A characterization of continuous images of arcs by their images of weight $\leq \aleph_1$

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Abstract. *The main purpose of this paper is to characterize the continuous images of arcs by their images of the weight* $\leq \aleph_1$. *More precisely, we will show that a compact space X is the continuous image of an arc if and only if every continuous image* $Y = f(X)$ *with* $w(Y) \leq \aleph_1$ *is a continuous image of an arc.*

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1. Introduction

An arc is a continuum with precisely two nonseparating points. A space X is said to be a *continuous image of an arc* if there exists an arc L and a continuous surjection f : L→X. Let X be a non-degenerate locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of a locally connected continuum is again a locally connected continuum. Let

$$\mathbf{L}_X = \{Y \subset X : \text{Y is a non- degenerate cyclic element of X}\}.$$

If Y is a closed subset of X, we let $\mathbf{K}$(X\Y) denote the family of all components of X\Y. Let X be a locally connected continuum. A subset Y of X is said to be a *T - set* if Y is closed and |Bd(J)| = 2 for each J∈$\mathbf{K}$(X\Y).

Theorem 1. [1, Theorem 1] *A Hausdorff locally connected continuum S is the continuous image of an arc if and only if each cyclic element of S is the continuous image of an arc.*

The following theorem is a part of Theorem 4.4 of [9].

Theorem 2. *If X is a locally connected continuum, then the following conditions are equivalent:*

1. *X is a continuous image of an arc,*

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2. *X is a continuous image of an ordered compactum,*

3. *for each* $Y \in \mathbf{L}_X$ *and any* $p,q,r \in Y$ *there exists a metrizable T - set Z in Y such that* $p,q,r \in Z$.

4. *For each* $Y \in \mathbf{L}_X$ *and each closed metrizable subset M of Y there exists a metrizable T - set A in Y such that* $M \subseteq A$.

In this paper we shall use the notion of *inverse systems* $\mathbf{X} = \{X_a, p_{ab}, A\}$ and their limits in the usual sense [2, p. 135].

The notion of *approximate inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ will be used in the sense of S. Mardešić [6]. See also [8].

Let τ be an infinite cardinal. We say that a partially ordered set A is τ*-directed* if for each $B \subseteq A$ with $\mathrm{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. If A is $\aleph_0$-directed, then we will say that A is σ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be τ*-directed* if A is τ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if A is σ-directed.

Lemma 1. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate* τ*-directed inverse system of compact spaces with surjective bonding mappings and with the limit X. Let Y be a compact space with* $w(Y) \leq \tau$. *For each surjective mapping* $f: X \to Y$ *there exists an* $a \in A$ *such that for each* $b \geq a$ *there exists a mapping* $g_b: X_b \to Y$ *such that* $f = g_b \circ p_b$, *where* p_b *is the natural projection of X onto* X_b.

Proof. Let $\mathcal{B}$ be a basis of Y, $\mathrm{card}(\mathcal{B}) = \tau$ and let $\mathcal{V}$ be the collection of all finite subfamilies of $\mathcal{B}$. Clearly, $\mathrm{card}(\mathcal{V}) = \tau$. We assume that τ is an initial ordinal number. Hence, $\mathcal{V} = \{\mathcal{V}_\alpha : \alpha < \tau\}$. For each $\mathcal{V}_\alpha$ $f^{-1}(\mathcal{V}_\alpha)$ is a covering of X. There exists an $a(\alpha) \in A$ such that for each $b \geq a(\alpha)$ there is a cover $\mathcal{V}_{\alpha b}$ of X_b such that $p_b^{-1}(\mathcal{V}_{\alpha b})$ refines $f^{-1}(\mathcal{V}_\alpha)$, i.e. $p_b^{-1}(\mathcal{V}_{\alpha b}) < f^{-1}(\mathcal{V}_\alpha)$. From the τ-directedness of A it follows that there is an $a \in A$ such that $a \geq a(\alpha)$, $\alpha < \tau$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ for $x_b \in X_b$ is degenerate. Suppose that there exists a pair u,v of distinct points of Y such that $u, v \in f(p_b^{-1}(x_b))$. Then there exists a pair x,y of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U,V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y \setminus \{u,v\}\}$. There exists a covering $\mathcal{V}_\alpha \in \mathcal{V}$ such that $\mathcal{V}_\alpha < \{U, V, X \setminus \{u,v\}\}$. We infer that there is a covering $\mathcal{V}_{\alpha b}$ of X_b such that $p_b^{-1}(\mathcal{V}_{\alpha b}) \prec f^{-1}(\mathcal{V}_\alpha)$. It follows that $p_b(x) \neq p_b(y)$ since x and y lie in disjoint members of the covering $f^{-1}(\mathcal{V}_\alpha)$. This is impossible since $x,y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define g_b: $X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is clear that $g_b p_b = f$. Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and p_b is quotient (as a closed mapping). □

The following theorem is Theorem 1.7 from [5].

