Some new estimates for the moments of guessing mappings

S. S. DRAGOMIR*

Abstract. *In this paper, by the use of some new analytic inequalities for arithmetic means, we point out new estimation for the moments of guessing mapping which complement in a natural way the recent results of Arikan [2], Boztas [3] and Dragomir, Van der Hoek [4]-[6].*

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1. Introduction

J.L. Massey in the paper [1] considered the problem of guessing the value of realization of a random variable X by asking questions of the form: "Is X equal to x?" until the answer is "Yes".

Let $G(X)$ denote the number of guesses required by a particular guessing strategy when $X = x$.

Massey observed that $E(G(X))$, the average number of guesses is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [2].

Let (X, Y) be a pair of random variable with X taking values in a finite set $\mathcal{X}$ of size n, Y taking values in a countable set $\mathcal{Y}$. Call a function $G(X)$ of the random variable X a *guessing function* in X if $G : \mathcal{X} \to \{1, ..., n\}$ is one-to-one. Call a function $G(X \mid Y)$ a *guessing function for X given Y* if for any fixed value $Y = y, G(X \mid y)$ is a guessing function for X . $G(X \mid y)$ will be thought of as the number of guessing required to determine X when the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X \mid Y)$ were proved by E. Arikan in the recent paper [2].

*School of Communications and Informatics, Victoria University of Technology, PO Box, 14428 MC Melbourne, 8001 Victoria, Australia, e-mail: sever@matilda.vu.edu.au

Theorem 1. *For an arbitrary guessing function $G(X)$ and $G(X \mid Y)$ and any $p \geq 0$, we have:*

$$E\left(G(X)^p\right) \geq (1+\ln n)^{-p}\left[\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1+p}}\right]^{1+p} \tag{1}$$

and

$$E\left(G(X \mid Y)^p\right) \geq (1+\ln n)^{-p}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}} P_{X,Y}(x,y)^{\frac{1}{1+p}}\right]^{1+p} \tag{2}$$

where $P_{X,Y}$ and P_X are probability distributions of (X,Y) and X, respectively.

In paper [6], S.S. Dragomir and J. van der Hoek have proved the following result for the moments of guessing mapping:

Theorem 2. *Assume that $P_M := \max\{p_i \mid i=1,...,n\}$ and $P_m := \min\{p_i \mid i=1,...,n\}$ and $P_M \neq P_m$. Then we have the estimates:*

$$\begin{aligned} & G_p(n)\left[P_m n^{p+1} + \tfrac{1}{(P_M-P_m)^p}(1-nP_m)^{p+1}\right] \\ \leq\ & E\left(G(X)^p\right) \leq G_p(n)\left[P_M n^{p+1} + \tfrac{1}{(P_M-P_m)^p}(nP_M-1)^{p+1}\right] \end{aligned}$$

for $p \geq 1$, where $G_p(n) := \frac{S_p(n)}{n^{p+1}}$ and $S_p(n) := \sum_{i=1}^{n} i^p$.

Corollary 1. *With the above assumption, we have:*

$$\begin{aligned} \tfrac{1}{2}\left(1+\tfrac{1}{n}\right)\tfrac{P_m P_M n^2 - 2nP_m + 1}{P_M - P_m} \leq\ & E(G(X)) \\ \leq\ & \tfrac{1}{2}\left(1+\tfrac{1}{n}\right)\tfrac{-P_M P_m n^2 + 2P_M n - 1}{P_M - P_m}. \end{aligned}$$

For other estimations of $E\left(G(X)^p\right)$ see the papers [4] - [6].

The main aim of this paper is to point out different estimations of the moments $E\left(G(X)^p\right)$ by the use of some new inequalities for arithmetic means which will be pointed out in the next section.

2. Some analytic inequalities

We shall start with the following lemma of "*summation by parts*" which is well known in the litarature (see for example [7, p. 26]):

Lemma 1. *Let $a_i, b_i \in \mathbf{R}$ $(i=1,...n)$ and denote $\Delta a_i := a_{i+1} - a_i$ $(i=1,...,n)$. Then we have the inequality:*

$$\sum_{i=1}^{n-1} b_i \Delta a_i = a_n b_n - a_1 b_1 - \sum_{i=1}^{n-1} a_{i+1}\Delta b_i. \tag{3}$$

The following corollary holds:

Corollary 2. *If* $t_i,\ z_i \in \mathbf{R}$ $(i = 1, ..., n)$ *and* $T_k = \sum_{i=1}^{k} t_i,\ T_0 := 0$ $(k = 1, ..., n)$, *then we have the identity:*

$$\sum_{i=1}^{n} t_i z_i = T_n z_n - \sum_{i=1}^{n} T_i \Delta z_i. \tag{4}$$

Now, let us consider the arithmetic means:

$$A_n(p, x) := \sum_{i=1}^{n} t_i x_i \text{ where } p_i \ge 0 \text{ and } \sum_{i=1}^{n} p_i = 1,$$

$$A_n(q, x) := \sum_{i=1}^{n} q_i x_i \text{ where } q_i \ge 0 \text{ and } \sum_{i=1}^{n} q_i = 1$$

and $x = (x_i)_{i=\overline{1,n}}, x_i \in \mathbf{R}$ $(i = 1, ..., n)$.

We are interested here to establish some estimations for the difference $A_n(p, x) - A_n(q, x)$ in terms of p, q and x.

