# Intersection properties of Brownian paths

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**Abstract**. *This review presents a modern approach to intersections of Brownian paths. It exploits the fundamental link between intersection properties and percolation processes on trees. More precisely, a Brownians path is intersect-equivalent to certain fractal percolation. It means that the intersection probabilities of Brownian paths can be estimated up to constant factors by survival probabilities of certain branching processes.*

**Key words:** *Brownian motion, stable processes, fractal percolation, intersect-equivalence, potential theory*

**AMS subject classifications:** 60J45, 60G17

Received March 14, 2000 Accepted May 11, 2000

## 1. Main results

In this review we present a modern proof due to ([13]) of Dvoretzky, Erdös, Kakutani and Taylor' classical results on intersections of Brownian paths ([4], [5], [8]). M. Aizenman ([1]) suggested that intersections of Brownian paths and percolation processes on trees should be closely related. However, he pointed out that attempting to establish a direct probabilistic link between the two settings runs into delicate dependence problems. The potential theory serves as a bridge in latter papers. In particular, the long-range intersection probabilities of Brownian paths can be estimated up to constant factors by survival probabilities of certain branching processes.

**Definition 1.** *Two random (Borel) sets $A$ and $B$ are* **intersect-equivalent** *on the open set $U$, if for any closed set $\Lambda \subset U$, we have*

$$\mathbf{P}(A \cap \Lambda \neq \emptyset) \asymp \mathbf{P}(B \cap \Lambda \neq \emptyset),$$

*i.e. the ratio of both sides is bounded above and below by positive constants which do not depend on $\Lambda$.*

**Fractal percolation.** Given $d \geq 3$ and $0 < p < 1$, consider the natural tiling of the unit cube $[0,1]^d$, by $2^d$ closed cubes of side $\frac{1}{2}$. Let $Z_1$ be a random subcollection

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of these cubes, where each cube belongs to $Z_1$ with probability $p$ and these events are mutually independent. In general, if $Z_k$ is a collection of cubes of sides $2^{-k}$, tile each cube $Q \in Z_k$ by $2^d$ closed cubes of side $2^{-k-1}$ and include each of these subcubes in $Z_{k+1}$ with probability $p$ (independently). Finally, define

$$Q_d(p) = \bigcap_{k=1}^{\infty} \cup_{Q \in Z_k} Q.$$

**Theorem 1.** *Let $B_d(t)$ denote $d-$dimensional Brownian Motion, started according to any fixed distribution with a bounded density for $B_d(0)$.*

*(i) If $d \geq 3$, then the range $[B_d] = (B_d(t) : t \geq 0)$ is intersect-equivalent to $Q_d(2^{2-d})$ in the unit cube.*

*(ii) Let $S(t)$ be the symmetric stable process of index $\alpha$, started according to any distribution with a bounded density. If $\alpha < d$, then the range $[S]$ is intersect-equivalent to $Q_d(2^{\alpha-d})$ in the unit cube.*

A proof of *Theorem 1* will be presented below. Our present goal is to derive the following famous result.

**Theorem 2. (Dvoretzky, Erdös, Kakutani and Taylor [4], [5], [8])** .

*(i) For any $d \geq 4$, two independent BM in $\mathbf{R}^d$ are disjoint a.s.*

*(ii) In $\mathbf{R}^3$, two independent BM intersect a.s., but three independent BM have no points of mutual intersection.*

*(iii) In $\mathbf{R}^2$, any finite number of independent BM have non-empty mutual intersection a.s.*

**Proof.** (i) It sufficies to consider $d = 4$ and check, that two independent BM $[B_4]$ and $[B_4']$ a.s. have no points of intersection in the unit cube, since countably many cubes cover $\mathbf{R}^4$. We use the following

**Lemma 1.** *Suppose that $A_1, \ldots, A_k, F_1, \ldots, F_k$ are independent random (Borel) sets, with $A_i$ intersect-equivalent to $F_i$ for all $1 \leq i \leq k$. Then $A_1 \cap A_2 \cap \ldots \cap A_k$ is intersect-equivalent to $F_1 \cap F_2 \cap \ldots \cap F_k$.*

