A note on compact operators and operator matrices

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Abstract. *In this note two properties of compact operators acting on a separable Hilbert space are discussed. In the first part a characterization of compact operators is obtained for bounded operators represented as tri-block diagonal matrices with finite blocks. It is also proved that one can obtain such a tri-block diagonal matrix representation for each bounded operator starting from any orthonormal basis of the underlying Hilbert space by an arbitrary small Hilbert-Schmidt perturbation.*

The second part is devoted to the so-called Hummel's property of compact operators: each compact operator has a uniformly small orthonormal basis for the underlying Hilbert space. The class of all bounded operators satisfying Hummel's condition is determined.

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Throughout this note H denotes a separable infinite-dimensional complex Hilbert space with a scalar product $(\cdot|\cdot)$. Let $B(H)$ and $K(H)$ denote the algebra of all bounded operators and the ideal of all compact operators on H, respectively. The spectrum of an operator $A \in B(H)$ we denote by $\sigma(A)$. The set of all natural numbers is denoted by $\mathbf{N}$.

1. Infinite block matrices

Let (e_n) be any orthonormal basis for a Hilbert space H. Each operator $A \in B(H)$ is uniquely determined by its matrix $A = (a_{ij}),\ a_{ij} = (Ae_j|e_i),\ i, j \in \mathbf{N}$. There is a number of properties of bounded operators on H which can be deduced from their matrices. In particular, there are several well-known compactness conditions for bounded operators formulated in terms of their matrix representations. Probably the most popular one is concerned with diagonal operators [9, Problem 132]. Further, it is proved in [7] and [12] (see also [2] and [3]) that a bounded operator $A \in B(H)$ is compact if and only if either of the following two conditions is satisfied for any orthonormal basis for H:

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1. $\lim_n Ae_n = 0$,
2. $\lim_n (Ae_n|e_n) = 0$.

It is important to emphasize that each of the above conditions is required for all orthonormal bases. Namely, there exist non-compact operators satisfying the same conditions but only for some suitable orthonormal basis.

On the other hand, there are compactness conditions which are concerned with a single matrix representation of an operator. The following theorem [1, pp. 80, 81] serves as a typical result of this kind. It provides the compactness criterion for generalized Jacobian matrices, i. e. for matrices with a finite number of non-zero diagonals.

Theorem 1. *Let $A \in L(H)$ be an operator and let (e_n) be an orthonormal basis for H. Assume that there exists an integer $r \geq 0$ such that $(Ae_n|e_m) = 0$, $\forall n, m$, $|n - m| > r$. Then A is compact if and only if*

$$\lim_n (Ae_{n+i}|e_n) = \lim_n (Ae_n|e_{n+i}) = 0,\ \forall i = 0, 1, \cdots, r.$$

Our aim is to generalize the above theorem to infinite block matrices with finite blocks. To obtain our theorem, we shall need two lemmas. The first one estimates the norm of an operator which is written as an infinite block matrix. The second lemma enables us to reduce the problem to tri-block diagonal matrices.

Our first lemma is concerned with infinite block matrices corresponding to an orthogonal decomposition of the underlying Hilbert space. Essentially, we repeat Kato's argument [11, p. 29] where the same estimate is proved for matrices with scalar entries. We also note that the alternative proof may be obtained by adapting the Schur test for infinite matrices.

Lemma 1. *Let $A \in B(H)$ be an operator represented as a block matrix*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} & \cdots \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \tag{1}$$

corresponding to an orthogonal decomposition $H = \oplus_{i=1}^{\infty} H_i$. Let us denote

$$\begin{array}{ll} R_i = \sum_{j=1}^{\infty} \|A_{ij}\|, & R = \sup\{R_i : i \in \mathbf{N}\}, \\ C_j = \sum_{i=1}^{\infty} \|A_{ij}\|, & C = \sup\{C_j : j \in \mathbf{N}\}. \end{array} \tag{2}$$

Then $\|A\| \leq \sqrt{R \cdot C}$.

Remark: Obviously, we may assume that $R \cdot C < \infty$, for otherwise there is nothing to prove. Also, one should observe that there is no restriction to dimensions of subspaces H_i. Consequently, A_{ij} may be infinite or finite and in the later case diagonal elements need not to be of the same size (which of course implies that off-diagonal elements are not necessarily square matrices).

