Heap – ternary algebraic structure*

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Abstract. *In this paper some classes of ternary algebraic structures (semi-heaps, heaps) are considered. The connection between heaps (laterally commutative heaps) and corresponding algebraic and geometric structures is presented. The equivalence of heap existence and the Desargues system on the same set is directly proved. It is the starting point for an analogous result about a laterally commutative heap and a parallelogram space.*

Key words: *semi-heap, heap, ternary operation, Desargues system, parallelogram space*

1. Introduction

The first traces of the algebraic heap theory can be found in the works of K. Suškevič (Teorija obobščennih grupp, 1937), H. Prüfer (Theorie der Abelschen Gruppen, 1924), and the serious research of heaps, semi-heaps and generalised heaps was done in a number of works by V.V. Vagner.

The motivation for the reaserch of such a ternary structure is the impossibility of defining the binary operations on the set of all binary relations between various sets A and B, where it is possible to define a ternary operation in a natural way. The ternary structure in question is later, providing it fulfills certain conditions, proved to be in a close relationship with groups and involutory semi-groups. The later research of algebraic and geometric structures such as groups, right solvable Ward groupoids, subtractive groupoids, medial Mal′cev functions, TST-spaces, showed a close connection of those structures and heaps.

2. Semi - heap

This section deals with a ternary algebraic structure, a semi-heap, and a connection between semi-heaps, and semi-groups with involution.

Semi-heap $(Q, [\ \ ])$ is a nonvoid set Q with ternary operation $[\ \ ] : Q^3 \to Q$ which satisfies the law of quasiassociativity, i.e. $[[abc]de] = [a[dcb]e] = [ab[cde]]$.

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Example 1. *Let Q be a set of all binary relations between elements of sets A and B ($Q = \mathcal{P}(A\times B)$). If a ternary operation $[\ \ ]$ on Q is defined by $[\rho_1\rho_2\rho_3] = \rho_3\circ\rho_2^{-1}\circ\rho_1$ ($\circ$ is the composition of relations), then $(Q, [\ \ ])$ is a semi-heap.*

Example 2. *The Cartesian product $\mathcal{P}(A \times B)\times\mathcal{P}(B \times A)$ is a semi-heap with respect to ternary operation $[(\rho_1, \sigma_1)(\rho_2, \sigma_2)(\rho_3, \sigma_3)] = \rho_3 \circ \sigma_2 \circ \rho_1, \sigma_1 \circ \rho_2 \circ \sigma_3$.*

A semi-heap $(Q, [\ \ ])$ is said to be laterally commutative if the identity $[abc] = [cba]$ holds.

Example 3. *Let Q be a set of all points in the affine plane and $[\ \ ] : Q^3 \to Q$ a ternary operation defined by the equivalence $[abc] = d \Leftrightarrow d$ is the fourth vertex of a parallelogram whose vertices are described cyclicaly as a, b, c and d.*

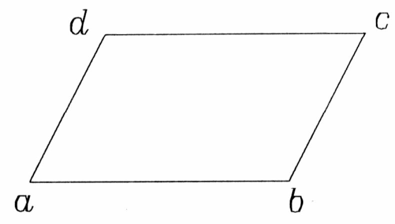


Figure 1.

The drawing illustrates the fulfillment of the law of quasiassociativity.

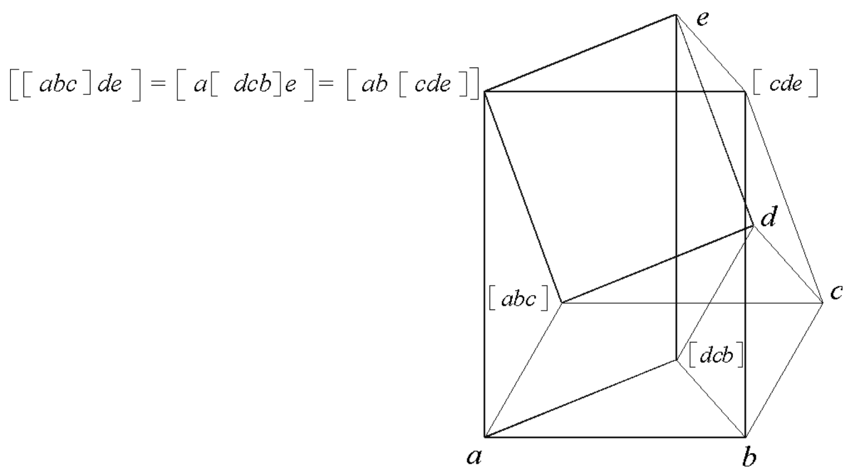


Figure 2.