Theorem 3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ *- directed inverse system of compact metrizable spaces and surjective bonding mappings. Then* $X = \lim \mathbf{X}$ *is metrizable if and only if there exists an* $a \in A$ *such that* $p_b: X \to X_b$ *is a homeomorphism for each* $b \geq a$.

2. The main theorems

We first establish the following theorem.

Theorem 4. *Let X be a compact Hausdorff space. The following are equivalent:*
a) *X is a continuous image of an arc,*
b) *If f : X→Y is a continuous surjection and w(Y)≤ $\aleph_1$, then Y is a continuous image of an arc.*

Proof. a)⇒ b) Obvious.
b)⇒ a) If w(X)≤ $\aleph_1$, then, by b) X is a continuous image of an arc since there exists the identity i : X→X and w(X)≤ $\aleph_1$. Let w(X)> $\aleph_1$. The proof consists of several steps.
(i) There exists an $\aleph_1$-directed inverse system $\mathbf{X} = \{X_\alpha, p_{\alpha\beta}, A\}$ *such that* $w(X_\alpha) \leq \aleph_1$ *and X is homeomorphic to lim*$\mathbf{X}$.

By [2, Theorem 2.3.23.] the space X is embeddable in $I^{w(X)}$. We identify the cardinal w(X) with an initial ordinal number Ω, i.e. with the set of all ordinal numbers of the cardinality <w(X). Consider the set A = {α, card(α) = $\aleph_1$} of all subsets of Ω of the cardinality $\aleph_1$ ordered by inclusion. It is obvious that A is $\aleph_1$-directed. For each α we have the cube I^α. It is clear that every I^α is a proper subcube of $I^{w(X)}$ since w(X)> $\aleph_1$. If α is a subset of β, let $P_{\alpha\beta}$ be the natural projection of I^β onto I^α. Arguing as in [2, 2.5.3. Example] we infer that $\mathbf{I} = \{I^\alpha, P_{\alpha\beta}, A\}$ is an inverse system with limit homeomorphic to $I^{w(X)}$. Let $P_\alpha : I^{w(X)} \to I^\alpha$, $\alpha \in$A, be the natural projection. For every $\alpha \in$A put $X_\alpha = P_\alpha(X)$. Every X_α has the weight $\leq \aleph_1$ and is a closed subspace of I^α since X is a closed subset of $I^{w(X)}$. Let p_α be the restriction of P_α on X. We have the inverse system $\mathbf{X} = \{X_\alpha, p_{\alpha\beta}, A\}$ whose limit is homeomorphic to X. Clearly, $\mathbf{X}$ is $\aleph_1$-directed since A is $\aleph_1$-directed.
(ii) The space X is a locally connected continuum.

By b) each X_α is a continuous image of an arc since $w(X_\alpha) \leq \aleph_1$. This means that each X_α is locally connected. Hence, X is a locally connected continuum since $\mathbf{X}$ is $\aleph_1$-directed and thus also σ-directed [3, Theorem 3].
(iii) There exists an $\aleph_1$-directed inverse system $\mathbf{Y} = \{Y_\alpha, q_{\alpha\beta}, A\}$ *of continuous images of arcs such that* $q_{\alpha\beta}$ *are monotone and X is homeomorphic to lim*$\mathbf{Y}$.

Let $\mathbf{X} = \{X_\alpha, p_{\alpha\beta}, A\}$ be as in (i) and let p_α be the natural projection of X onto $X_\alpha \in \mathbf{X}$. Applying the monotone-light factorization [13] to p_α, we get the compact spaces Y_α, monotone surjection $m_\alpha : X \to Y_\alpha$ and the light surjection $l_\alpha : Y_\alpha \to X_\alpha$ such that $p_\alpha = l_\alpha \circ m_\alpha$. By [7, Lemma 8] there exists a monotone surjection $q_{\alpha\beta} : Y_\beta \to Y_\alpha$ such that $q_{\alpha\beta} \circ m_\beta = m_\alpha$, $\alpha \leq \beta$. It follows that $\mathbf{Y} = \{Y_\alpha, q_{\alpha\beta}, A\}$ is an inverse system such that X is homeomorphic to lim$\mathbf{Y}$. Every Y_α is locally connected since X is locally connected. Moreover, by [7, Theorem 1] it follows that $w(Y_\alpha) = w(X_\alpha) \leq \aleph_1$. By b) we infer that every Y_α is a continuous image of an arc. The proof of (iii) is completed.

