2}t_1\left(\frac{b}{2} - s\right),\quad \int_0^s \frac{S_y^*}{t}ds = -\frac{h}{4}s\left(b - s\right) - \frac{h^2\left(6A_1 + A_0\right)}{24t_0} \qquad \left(0 \le s \le \frac{b}{2}\right)$$

$$A^* = bt_1 + \left(\frac{h}{2} - s\right)t_0,\quad S_y^* = \frac{h}{2}A_1 + \frac{t_0}{2}\left(\frac{h^2}{4} - s^2\right),\quad \int_0^s \frac{S_y^*}{t}ds = \frac{s}{t_0}\left[\frac{hA_1}{2} + \frac{t_0}{24}\left(3h^2 - 4s^2\right)\right] \qquad \left(0 \le s \le \frac{h}{2}\right)$$

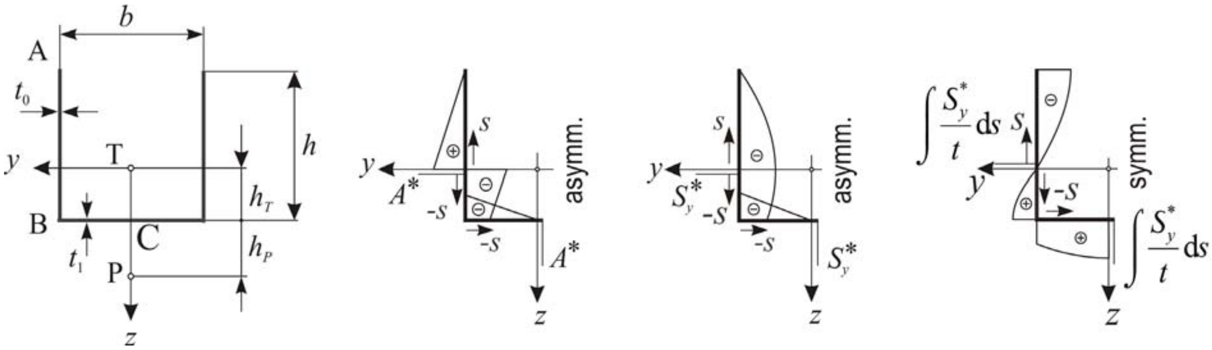

Fig. A2 Symmetrical U-section: Cross-section functions

Cross-section functions for the symmetrical U-section (Fig. A2) are:

$$A^* = \left(\frac{b}{2} - s\right)t_1,\quad S_y^* = \left(\frac{b}{2} - s\right)t_1 h_T,\quad \int_0^s \frac{S_y^*}{t}\mathrm{d}s = \frac{h_T s}{2}\left(b - s\right) + \frac{h_T}{2}\left[\left(h - h_T\right)^2 - \frac{h_T^3}{3}\right] \qquad \left(0 \le s \le \frac{b}{2}\right)$$

$$A^* = \frac{A_1}{2} + \left(h_T - s\right)t_0,\quad S_y^* = \frac{t_0}{2}\left[\left(h - h_T\right)^2 - s^2\right],\quad \int_0^s \frac{S_y^*}{t}\mathrm{d}s = \frac{s}{2}\left[\left(h - h_T\right)^2 - \frac{s^2}{3}\right] \qquad \left(0 \ge s \ge +h_T\right)$$

$$A^* = \left(-h + h_T - s\right)t_0,\quad S_y^* = \frac{t_0}{2}\left[\left(h - h_T\right)^2 - s^2\right],\quad \int_0^s \frac{S_y^*}{t}\mathrm{d}s = \frac{s}{2}\left[\left(h - h_T\right)^2 - \frac{s^2}{3}\right] \qquad \left(0 \ge s \ge -h + h_T\right)$$