**Proof.** By induction reduce to the case $k = 2$ It clearly suffices to show that $A_1 \cap A_2$ is intersect-equivalent to $F_1 \cap A_2$ :

$$\begin{aligned} \mathbf{P}(A_1 \cap A_2 \cap \Lambda \neq \emptyset) = \quad & \mathbf{E}[\mathbf{P}(A_1 \cap A_2 \cap \Lambda \neq \emptyset \mid A_2)] = \mathbf{E}[\mathbf{P}(F_1 \cap A_2 \cap \Lambda \neq \emptyset \mid A_2)] \\ = \quad & \mathbf{P}(F_1 \cap A_2 \cap \Lambda \neq \emptyset). \end{aligned}$$

□

Now observe that 1) for any $0 < p, q < 1$, if $Q_d(p)$ and $Q_d'(q)$ are statistically independent, then their intersection $Q_d(p) \cap Q_d'(q)$ has the same distribution as $Q_d(pq)$; 2) the cardinalities $|Z_k|$ of $Z_k$ form a *Galton-Watson branching process* which extincts a.s. in the critical case $\mathbf{E}|Z_1| = 1$.

For any $\epsilon > 0$ the distribution of $B_4(\epsilon)$ has a bounded density, so by *Theorem 1* and *Lemma 1*

$$\begin{aligned}\mathbf{P}(B_4(t): t \geq \epsilon) \cap (B_4'(s): s \geq \epsilon) \cap [0,1]^4 \neq \emptyset) &\asymp \mathbf{P}(Q_4(1/4) \cap Q_4'(1/4) \neq \emptyset) \\ &= \mathbf{P}(Q_4(1/16) \neq \emptyset)\end{aligned}$$

But $Q_4(1/16) = \emptyset$ a.s. because critical branching processes die out. Similar arguments provide a proof of (ii). □

## 2. Potential theory background

We need some basic facts of the classical potential theory to proceed with the proof of *Theorem 1*.

$K-$**capacity** Let $\Lambda-$ be a metric space with the metric $|x-y|$ and $K : \Lambda \times \Lambda \to [0,\infty)-$ be a Borel function. Define $K-$energy of a finite Borel measure $\mu$ on $\Lambda$ by

$$I_K(\mu) = \int_\Lambda \int_\Lambda K(x,y) d\mu(x) d\mu(y),$$

In the particular case $K(x,y) = f(|x-y|)$, where $f$ is a non-increasing function we use the notation $I_f(\mu)$; if $f = |x-y|^{-\beta}$ then

$$I_\beta(\mu) = \int_\Lambda \int_\Lambda |x-y|^{-\beta} d\mu(x) d\mu(y).$$

Define $K-$capacity ($f-$capacity, $\beta-$capacity) by

$$Cap_K(\Lambda) = [inf_\mu I_K(\mu)]^{-1}, \qquad Cap_\beta(\Lambda) = [inf_\mu I_\beta(\mu)]^{-1}$$

where the infimum is over probability measures $\mu$ on $\Lambda$.

It is well-known ([6]) that the range of $d-$dimensional Brownian motion, $d \geq 3$, has Hausdorff dimension 2. This fact admits a nice interpretation in viewpoint of fractal percolation. We slightly generalize the construction as above: let $l \geq 2$ and $(q_k, 0 \leq k \leq l^d)$ be a probabilistic distribution with mean value $M$. Consider the natural tiling of the unit cube $[0,1]^d$, by $l^d$ closed cubes of side $\frac{1}{l}$. Select $k$ small cubes with probability $q_k$ (their location is not relevant) and iterate this procedure. This recursive construction defines a fractal with Hausdorff dimension $dim_H(\Lambda) = \log_b M$ a.s. ([7]). In the case of Bernoulli percolation (cf. *Theorem 1*) $M = p2^d, p = 2^{2-d}$ and $dim_H(\Lambda) = \log_2 p2^d = 2$.

The following classical theorem characterizes the Hausdorff dimension as the critical parameter for positivity of Riesz-type capacity.