In the following step we shall represent every cyclic element of X as the limit of some inverse system of cyclic elements of Y_a, a∈A.
(iv) For each nondegenerate cyclic element W of X there exists an $\aleph_1$-directed inverse system $\mathbf{W} = \{W_a, P_{ab}, A^*\}$ *such that* W_a *is a nondegenerate cyclic element of some* X_α, P_{ab} *are monotone and* A^* *is a cofinal subset of A.*

By (iii) X is the limit of $\mathbf{Y} = \{Y_\alpha,\ q_{\alpha\beta},\ A\}$. Let $q_\alpha : X \to Y_\alpha$ be the natural projection. Every $q_\alpha(W)$ is a locally connected continuum because it is the image of the locally connected continuum W [12, p. 70, Lemma 1.5]. Moreover, every $q_\alpha(W)$ is the image of an ordered compactum since every Y_α is the continuous image of an arc. By *Theorem 2.* it follows that every $q_\alpha(W)$ is the continuous image of an arc. It easily follows that $W = \lim\{q_\alpha(W),\ q_{\alpha\beta}|q_\beta(W),\ A\}$. Define $r_{\alpha\beta} = q_{\alpha\beta}|q_\beta(W)$. As in the proof of Theorem 5.1 of [9] we infer that there exists an α_0 in A and a non-degenerate cyclic element W_{α_0} of $q_{\alpha_0}(W)$. Let $A^* = \{\alpha : \alpha \geq \alpha_0\}$. For each $\alpha \geq \alpha_0$ there exists a non-degenerate cyclic element W_α of $q_\alpha(W)$ such that $r_{\alpha_0\alpha}(W_\alpha) \supseteq W_{\alpha_0}$ ([9, Lemma 2.3]) since the restrictions $q_\alpha|W : W \to q_\alpha(W)$ are monotone ([9, Lemma 2.2]). Let $\rho_\alpha : q_\alpha(W) \to W_\alpha$ be the canonical retraction [9, p. 5]. We define $P_{\alpha\beta} = \rho_\alpha \circ r_{\alpha\beta}$ for each pair α, β such that $\alpha_0 \leq \alpha \leq \beta$. As in the proof of Theorem 5.1 of [9, p. 25] it follows that $\{W_\alpha,\ P_{\alpha\beta},\ A^*\}$ is an $\aleph_1$-directed inverse system with monotone bonding mappings $P_{\alpha\beta}$ whose limit is W. The proof of (iv) is complete.

(v) Every non-degenerate cyclic element W of X is a continuous image of an arc.

Let x, y and z be points of W. By (3) of *Theorem 2.* it suffices to prove that there exists a metrizable T-set of W which contains x, y and z. By (iv) W is the limit of $\mathbf{W} = \{W_\alpha,\ P_{\alpha\beta},\ A^*\}$. For each $\alpha \in A^*$ there exists a minimal metrizable T-set T_α containing $x_\alpha = P_\alpha(x)$, $y_\alpha = P_\alpha(y)$ and $z_\alpha = P_\alpha(z)$ (*Theorem 2.*). For every $\alpha \in A^*$ consider the family $\mathcal{T}_\alpha = \{P_{\alpha\beta}(T_\beta) : \beta \geq \alpha\}$. It is clear that $\mathcal{T}_\alpha$ is $\aleph_1$-directed by inclusion. Let us prove that $N_\alpha = \bigcup\{P_{\alpha\beta}(T_\beta) : \beta \geq \alpha\}$ is a compact metrizable space. This follows from the next claim.

Claim 1. Let $\mathcal{M} = \{M_\mu : \mu \in M\}$ be a family of compact metric subspaces M_μ of a space M partially ordered by inclusion $\subseteq$. If it is $\aleph_1$-directed, then $N = \bigcup\{M_\mu : \mu \in M\}$ is a compact metrizable subspace of M.