This important example will be denoted by $(Q, [\ \ ]_p)$ whenever it appears in this work. $(Q, [\ \ ]_p)$ is a laterally commutative semi-heap.

There is a close connection between a semi-heap and a semi group with involution. A semi-group $(Q, \cdot)$ is said to be a semi-group with involution if there exists an involutory mapping $i : Q \to Q$ which is an antiautomorphism, i.e. $i^2 = id,\ i(ab) = i(b)i(a)$.

Obviously, there is a semi-group which is not involutory. The existence of an antiautomorphism sets a number of conditions on this semi-group.

Theorem 1. *([8]) If $(Q,\cdot)$ is a semi-group with involution and ternary operation $[\ \ ]$ on Q defined by $[abc]=ai(b)c$, then $(Q,[\ \ ])$ is a semi-heap.*

If we want to observe a converse, we must define a biunitary element of a semi-heap. Let $(Q,\cdot)$ be a semi-heap. An element $e\in Q$ is said to be biunitary in Q if identity $[eea]=[aee]=a$ holds, for all $a\in Q$. The following theorem holds.

Theorem 2. *([8]) Let $(Q,[\ \ ])$ be a semi-heap with biunitary element e and $\cdot : Q^2\to Q$ a binary operation defined by $ab=[aeb]$, and $i:Q\to Q$ involutory mapping defined by $i(a)=[eae]$. Then $(Q,\cdot)$ is a semi-group with involution and $[abc]=ai(b)c$.*

Semi-groups with involution obtained from a semi-heap by fixing its various elements are isomorphic.

3. Heap

The previous section dealt with a semi-heap, a set with ternary operation in which the law of quasiassociativity is satisfied. Now, we shall define a more complex ternary algebraic structure.

Definition 1. *A heap is a semi-heap if all its elements are biunitary.*

In [8] V.V. Vagner proved the following theorem.

Theorem 3. *A nonvoid set Q with ternary operation $[\ \ ]:Q^3\to Q$ which satisfies*

(Q1) $$[[abc]de]=[ab[cde]]$$

(Q2) $$[abb]=[bba]=a$$

for all $a,b,c,d,e\in Q$ is a heap.

As a consequence of the previous theorem it follows that we can define a heap as a nonvoid set with a ternary operation which satisfies (Q1), (Q2).

Example 4. *A semi-heap $(Q,[\ \ ])$ in* Example 1 *is not a heap. $\rho_2\circ\rho_2^{-1}$ is not always an identical relation and consequently $[\rho_1\rho_2\rho_2]=\rho_2\circ\rho_2^{-1}\circ\rho_1$ is not always equal to ρ_1. But, if Q is the set of all binary relations which define one-to-one mapping of set A into set B (partial mapping) and ternary operation defined by $[\rho_1\rho_2\rho_3]=\rho_3\circ\rho_2^{-1}\circ\rho_1$, then $(Q,[\ \ ])$ is a heap. $(Q,[\ \ ])$ is obviously not a laterally commutative heap.*

Example 5. *A semi-heap $(Q,[\ \ ]_p)$ is a laterally commutative heap.*

2. Heaps (laterally commutative heaps) and their corresponding algebraic and geometric structures

In this section we shall examine the connection between heaps (laterally commutative heaps) and some algebraic and geometric structures. Firstly, we shall consider the connection between heaps and groups.

Analogously to the *Theorem 2* for semi-heaps and semi-groups with involution the following theorem holds.

Theorem 4. *Let* $(Q,[\ \ ])$ *be a heap and* $\cdot : Q \to Q$ *is a binary operation defined by* $ab = [aeb]$ *where* $e \in Q$ *is a given element. Then* $(Q,\cdot)$ *is a group with the right unit* e *and* $[abc] = ab^{-1}c$*. If* $(Q,[\ \ ])$ *is a laterally commutative heap, then* $(Q,\cdot)$ *is a commutative group.*

Groups obtained from a heap by fixing its various elements are isomorphic. Converse of *Theorem 4* is also true.