Proof. Let us take $x \in H,\ y = Ax$ and write $x = \sum_{i=1}^{\infty} x_i,\ y = \sum_{i=1}^{\infty} y_i,\ x_i, y_i \in H_i,\ \forall i.$ Then

$$y_i = \sum_{j=1}^{\infty} A_{ij} x_j \text{ and } \frac{\|y_i\|}{R_i} \leq \sum_{j=1}^{\infty} \frac{\|A_{ij}\|}{R_i} \|x_j\|.$$

Now, since $t \mapsto t^2$ is a convex function, the last inequality together with $\sum_{j=1}^{\infty} \frac{\|A_{ij}\|}{R_i} = 1$ implies

$$(\frac{\|y_i\|}{R_i})^2 \leq \sum_{j=1}^{\infty} \frac{\|A_{ij}\|}{R_i} \|x_j\|^2,$$

therefore

$$\|y_i\|^2 \leq R_i \sum_{j=1}^{\infty} \|A_{ij}\| \cdot \|x_j\|^2 \leq R \sum_{j=1}^{\infty} \|A_{ij}\| \cdot \|x_j\|^2 \Rightarrow$$

$$\|y\|^2 \leq R \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|A_{ij}\| \cdot \|x_j\|^2 \leq R \cdot C \sum_{j=1}^{\infty} \|x_j\|^2 = R \cdot C \|x\|^2.$$

□

The following lemma will focus our attention to tri-block diagonal matrices. It states that one can obtain a tri-block diagonal matrix for each bounded operator (starting from any basis) by extracting the appropriate Hilbert-Schmidt operator of an arbitrary small Hilbert-Schmidt norm (denoted by $\|\cdot\|_2$). We also note that similar construction appears in Lemma 5 from [4].

Lemma 2. *Let $T \in B(H),\ \varepsilon > 0$, and let (e_n) be any orthonormal basis for H. There exists a sequence $(p(n))$ of natural numbers ($p(0) = 0$) such that T has the form $T = A + K$, where K is a Hilbert-Schmidt operator with $\|K\|_2 < \varepsilon$ and A is a tri-block diagonal according to the decomposition $H = \oplus_{n=1}^{\infty}$ with $H_n = [\{e_{p(n-1)+1}, e_{p(n-1)+2}, \ldots, e_{p(n)}\}]$.*

Proof. Let us take a sequence (ε_n) of positive numbers such that $\varepsilon^2 = \sum_{n=1}^{\infty} \varepsilon_n^2$. Furthermore, denote by $P_k \in B(H)$ the orthogonal projection onto the subspace $[\{e_1, \ldots, e_k\}],\ \forall k \in \mathbf{N}$ and observe that $P_k \to I$ in the strong operator topology. Let $p(0) = 0$ and let $p(1)$ be an arbitrary natural number.

Since $P_k T e_i \to T e_i$ and also $P_k T^* e_i \to T^* e_i,\ \forall i = 1, 2, \ldots, p(1)$ there exists a natural number $p(2)$ such that

$$\|Te_i\|^2 - \|P_{p(2)} T e_i\|^2 < \frac{\varepsilon_1^2}{2(p(1) - p(0))},$$

and also

$$\|T^* e_i\|^2 - \|P_{p(2)} T^* e_i\|^2 < \frac{\varepsilon_1^2}{2(p(1) - p(0))},\ \forall i = 1, \ldots, p(1).$$

It follows

$$\sum_{i=p(0)+1}^{p(1)} \sum_{j=p(2)+1}^{\infty} |(Te_i|e_j)|^2 = \sum_{p(0)+1}^{p(1)} (\|Te_i\|^2 - \|P_{p(2)} T e_i\|^2) < \frac{\varepsilon_1^2}{2}.$$

Exactly in the same way we get

$$\sum_{i=p(0)+1}^{p(1)} \sum_{j=p(2)+1}^{\infty} |(Te_j|e_i)|^2 = \sum_{i=p(0)+1}^{p(1)} \sum_{j=p(2)+1}^{\infty} |(T^*e_i|e_j)|^2 < \frac{\varepsilon_1^2}{2}.$$

We continue with this procedure inductively, the inductive step being as above with $p(n-1), p(n), p(n+1)$ and ε_n in the role of $p(0), p(1), p(2)$ and ε_1.