**Theorem 3. (Frostman, 1935)** *For any Borel set $\Lambda$ in $\mathbf{R}^d$, the Hausdorff dimension $dim_H(\Lambda)$ is exactly $inf[\beta > 0 : Cap_\beta(\Lambda) = 0]$.*

**Theorem 4. (Hunt and Doob after Kakutani, 1944)** *Let $(S_t)-$ be a symmetric stable process of index $\alpha < d$ in $\mathbf{R}^d$, and the initial distribution $\pi$ has a bounded density on the unit cube, then*

$$\mathbf{P}_\pi(\exists t \geq 0 : S_t \in \Lambda) \asymp Cap_{d-\alpha}(\Lambda)$$

**Proof.** There exists a finite measure $\nu$ on $\Lambda$, such that $\forall x$

$$\mathbf{P}_x(\exists t \geq 0 : S_t \in \Lambda) = \int_\Lambda G(x,y) d\nu(y)$$

and

$$\nu(A) = Cap_G(\Lambda) = Cap_{d-\alpha}(\Lambda).$$

In this case $G(x,y) = |x-y|^{\alpha-d}$ and straightforward integration yields

$$C_1 Cap_G(\Lambda) \leq \mathbf{P}_\pi(\exists t \geq 0 : S_t \in \Lambda) \leq C_2 Cap_G(\Lambda).$$

□

## 3. Independent percolation on trees

The second cornerstone of the proof is a fundamental result of ([11]) concerning percolation on trees.

Let $T-$ be a finite or infinite rooted tree; $\partial T$ be its **boundary**, i.e. the set of maximal self-avoiding paths emanated from the roof $\rho$ of $T$ and called **rays**. The distance between two (infinite) rays $\xi$ and $\eta$ is defined to be $|\xi - \eta| = 2^{-\kappa}$ where $\kappa = \kappa(\xi, \eta) = |\xi \wedge \eta|$ is the number of edges that these two rays have in common. Here $\xi \wedge \eta$ is the edge farthest from the root which is common to both $\xi$ and $\eta$ (or the path from the root to this edge). In analogy with $\beta-$capacity we define

$$Cap_\beta(\partial T) = [inf_\mu I_\beta(\mu)]^{-1}$$

where

$$I_\beta(\mu) = \int\int 2^{\beta\kappa(\xi,\eta)} d\mu(\xi) d\mu(\eta).$$

Let $0 < p < 1$. We say that a path $\xi$ *survives the percolation* with parameter $p$ if all the edges on $\xi$ are retained (each edge of $T$ is retained with probability $p$ and deleted with probability $1-p$ independently). We say that the tree boundary $\partial T$ survives if some ray on $T$ survives the percolation.

**Theorem 5. ([11])** *Let $\beta > 0$. If percolation with parameter $p = 2^{-\beta}$ is performed on a rooted tree $T$, then*

$$Cap_\beta(\partial T) \leq \mathbf{P}[\partial T \text{ survives the percolation}] \leq 2 Cap_\beta(\partial T)$$

**Theorem 5'. ([11])** *In the model with different surviving probabilities $p_e$ for different edges we define*

$$K(x,y) = \prod p_e^{-1} : e \in x \wedge y.$$

*Then*

$$Cap_F(\partial T) \leq \mathbf{P}[\partial T \text{ survives the percolation}] \leq 2 Cap_F(\partial T)$$

*Sketch of original proof of* Theorem 5. The relations between random walks, electrical networks and percolation on trees are well-known ([10]). In particular, the conductance of an edge $\sigma$ $C_\sigma = (1-p)^{-1} p^{|\sigma|}$ where $|\sigma|$ is the number of edges between $\sigma$ and the root and $p$ is the percolation probability. One can easily check ([11]) that

$$Cap_\beta(\partial T) = [1 + \mathcal{G}(0 \to \partial T)^{-1}]^{-1},$$

where $\mathcal{G}(0 \to \partial T)$ is the effective conductance of electrical network between the root and $\partial T$. The proof of *Theorem 5* follows from the following estimate.