Theorem 5. *If* $(Q,\cdot)$ *is a group and* $[\ \ ] : Q^3 \to Q$ *a ternary operation defined by* $[abc] = ab^{-1}c$*, then* $(Q,\cdot)$ *is a heap. If* $(Q,\cdot)$ *is a commutative group, then* $(Q,[\ \ ])$ *is a laterally commutative heap.*

The concept of heap was introduced in the study of the above ternary operation on a commutative group. In [3] Certaine defined heap by the set of postulates in a weakened form of a set given by V.V.Vagner in his paper [8]. In the next part of this section some algebraic and geometric structures are defined, and the structure of a heap in terms of these structures is described.

A groupoid $(Q,\cdot)$ is a right transitive groupoid if it satisfies the identity of right transitivity $ac \cdot bc = ab$. This groupoid is also called a Ward groupoid (see [2]).

A groupoid $(Q,\cdot)$ is said to be a right solvable if for any $a,b \in Q$ there is an element $x \in Q$ such that $ax = b$. It can be proved that in any right solvable Ward groupoid $(Q,\cdot)$ there is a uniquely determined element $e \in Q$ such that the following identities hold: $aa = e,\ \ ae = a$.

In [12] the following theorem is proved.

Theorem 6. *There exists a heap* $(Q,[\ \ ])$ *if and only if there exists a right solvable Ward groupoid* $(Q,\cdot)$ *where the formula* $[abc] = ab \cdot ec$ *defines a ternary operation by means of a multiplication, where* $e \in Q$ *is the right unit for* $(Q,\cdot)$*, and the formula* $ab = [abe]$ *defines a multiplication by means of ternary operation* $[\ \ ]$*, where* $e \in Q$ *is the selected element.*

If a Ward groupoid $(Q,\cdot)$ satisfies the identity $a \cdot ab = b$ then $(Q,\cdot)$ is the so-called subtractive groupoid (see [4]), which obviously is right solvable. Analogously to *Theorem 6* it can be proved that the existence of a laterally commutative heap is equivalent to the existence of a subtractive groupoid on the same set.

Example 6. *Let* $(Q,[\ \ ]_p)$ *be a laterally commutative heap (*Example 3*),* $e \in Q$ *given element and* $\cdot : Q^2 \to Q$ *binary operation defined by* $ab = [abe]$*. (*ab *is the fourth vertex of the parallelogram whose vertices are described cyclicaly as* a, b, e *and* ab*). Then* $(Q,\cdot)$ *is a subtractive groupoid.*

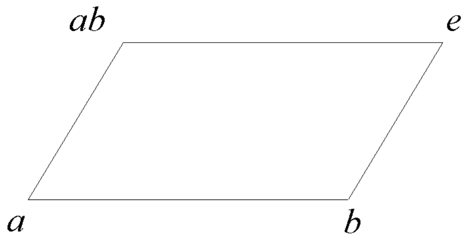


Figure 3.

The following drawing shows that $a \cdot ab = b$.

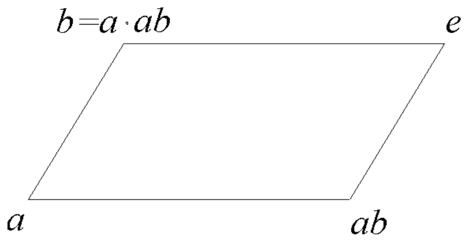


Figure 4.

The fulfillment of the right transitivity ($ac \cdot bc = ab$):

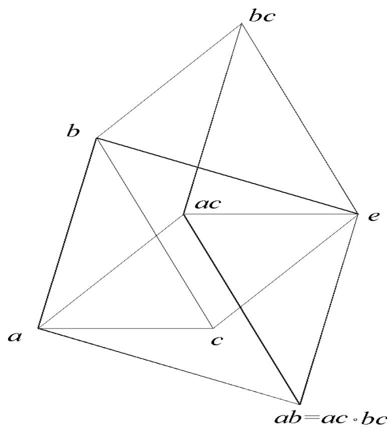


Figure 5.

In [6] the notation of a TST-space is defined. A TST-space (Q, S) is a nonvoid set Q with a family S of its involutory mappings which are called symmetries such that S acts transitively on Q and from $\sigma_1, \sigma_2, \sigma_3 \in S$ it follows $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$. In [10] the next theorem is proved.