Suppose that $w(N) \geq \aleph_1$. By virtue of [4] (or [10, Theorem 1.1]), for $\lambda = \aleph_1$, there exists a subspace $N_{\aleph_1}$ of N such that $\mathrm{card}(N_{\aleph_1}) \leq \aleph_1$ and $w(N_{\aleph_1}) \geq \aleph_1$. For each $x \in N_{\aleph_1}$ there exists an $M_\mu(x) \in \mathcal{M}$ such that $x \in M_\mu(x)$. The family $\mathcal{M}_1 = \{M_\mu(x) : x \in N_{\aleph_1}\}$ has the cardinality $\leq \aleph_1$. By the $\aleph_1$-directedness of $\mathcal{M}$ there exists an $M_\nu \in \mathcal{M}$ such that $M_\nu \supseteq M_\mu(x)$ for each $x \in N_{\aleph_1}$. This means that $N_{\aleph_1} \subseteq M_\nu$. We infer that $w(N_{\aleph_1}) \leq \aleph_0$ since M_ν is a compact metric subspace of X. This contradicts the assumption $w(N_{\aleph_1}) \geq \aleph_1$. Hence, $w(N) \leq \aleph_0$. There exists a countable dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of N. For each z_n there is an $M_\mu(n) \in \mathcal{M}$ such that $z_n \in M_\mu(n)$. It is clear that $L = \bigcup\{M_\mu(n) : n \in \mathbb{N}\}$ is dense in N. By virtue of the $\aleph_1$-directedness of $\mathcal{M}$ there exists an $M_\nu \in \mathcal{M}$ such that $M_\nu \supseteq M_\mu(n)$ for each n. We infer that $M_\nu \supseteq L$ and, consequently, M_ν is dense in N. From the compactness of M_μ it follows that $N = M_\mu$. Hence, N is a compact metrizable subspace of M. The proof of *Claim 1* is complete.

It is obvious that the collection $\mathcal{N} = \{N_\alpha,\ P_{\alpha\beta}|N_\beta,\ A^*\}$ is an inverse system. Every N_α is a T-set in W_α [9, Theorem 3.1]. By [9, Theorem 3.13] $N = \lim\mathcal{N}$ is a T-set in W which contains x, y and z. It remains to prove that N is metrizable. This is established by the following *Claim 2.*

Claim 2. Let $\mathbf{Z} = \{Z_a,\ p_{ab},\ A\}$ be an $\aleph_1$-directed inverse system of compact metric spaces Z_a and surjective bonding mappings. Then $Z = \lim\mathbf{Z}$ is a compact metrizable space.

By virtue of *Theorem 3.* it suffices to prove that there exists an a∈A such that p_{ab}: $Z_b \to Z_a$ is a homeomorphism for each b≥a. Suppose that this is not true, i.e. that for each a∈A there exists a b≥a such that p_{ab}: $Z_b \to Z_a$ is not a homeomorphism. Let a_1 be any element of A. By assumption, there exists an a_2 ∈A such that $a_2 \geq a_1$ and $p_{a_1a_2}$: $Z_{a_2} \to Z_{a_1}$ is not a homeomorphism. Suppose that for each ordinal number $\alpha < \beta < \omega_1$ the element a_α is defined. Let us define a_β. If there exists β - 1, then we define a_β so that $p_{a_{\beta-1}a_\beta} : Z_{a_\beta} \to Z_{a_{\beta-1}}$ is not a homeomorphism. If β is a countable limit ordinal, then there exists an a_β such that $a_\beta \geq a_\alpha$ for each $\alpha < \beta$ since **Z** is $\aleph_1$-directed. Now, we have the transfinite sequence $\Omega = \{a_\alpha : \alpha < \omega_1\}$ and a well-ordered inverse system $\mathbf{Z}_\Omega = \{Z_\alpha, p_{\alpha\beta}, \Omega\}$. Let Y = lim$\mathbf{Z}_\Omega$. We shall prove that Y is metrizable. By virtue of the $\aleph_1$-directedness of A there exists an a∈A such that $a \geq a_\alpha$ for each $\alpha < \omega_1$. It is clear that there exists a mapping q : $X_a \to$Y induced by the mappings $P_{a_\alpha a}$. This means that Y is metrizable since X_a is metrizable. By *Theorem 3.* there exists an α_0 such that $p_{a_\beta a_\gamma} : Z_{a_\gamma} \to Z_{a_\beta}$ is a homeomorphism, $\alpha_0 < \beta < \gamma < \omega_1$. This contradicts the construction of $\Omega = \{a_\alpha : \alpha < \omega_1\}$ and the well-ordered inverse system $\mathbf{Z}_\Omega = \{W_\alpha, P_{\alpha\beta}, \Omega\}$. Hence, Z = lim**Z** is a compact metrizable space.