**Lemma 2.** *For any finite tree $T$*

$$\frac{\mathcal{G}(0 \to \partial T)}{1 + \mathcal{G}(0 \to \partial T)} \leq \mathbf{P}[\partial T \text{ survives the percolation}] \leq 2\frac{\mathcal{G}(0 \to \partial T)}{1 + \mathcal{G}(0 \to \partial T)}.$$

**Proof.** One can easily deduce these inequalities from the usual series-parallel circuit laws

$$\mathcal{G}(0 \to \partial T) = \sum_{|\sigma|=1} (C_\sigma^{-1} + \mathcal{G}(\sigma \to \partial T)^{-1})^{-1},$$

where $\mathcal{G}(\sigma \to \partial T)$ the effective conductance of electrical network between $\sigma$ and $\partial T$. □

A general estimate of capacities for a Markov chain on countable state space yields a short proof of *Theorem 5* and *5'*.

**Theorem 6. ([3])** *Let $X$ be a transient Markov chain on the countable state space $Y$ with initial state $\rho$ and transitional probabilities $p(x, y)$. Let*

$$G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y)$$

*be the Green function. Define the kernal $F(x, y) = K(x, y) + K(y, x), K(x, y) = \frac{G(x,y)}{G(\rho,y)}$, and the average Green function $G(\rho, y)$ with respect to initial state $\rho$. Then for any $\Lambda \subset Y$*

$$Cap_F(\Lambda) \leq \mathbf{P}_\rho(\exists n \geq 0 : X_n \in \Lambda) \leq 2Cap_F(\Lambda).$$

*Proof of Theorem 5'.* The result follows from similar estimates on finite trees. We construct a Markov chain on $\partial T \cup \rho, \delta$ where $\rho$ is the root and $\delta$ is a formal absorbing cementry. Indeed, enumerate all leaves on $T$ that survive the percolation from left to right as $V_1, V_2, \ldots, V_r$. The key observation is that the random sequence $\rho, V_1, V_2, \ldots, V_r, \delta, \delta, \ldots$ is a Markov chain. Indeed, given that $V_k = x$ conditional probabilities that parths on the right of $x$ survive the percolation do not depend on $V_1, \ldots, V_{k-1}$. One can easily check that $G(\rho, y) = \prod_{e \in y} p_e$ and, if $x$ is to the left of $y$, then

$$G(x, y) = \prod_{e \in y \setminus x} p_e.$$

This equality yields that

$$K(x, y) = \frac{G(x, y)}{G(\rho, y)} = \prod_{e \in y \wedge x} p_e^{-1}.$$

□

*Proof of* Theorem 6. (i) Let $\tau$ be the first hitting time of $\Lambda$ and $\nu(x) = \mathbf{P}_\rho[X_\tau = x]$. Then

$$\nu(\Lambda) = \mathbf{P}_\rho(\exists n \geq 0 : X_n \in \Lambda).$$

Observe that $\forall y \in \Lambda$

$$\int G(x,y)d\nu(x) = \sum_{x\in\Lambda} \mathbf{P}_\rho[X_\tau = x]G(x,y) = G(\rho,y).$$

Hence $\int K(x,y)d\nu(x) = 1$ and

$$I_F(\frac{\nu}{\nu(\Lambda)}) = \frac{2}{\nu(\Lambda)}.$$

Consequently $\nu(\Lambda) \leq Cap_F(\Lambda)$, this proves the right-hand side inequality.
(ii) Let $\mu$ be a probability measure on $\Lambda$. Consider the random variable

$$Z = \int_\Lambda G(\rho,y)^{-1} \sum_{n=0}^{\infty} \mathbf{1}_{X_n=y} d\mu(y).$$

By Cauchy-Schwartz inequality

$$\mathbf{P}_\rho(\exists n \geq 0 : X_n \in \Lambda) \geq \mathbf{P}_\rho(Z > 0) \geq \frac{(\mathbf{E}_\rho Z)^2}{\mathbf{E}_\rho Z^2}.$$

One can easily check that $\mathbf{E}_\rho Z = 1$, hence the left-hand side inequality follows from the following estimate $\mathbf{E}_\rho Z^2 \leq I_F(\mu)$. Let us check that

$$\mathbf{E}_\rho Z^2 \leq 2\int_\Lambda\int_\Lambda G(\rho,y)^{-1}G(\rho,x)^{-1}\Sigma_\rho d\mu(x)d\mu(y),$$

$$\Sigma_\rho = \sum_m \mathbf{E}_\rho[\sum_{n=m}^{\infty} \mathbf{1}_{X_m=x,X_n=y}] = G(\rho,x)G(x,y).$$