Theorem 7. *There exists a TST-space (Q, S) if and only if there exists a laterally commutative heap $(Q, [\ \ ])$ where the formula $\sigma_{a,b}(c) = [acb]$ defines symmetry by means of a ternary operation, and vice versa, a ternary operation $[\ \ ]$ by symmetries.*

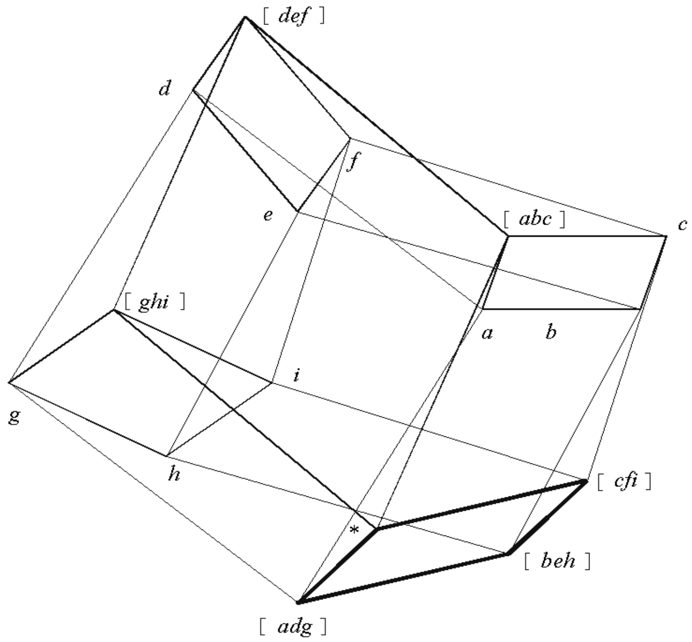
In their paper [1] W. Bos and G. Wolff introduced the notation of a semiaffine space as an ordered pair $(Q, [\ \ ])$ with the following properties $[abc] = [cba], [abb] = a$, $[[abc]bd] = [ab[cbd]]$, $[[abc]db] = [adc]$. In [13] the following theorem is proved.

Theorem 8. *There exists a laterally commutative heap* $(Q, [\ \ ])$ *if and only if there exists a semiaffine space* $(Q, [\ \ ])$.

A ternary operation $[\ \ ]$ on the set Q is said to be medial if the identity $[[abc][def][ghi]] = [[adg][beh][cfi]]$ holds. We say that $[\ \ ]$ is a Mal'cev function if the identities $[abb] = a$, $[bba] = a$ hold. In [11] the next theorem is proved.

Theorem 9. *There exists a laterally commutative heap if and only if there exists a medial Mal'cev function* $[\ \ ]$ *on the set* Q.

Example 7. *The following drawing illustrates the mediality of a ternary operation in* $(Q, [\ \ ]_p)$.



$* \cdots [[abc][def][ghi]] = [[adg][beh][cfi]]$

Figure 6.

In [9] V.V. Vakarelov proved that the existence of a heap is equivalent to the existence of a Desargues system defined on the same set, using the connection of these structures with groups. We shall prove this statement directly.

Definition 2. *([9]) A Desargues system* (Q, P) *is a nonvoid set* Q *with a quaternary relation* $P \subset Q^4$ *such that the following conditions are satisfied:*

(D1) $P(x, a, b, y)$, $P(x, c, d, y)$ *implies* $P(c, a, b, d)$ *for all* $a, b, c, d, x, y \in Q$

(D2) $P(b,a,x,y),\ P(d,c,x,y)$ *implies* $P(b,a,c,d)$ *for all* $a,b,c,d,x,y \in Q$

(D3) For any three $a,b,c \in Q$ *there is exactly one element* $d \in Q$ *such that* $P(a,b,c,d)$

Lemma 1. *If* (Q,P) *is a Desargues system then the following statements are valid.*

(a) $P(a,a,b,b), P(a,b,b,a)$ for all $a,b \in Q$

(b) $P(a,b,c,d)$ implies $P(b,a,d,c), P(d,c,b,a)$ for all $a,b,c,d \in Q$.

Ostermann and Schmidt introduced the notation of a parallelogram space in [5].

Definition 3. *([5]) A parallelogram space* (Q,Par) *is a nonvoid set* Q *with a quaternary relation* $Par \subset Q^4$ *which satisfies*

(P1) $Par(a,b,c,d)$ implies $Par(a,c,b,d)$ for all $a,b,c,d \in Q$

(P2) $Par(a,b,c,d)$ implies $Par(c,d,a,b)$ for all $a,b,c,d \in Q$

(P3) $Par(a,b,c,d),\ Par(c,d,e,f)$ implies $Par(a,b,e,f)$ for all $a,b,c,d,e,f \in Q$

(P4) For any three $a,b,c \in Q$ there is exactly one element $d \in Q$ such that $Par(a,b,c,d)$

In [7] it is proved that a Desargues system is a generalization of a parallelogram space. Namely, the following theorem is valid.