Clearly, a tri-block diagonal operator A arising from our construction (the building blocks are $A_{n,n}, A_{n,n+1},\ \forall n$ and $A_{n-1,n},\ \forall n \geq 2$,) has the required property, namely, $T - A$ is a Hilbert-Schmidt operator and $\|T-A\|_2^2 < \sum_{n=1}^{\infty} \varepsilon_n^2 = \varepsilon^2$. $\square$

Theorem 2. *Let $A \in B(H)$ be an operator having a tri-block diagonal matrix*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & & & & \\ A_{2,1} & A_{2,2} & A_{2,3} & 0 & & & \\ 0 & A_{3,2} & \ddots & \ddots & \ddots & & \\ & 0 & \ddots & A_{n,n} & A_{n,n+1} & 0 & \\ & & \ddots & A_{n+1,n} & A_{n+1,n+1} & \ddots & \ddots \\ & & & 0 & \ddots & \ddots & \\ & & & & \ddots & & \end{bmatrix} \tag{3}$$

according to the decomposition $H = \oplus_{n=1}^{\infty} H_n$ onto finite dimensional subspaces H_n. Then A is a compact operator if and only if

$$\lim_n \|A_{n,n}\| = \lim_n \|A_{n+1,n}\| = \lim_n \|A_{n,n+1}\| = 0.$$

Proof. First we prove necessity. Let

$$B_n = A_{n-1,n}^* A_{n-1,n} + A_{n,n}^* A_{n,n} + A_{n+1,n}^* A_{n+1,n} \in L(H_n),\ \forall n \in \mathbf{N}.$$

Then each B_n is a positive finite rank operator, so there exists a unit vector $x_n \in H_n$ such that $\|B_n\| = \|B_n x_n\|$, $\forall n$. The orthonormal sequence (x_n) obtained in this way has the property $\|Ax_n\|^2 = \|B_n x_n\| = \|B_n\|$, $\forall n$, and because A is compact this implies $\lim_n \|B_n\| = 0$, hence

$$\lim_n \|A_{n,n}\| = \lim_n \|A_{n+1,n}\| = \lim_n \|A_{n,n+1}\| = 0.$$

To prove the converse, let us denote by $P_n \in B(H)$ the orthogonal projection of the finite rank onto $\oplus_{i=1}^{n} H_i$. Consequently, $A_n = P_n A P_n$ is also a finite rank operator and

$$A - A_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & & & \\ 0 & \ddots & \ddots & \ddots & & & \\ 0 & \ddots & 0 & 0 & 0 & & \\ \vdots & \ddots & 0 & 0 & A_{n,n+1} & 0 & \\ & & 0 & A_{n+1,n} & A_{n+1,n+1} & \ddots & \ddots \\ & & & 0 & \ddots & \ddots & \\ & & & & \ddots & & \end{bmatrix}.$$

Now, we are going to apply *Lemma 1*. By hypothesis is $\lim_n \|A_{n,n}\| = \lim_n \|A_{n+1,n}\| = \lim_n \|A_{n,n+1}\| = 0$, so one can find n large enough to make the estimators R and C (with the same meaning as in *Lemma 1*) for the operator $A - A_n$ arbitrary small. Then, by *Lemma 1*, $\|A - A_n\| \leq \sqrt{R \cdot C}$, and since A_n has a finite rank, A must be compact. □

We proceed with a few comments and remarks. First, we note that the necessity part of the above theorem remains true even if the subspaces H_n in the above decomposition of H are infinite-dimensional.

Secondly, one should observe that the role of *Lemma 2* is not in proving, but in applying *Theorem 2*: each operator given in any orthonormal basis has the suitable Hilbert-Schmidt perturbation which enables us to apply *Theorem 2*.

It is perhaps worth noting that *Theorem 2* does not depend on the main topological property of compact operators (i. e. maping the unit sphere onto the relatively compact set). Actually, the proof takes advantage of the simple but remarkable property of compact operators: they convert orthonormal sequences into sequences strongly converging to 0.

Finally, let us note that *Theorem 1* can easily be derived from our *Theorem 2*: it is enough to observe that each generalized Jacobian matrix is in fact a tri-block diagonal matrix with finite blocks of the same size.