Hence

$$\mathbf{E}_\rho Z^2 \leq 2\int_\Lambda\int_\Lambda G(\rho,y)^{-1}G(x,y)d\mu(x)d\mu(y) = I_F(\mu).$$

□

Next we define a canonical map $\mathcal{R}$ from the boundary of $2^d-$ary (each vertex has $2^d$ children) tree $\mathcal{T}^d$ to the cube $[0,1]^d$. Formally, label the edges from each vertex to its children with the vectors in $\Omega^{\mathbf{Z}_+} = (0,1)^d$. Then define

$$\mathcal{R}(\omega_1,\omega_2,\ldots) = \sum_{i=1}^{\infty} 2^{-n}\omega_n.$$

Similarly, a vertex $\sigma \in \mathcal{T}^d, |\sigma| = k$ is identified with a finite sequence $\Omega_k = (\omega_1,\omega_2,\ldots,\omega_k)$. Let $\mathcal{R}(\sigma)$ be the cube with the side $2^{-k}$ containing the images under the mapping $\mathcal{R}$ of all sequences with the prefix $\Omega_k$.

**Theorem 7. ([2],[14])** *Let $T$ be a subtree of the regular $2^d-$ary tree $\mathcal{T}^d$. Then*

$$Cap_\beta(\partial T) \asymp Cap_\beta(\mathcal{R}(\partial T))$$

.

**Proof.** We shall check that for $f(n) = g(2^{-n})$ and any probability measure $\mu$ on $\partial T$

$$I_f(\mu) \asymp I_g(\mu\mathcal{R}^{-1}).$$

**Step 1. Computation of energy**

$$\begin{aligned} I_f(\mu) = & \int\int f(|x \wedge y|)d\mu(x)d\mu(y) = \int\int \sum_{\sigma \leq x\wedge y} [f(|\sigma|) - f(|\sigma| - 1)]d\mu(x)d\mu(y) \\ = & \sum_{\sigma\in T} [f(|\sigma|) - f(|\sigma| - 1)] \int\int \mathbf{1}_{x,y\geq\sigma} d\mu(x)d\mu(y) \\ = & \sum_{\sigma\in T} [f(|\sigma|) - f(|\sigma| - 1)]\mu(\sigma)^2 \\ = & \sum_{k=1}^{\infty} h(k)S_k(\mu). \end{aligned}$$

Here $\mu(\sigma) = \mu y \in \partial T : \sigma \in y$, $h(k) = f(k) - f(k-1), f(-1) = 0$, $S_k(\mu) = \sum_{|\sigma|=k} \mu(\sigma)^2$.

**Step 2. Estimate from above**

$$I_g(\mu\mathcal{R}^{-1}) \leq \sum_{k=1}^{\infty} h(k)\mathcal{V}(k),$$

here

$$\mathcal{V}(k) = (\mu\mathcal{R}^{-1}) \times \mu\mathcal{R}^{-1})[(x,y) : |x-y| \leq 2^{1-k}].$$

Next we check that $\mathcal{V}(k) \leq 6^d S(k)$. Indeed, let

$$\mathbf{I} = \mathbf{I}_{\mathcal{R}(\sigma)\cap\mathcal{R}(\tau)\neq\emptyset}, A(k-1) = (\sigma,\tau) : |\sigma| = |\tau| = k-1, \mathbf{I} > 0.)$$

If $|x-y| \leq 2^{1-k}, x, y \in \mathcal{R}(\partial T)$, then

$$\exists(\sigma,\tau) \in A(k-1) : x \in \mathcal{R}(\sigma), y \in \mathcal{R}(\tau).$$