Theorem 10. (Q,Par) *is a parallelogram space if and only if* (Q,P) *is a Desargues system in which the following condition holds*

(D4) $P(a,b,c,d)$ *implies* $P(a,d,c,b)$ *for all* $a,b,c,d \in Q$

and $Par \subset Q^4$ *is defined by the equivalence* $Par(a,b,c,d) \Leftrightarrow P(a,b,d,c)$.

Let us prove that the existence of a heap is equivalent to the existence of a Desargues system on the same set.

Theorem 11. *If* (Q,P) *is a Desargues system and* $[\ \ ] : Q^3 \to Q$ *a ternary operation defined by the equivalence* $[abc] = d \Leftrightarrow P(a,b,c,d)$*, then* $(Q,[\ \ ])$ *is a heap.*

Proof. Firstly, we shall prove that identity $[[abc]de] = [ab[cde]]$ holds.
For $a,b,c,d,e \in Q$ let $[abc] = x,\ [xde] = y,\ [cde] = z$.
It implies $P(a,b,c,x),\ P(x,d,e,y),\ P(c,d,e,z)$. Since $P(x,d,e,y),\ P(c,d,e,z)$, it follows $P(d,x,y,e)\ P(d,c,z,e)$ by *Lemma 1 (b)*. Now from $P(d,x,y,e)\ P(d,c,z,e)$ it follows $P(c,x,y,z)$ by (D1). From $P(c,x,y,z)$ it follows $P(z,y,x,c)$ by *Lemma 1 (b)*, and again by *Lemma 1 (b)* it follows $P(y,z,c,x)$. Further, since $P(a,b,c,x),\ P(y,z,c,x)$ it follows $P(a,b,z,y)$ by (D2), i.e. $[abz] = y$.
According to *Lemma 1 (a)*, it follows that all elements of Q are biunitary, i.e. $[abb] = a,\ [bba] = a$. □
If (Q,P) is a Desargues system in which (D4) holds, then the following is true.

Since $P(a,b,c,d)$, it follows $P(a,d,c,b)$ by (D4) and therefore $P(d,a,b,c)$ by *Lemma 1 (b)*, and $P(c,b,a,d)$ again by *Lemma 1 (b)*. Hence $[abc] = d$ implies $[cba] = d$ i.e. $[abc] = [cba]$.

We have proved the following theorem.

Theorem 12. *If* (Q, Par) *is a parallelogram space and* $[\ ] : Q^3 \to Q$ *a ternary operation defined by the equivalence* $[abc] = d \Leftrightarrow Par(a, b, d, c)$*, then* $(Q, [\ ])$ *is a laterally commutative heap.*

Converse of *Theorem 10* also holds.

Theorem 13. *If* $(Q, [\ ])$ *is a heap and* $P \subset Q^4$ *quaternary relation defined by the equivalence* $P(a, b, c, d) \Leftrightarrow [abc] = d$ *then* (Q, P) *is a Desargues system.*

Proof. (D1) Let $P(x, a, b, y)$, $P(x, c, d, y)$ i.e. $[xab] = y$, $[xcd] = y$.
Since $[xab] = [xcd]$, it follows $[cx[xab]] = [cx[xcd]]$.
Since $(Q, [\ ])$ is a heap, it follows $[[cxx]ab] = [[cxx]cd]$ i.e. $[cab] = [ccd] = d$. Hence $P(c, a, b, d)$ and (D1) is proved.
The (D2) can be proved analogously.
(D3) is valid because of the definition of the ternary operation.
Therefore, (Q, P) is a Desargues system. □
If in the previous theorem $(Q, [\ ])$ is laterally commutative heap, then the identity $[abc] = [cba]$ holds, i.e. if $P(a, b, c, d)$, then we have $P(c, b, a, d)$ and then $P(b, c, d, a)$ by *Lemma 1 (b)*. From $P(b, c, d, a)$ it follows that $P(a, d, c, b)$ by *Lemma 1 (b)* and (D4) is proved.

We have proved the theorem.

Theorem 14. *If* $(Q, [\ ])$ *is a laterally commutative heap and* $[\ ] : Q^3 \to Q$ *a ternary operation defined by the equivalence* $Par(a, b, d, c) \Leftrightarrow [abc] = d$ *then* (Q, Par) *is a parallelogram space.*

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