Finally, the proof of (v) is complete.

(vi) X is the continuous image of an arc. This follows from (v) and *Theorem 1.* □

Corollary 1. *Let X be a locally connected continuum. The following are equivalent:*
a) *X is a continuous image of an arc,*
b) *If f : Z→Y is a continuous surjection, where Z is a cyclic element of X and w(Y)≤ $\aleph_1$, then Y is a continuous image of an arc.*

Proof. a)⇒ b) If X is a continuous image of an arc, then every cyclic element Z of X is a continuous image of an arc (*Theorem 1.*). We infer that Y is a continuous image of an arc since there exists a surjection f : Z→Y.

b)⇒ a) Let Z be any cyclic element of X. Applying *Theorem 4.* for Z, we infer that Z is a continuous image of an arc. By *Theorem 1.* we infer that X is a continuous image of an arc since every cyclic element Z of X is a continuous image of an arc. □

A space X is said to be *rim-finite* (*rim-countable*) if it has a basis $\mathcal{B}$ such that card(Bd(U))< $\aleph_0$ (card(Bd(U))≤ $\aleph_0$) for each U∈ $\mathcal{B}$. Equivalently, a space X is rim-finite (rim-countable) if and only if for each pair F,G of disjoint closed subsets of X there exist a finite (countable) subset of X which separates F and G. This follows from the fact that if $\{A_s\}$ is a locally finite family of subsets of X, then [2, p. 46]

$$Bd(\bigcup A_s) \subseteq \bigcup Bd(A_s).$$

Every rim-finite continuum is a continuous image of an arc [11]. Hence, every rim-finite continuum is locally connected and hereditarily locally connected.

Lemma 2. *Let f : X→Y be a monotone surjection. If X is rim-finite (rim-countable), then Y is rim-finite (rim-countable).*

Theorem 5. [9, Theorem 9.9] *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a σ-directed inverse system of rim-finite continua. Then* $X = lim\mathbf{X}$ *is rim-finite.*

Now we are ready to prove the following characterization theorem for rim-finite continua.

Theorem 6. *Let X be a continuum. The following are equivalent:*

a) *X is rim-finite,*

b) *If f : X→Y is a monotone surjection and if Y is metrizable, then Y is rim-finite.*

Proof. a)⇒b). Apply *Lemma 2.*

b)⇒a). By virtue of [9, Theorem 9.5] there exists a σ-directed monotone inverse system $\mathbf{X} = \{X_\alpha, p_{\alpha\beta}, A\}$ such that $w(X_\alpha) \leq \aleph_0$ and X is homeomorphic to lim**X**. From b) it follows that each X_a is rim-finite. By *Theorem 5.* we infer that X is rim-finite. □

3. Applications

In this section some applications of *Theorems 4.* and *6.* are given.

Theorem 7. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate* $\aleph_1$*-directed inverse system of continuous images of arcs. Then X = lim***X** *is the continuous image of an arc.*

Proof. By *Theorem 4.* it suffices to prove that if f : X→Y is a continuous surjection and $w(Y) \leq \aleph_1$, then Y is a continuous image of an arc. Using *Lemma 1.*, for $\tau = \aleph_1$, we will find an a∈A and a continuous surjection $g_a : X_a \to Y$ such that $f = g_a p_a$. Hence, Y is a continuous image of an arc since X_a is a continuous image of an arc. By *Theorem 4.* we infer that X is a continuous image of an arc. □

Corollary 2. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an* $\aleph_1$*-directed inverse system of continuous images of arcs. Then X = lim***X** *is the continuous image of an arc.*

Remark 1. *Let us observe that mappings* p_{ab} *in* Theorem 7. *and* Corollary 2. *are not necessarily monotone.*

Theorem 8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate* σ*-directed inverse system of rim-finite continua. Then X = lim***X** *is a rim-finite continuum.*

Proof. By *Theorem 6.* it suffices to prove that if f: X→Y is a monotone surjection onto a metrizable space Y, then Y is rim-finite. By *Lemma 1.* there exists an a∈A and a continuous mapping $g_a : X_a \to Y$ such that $f = g_a p_a$. It follows that g_a is monotone since f is monotone. From *Lemma 2.* it follows that Y is rim-finite. Hence, X is rim-finite (*Theorem 6.*). □

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