Therefore

$$\mathcal{V}(k) \leq \sum_{A(k-1)} \theta(\sigma)\theta(\tau).$$

Using the estimate

$$\theta(\sigma)\theta(\tau) \leq \frac{\theta(\sigma)^2 + \theta(\tau)^2}{2}$$

and observing that the number of $\sigma$ for any fixed $\tau$ (and the number of $\tau$ for any fixed $\sigma$) in $A(k-1)$ is bounded from above by $3^d$, we get $\mathcal{V}(k) \leq 3^d S_{k-1}$. Finally, we can easily check that $S_{k-1} \leq 2^d S_k : \forall |\sigma| = k-1$

$$\theta(\sigma)^2 = (\sum_{\tau\geq\sigma,|\tau|=k} \theta(\tau))^2 \leq 2^d \sum_{\tau\geq\sigma,|\tau|=k} \theta(\tau)^2.$$

**Step 3. Estimate from below**

$$I_g(\mu\mathcal{R}^{-1}) \geq \sum_{k=1}^{\infty} h(k) S_{k+l}(\mu),$$

where $2^l \geq d^{\frac{1}{2}}$. Therefore

$$(x, y : |x-y| \leq 2^{-n}) \supseteq \cup_{|\sigma|=n+l}[\mathcal{R}(\sigma) \times \mathcal{R}(\sigma)].$$

Finally, observe that $S_k \geq 2^{-d} S_{k-1}$, yields the inequality

$$I_g(\mu\mathcal{R}^{-1}) \geq 2^{-dl} I_f(\mu).$$

□

**Corollary 1.** *For any closed set* $\Lambda$ *in the cube* $[0,1]^d$

$$\mathbf{P}(Q_d(2^{-\beta}) \cap \Lambda \neq \emptyset) \asymp Cap_\beta(\Lambda).$$

**Proof.** Any closed set $\Lambda$ is the image of the boundary $\mathcal{R}(\partial T)$ of a subtree imbedded into the regular $2^d-$ary tree $\mathcal{T}^d$. Consider a percolation with parameter $p = 2^{-\beta}$. Then

$$\mathbf{P}[Q_d(p) \text{ intersect } \Lambda] = \mathbf{P}[\partial T \text{ survives the percolation }] \asymp Cap_\beta(\partial T) \asymp Cap_\beta(\Lambda).$$

□

**Corollary 2. ([7],[10])** *Let* $p = 2^{-\beta}$. *For any (Borel) set* $\Lambda \subset [0,1]^d$

*(i) If* $dim_H(\Lambda) < \beta$, *then the intersection* $Q_d(p) \cap \Lambda$ *is a.s. empty.*

*(ii) If* $dim_H(\Lambda) > \beta$, *then* $\Lambda$ *intersects* $Q_d(p)$ *with positive probability.*

**Proof.** It follows immediately from *Corollary 1* and *Theorem 3* connecting Haudorff dimension and capacity. □

**Proof of *Theorem 1.*** We check (ii) because (i) is its special case $\alpha = 2$. *Theorem 4* and *Corollary 1* imply that for $p = 2^{\alpha-d}$

$$\mathbf{P}_\pi(\exists t \geq 0 : S_t \in \Lambda) \asymp Cap_{d-\alpha}(\Lambda) \asymp \mathbf{P}[Q_d(p) \text{ intersect } \Lambda].$$

□

## 4. Capacity of Brownian paths

We have mentioned in *Section 2* that the image of $d-$dimensional Brownian motion, $d \geq 3$, has Hausdorff dimension 2. A more precise version of this result was recently proved ([15]).

**Theorem 8.** *For* $d \geq 3$, *the Brownian trace* $B[0,1]$ *is a.s. capacity-equivalent* $[0,1]^2$, *i.e. with probability 1* $\exists$ *random constants* $C_1, C_2 > 0$ *such that*

$$C_1 Cap_f([0,1]^2) \leq Cap_f(B[0,1]) \leq C_2 Cap_f([0,1]^2)$$

*for all non-increasing functions* $f$ *simultaneously.*

**Proof.** Let $\mathcal{D}_n$ be a partition of $[0,1]^2$ on dyadic cubes with a side $2^{-n}$ and $N_n(\Lambda)-$ be a number of dyadic cubes $Q \in \mathcal{D}_n$ that intersect a random (Borel) set $\Lambda$. We use the strong law of large numbers ([9])