2. Uniformly small orthonormal bases

In 1957 Hummel [10] proved that each compact operator A on a separable infinite dimensional Hilbert space H has a uniformly small orthonormal basis, i.e. A satisfies the following condition:

$$\forall \varepsilon > 0 \;\; \exists \text{ orthonormal basis } (e_n) \text{ for } H \text{ such that } \|Ae_n\| < \varepsilon, \; \forall n. \qquad (h)$$

The original Hummel's proof uses the theory of analytic functions in the complex plane. Later on, Fujii, Izumino and Nakamura [8] demonstrated a simple proof of Hummel's theorem.

However, there also exist non-compact bounded operators which satisfy Hummel's condition (h). An easy example is provided by the orthogonal projection whose matrix in some orthonormal basis is given by

$$P = \begin{bmatrix} 1 & 0 & & 0 & & & \dots \\ 0 & \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{matrix} & & 0 & & & \dots \\ 0 & 0 & & \begin{matrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{matrix} & & & \\ \vdots & \vdots & & & & & \ddots \end{bmatrix}.$$

A natural question arises, namely, which operators, besides the compacts, satisfy the above Hummel's condition (h).

Let us denote the set of all bounded operators which satisfy the Hummel's condition by $\mathcal{L}$. In the first theorem some of its elementary, but useful properties are collected.

Theorem 3. *The set $\mathcal{L}$ has the following properties:*

(a) $TA \in \mathcal{L}, \quad \forall T \in B(H)$ *and* $\forall A \in \mathcal{L}$.

(b) *$\mathcal{L}$ is (norm) closed.*

(c) *$\mathcal{L}$ contains no regular operators.*

(d) *Let $A \in \mathcal{L}$ be a normal operator and let f be an arbitrary continuous function on $\sigma(A)$ with $f(0) = 0$. Then $f(A) \in \mathcal{L}$.*

(e) $A \in \mathcal{L} \Leftrightarrow A^*A \in \mathcal{L}$.

Proof. The first three statements are obvious. To prove (d) it is enough to approximate f uniformly on $\sigma(A)$ by the polynomial p which also satisfies $p(0) = 0$. Then $p(A)$ must be of the form $p(A) = TA$ for some bounded operator T, so by (a) $p(A) \in \mathcal{L}$ and it remains to apply (b).

The last statement follows from (a), and in the opposite direction, from (d) $(f(t) = \sqrt{t})$ and (a) using the polar decomposition. □

In our next theorem the set $\mathcal{L}$ is completely determined. A related tool is the essential spectrum of an operator. The essential spectrum $\sigma_e(A)$ (resp. the left essential spectrum $\sigma_{le}(A)$) of an operator $A \in B(H)$ consists of all complex numbers λ such that $A - \lambda I$ does not have the inverse (resp. the left inverse) in the Calkin algebra $B(H)/K(H)$.

Theorem 4. *A bounded operator $A \in (H)$ on a separable Hilbert space H satisfies the Hummel's condition (h) if and only if its left essential spectrum contains 0.*

Proof. Obviously, an operator satisfying (h) can not have the left essential inverse.

Conversely, suppose $0 \in \sigma_{le}(A)$. Then by [6, Theorem 1.1] there exists an orthonormal sequence (b_n) such that $Ab_n \to 0$, or equivalently, $(A^*Ab_n|b_n) \to 0$. But then by [6, Theorem 5.1] the essential numerical range of A^*A contains 0. Now we can apply Theorem 3 from [5]: A^*A can be approximated by the set of zero-diagonal operators. Precisely, for $\varepsilon > 0$ there exists an operator Z possessing the orthonormal basis (e_n) for H such that $(Ze_n|e_n) = 0$, $\forall n$ and $\|A^*A - Z\| < \varepsilon$. This implies $(A^*Ae_n|e_n) < \varepsilon$, $\forall n$, or, in other words $\|Ae_n\|^2 < \varepsilon$, $\forall n$. Hence $A \in \mathcal{L}$. $\square$

To conclude, let us note that a related characterization of the essential numerical range of an operator is obtained in Theorem 2.3. from [13]: a bounded operator T has 0 in its essential numerical range if and only if for any $\varepsilon > 0$ there is an orthonormal basis so that all entries of the matrix of T have absolute value less than ε.

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