$$C_1 \leq \frac{N_n(B[0,1])}{4^n} \leq C_2, C_1, C_2 > 0.$$

Using the expression for $I_f(\mu)$ (cf. Step 1 in the proof of *Theorem 6*) one can easily check that for any measure $\mu$ supported by the random set $\Lambda$

$$\begin{aligned} I_f(\mu) &\asymp \sum_{n=0}^{\infty}(f(2^{-n}) - f(2^{1-n}))\sum Q \in \mathcal{D}_n \mu(Q)^2 \\ &\geq c\sum_{n=0}^{\infty}(f(2^{-n}) - f(2^{1-n}))N_n(\Lambda)^{-1}, \end{aligned}$$

i.e.

$$Cap_f(B[0,1]) \leq c^{-1}[\sum_{n=0}^{\infty}(f(2^{-n}) - f(2^{1-n}))N_n(\Lambda)^{-1}]^{-1},$$

Moreover, this estimate is sharp (up to a constant factor independent of $f$) if the set $\Lambda$ carries a positive measure $\mu$ such that $\mu(Q) \leq cN_n(\Lambda)^{-1}$. Finally, we use the strong law of large numbers cited above and observe that

$$Cap_f([0,1]^2) \asymp [\int_0^1 f(r)rdr]^{-1}.$$

□

Finally, we present estimates of hitting probabilities for Brownian motion (cf. *Theorem 6*).

**Theorem 9. (([3]))** *Let $B_d(t), d \geq 3$, denote standard Brownian motion with $B_d(0) = 0$ and $\Lambda \subset \mathbf{R}^d$ is a closed set. Then*

$$Cap_F(\Lambda) \leq \mathbf{P}(\exists t > 0 : B_d(t) \in \Lambda) \leq 2Cap_F(\Lambda),$$

*where $F(x,y) = \frac{|y|^{d-2}}{|x-y|^{d-2}} + \frac{|x|^{d-2}}{|x-y|^{d-2}}$ and $|x-y|$ is the Euclidean distance.*

**Proof.** Proof follows the scheme of that for *Theorem 6*. Let $\tau = min[t > 0 : B_d(t) \in \Lambda]$ and

$$\nu(\Lambda) = \mathbf{P}(\tau < \infty) = \mathbf{P}(\exists t \geq 0 : B_d(t) \in \Lambda).$$

Now recall the standard formula, valid when $0 < \epsilon < |y|$ :

$$\mathbf{P}[|B_d(t) - y| < \epsilon] = \frac{\epsilon^{d-2}}{|y|^{d-2}}.$$

This probability is bounded from below by

$$\mathbf{P}[|B_d(\tau) - y| > \epsilon \text{and} \exists t > \tau : B_d(t) - y| < \epsilon] = \int_{x:|x-y|\geq\epsilon} \frac{\epsilon^{d-2}}{|y|^{d-2}} d\nu(x).$$

This inequality implies

$$\int_\Lambda \frac{d\nu(x)}{|x-y|^{d-2}} \leq \frac{1}{|y|^{d-2}}$$

and an upper bound (cf. *Theorem 6*)

$$2Cap_F(\Lambda) \geq \nu(\Lambda).$$

To prove a lower bound, a second order estimate is used. Given a probability measure $\mu$ on $\Lambda$ and $\epsilon > 0$, consider the random variable

$$Z_\epsilon = \int_\Lambda \mathbf{1}_{\exists t \geq 0: B_d(t) \in D(y,\epsilon)} h_\epsilon(|y|)^{-1} d\nu(x) d\mu(y).$$

Here $D(y,\epsilon)$ is the Euclidean ball of radius $\epsilon$ and $h_\epsilon(r) = (\frac{\epsilon}{r})^{d-2}$ if $r > \epsilon$ and 1 otherwise. Clearly, $\mathbf{E}Z_\epsilon = 1$ and the result follows (cf. *Theorem 6*) from the estimate

$$\lim_{\epsilon \to 0} \mathbf{E}Z_\epsilon \leq I_F(\mu).$$

This is a straightforward calculation which we omit for the sake of brevity. □

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