Theorem 3. *With the above assumptions for the sequences* $p_i,\ q_i,\ x_i$ $(i = 1, ..., n)$ *we have the inequality:*

$$|A_n(p, x) - A_n(q, x)|$$

$$\le \begin{cases} \max\limits_{i=\overline{1,n-1}} |x_{i+1} - x_i| \sum\limits_{i=1}^{n-1} |P_i - Q_i|, \\ \left(\sum\limits_{i=1}^{n-1} |P_i - Q_i|^s\right)^{\frac{1}{s}} \left(\sum\limits_{i=1}^{n-1} |x_{i+1} - x_i|^l\right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty, \\ \max\limits_{i=\overline{1,n-1}} |P_i - Q_i| \sum\limits_{i=1}^{n-1} |x_{i+1} - x_i|, \end{cases} \tag{5}$$

where

$$P_i := \sum_{k=1}^{i} p_k \text{ and } Q_i := \sum_{k=1}^{i} q_k \text{ for } i \in \{1, ..., n\}.$$

Proof. Using the identity (4) we have :

$$\begin{aligned} |A_n(p, x) - A_n(q, x)| = & \left|\sum_{i=1}^{n} (p_i - q_i) x_i\right| = \left|x_n \sum_{i=1}^{n} (p_i - q_i) - \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i\right| \\ = & \left|\sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i\right| \le \sum_{i=1}^{n-1} |P_i - Q_i| |x_{i+1} - x_i|. \end{aligned}$$

Now, let us remark that the first and the last inequality in (5) are obvious.
The second inequality follows by the discrete Hölder's inequality.
We shall omit the details. □

Corollary 3. *With the above assumptions we have that*

$$|A_n(p,x) - A_n(x)| \leq \begin{cases} \max\limits_{i=\overline{1,n-1}} |x_{i+1}-x_i| \sum\limits_{i=1}^{n-1} \left|P_i - \frac{i}{n}\right|, \\ \left(\sum\limits_{i=1}^{n-1} \left|P_i - \frac{i}{n}\right|^s\right)^{\frac{1}{s}} \left(\sum\limits_{i=1}^{n} |x_{i+1}-x_i|^l\right)^{\frac{1}{l}}, \frac{1}{s}+\frac{1}{l}=1, 1<s<\infty, \\ \max\limits_{i=\overline{1,n-1}} \left|P_i - \frac{i}{n}\right| \sum\limits_{i=1}^{n-1} |x_{i+1}-x_i|, \end{cases} \tag{6}$$

where by $A_n(x)$ we denote the unweighted arithmetic mean, i.e.,

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Another type of estimation can be found in the following theorem too.

Theorem 4. *Under the assumption of* Theorem 3*, we have*

$$|A_n(p,x) - A_n(x) - X_n(p_n-q_n)| \leq \begin{cases} \max\limits_{i=\overline{1,n-1}} |\Delta(p_i-q_i)| \sum\limits_{i=1}^{n-1} |X_i|; \\ \left(\sum\limits_{i=1}^{n-1} |\Delta(p_i-q_i)|^s\right)^{\frac{1}{s}} \left(\sum\limits_{i=1}^{n-1} |X_i|^l\right)^{\frac{1}{l}}, \frac{1}{s}+\frac{1}{l}=1, 1<s<\infty; \\ \max\limits_{i=\overline{1,n-1}} |X_i| \sum\limits_{i=1}^{n-1} |\Delta(p_i-q_i)|; \end{cases} \tag{7}$$

where X_i is given by

$$X_i := \sum_{k=1}^{i} x_k, i \in \{1,...,n\}.$$

Proof. Using the identity (4) we can write :

$$|A_n(p,x) - A_n(q,x)| = \sum_{i=1}^{n} (p_i-q_i) x_i = X_n(p_n-q_n) - \sum_{i=1}^{n-1} X_i \Delta(p_i-q_i)$$

from where we get

$$A_n(p,x) - A_n(q,x) - X_n(p_n-q_n) = -\sum_{i=1}^{n-1} X_i \Delta(p_i-q_i)$$

and then we have the estimation :

$$|A_n(p,x) - A_n(q,x) - X_n(p_n-q_n)| \leq \sum_{i=1}^{n-1} |X_i| |\Delta(p_i-q_i)|$$

which, as above, imply the desired inequality (7). □

Corollary 4. *With the above assumption, we have that:*

$$\left|A_n(p,x) - A_n(x) - X_n\left(p_n - \tfrac{1}{n}\right)\right| \leq \begin{cases} \max\limits_{i=\overline{1,n-1}} |\Delta p_i| \sum\limits_{i=1}^{n-1} |X_i|, \\ \left(\sum\limits_{i=1}^{n-1} |\Delta p_i|^s\right)^{\frac{1}{s}} \left(\sum\limits_{i=1}^{n-1} |X_i|^l\right)^{\frac{1}{l}}, \frac{1}{s}+\frac{1}{l}=1, 1<s<\infty, \\ \max\limits_{i=\overline{1,n-1}} |X_i| \sum\limits_{i=1}^{n-1} |\Delta p_i|. \end{cases} \tag{8}$$

Now, we state another inequality which is the discrete version of Grüss' integral inequality:

Lemma 2. *Let $a_i, b_i \in \mathbf{R}$ $(i = 1, ..., n)$ be so that*

$$a \leq a_i \leq A, b \leq b_i \leq B$$

for all $i = 1, ..., n$. Then we have the inequality:

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right| \leq \frac{1}{4}(A-a)(B-b). \tag{9}$$

Proof. For the sake of completeness, we shall give here a short proof (see also [5]). We use Grüss' integral inequality which states that:

If $h, g : [a,b] \to \mathbf{R}$ are two integrable functions so that $m_1 \leq g(x) \leq M_1$, $m_2 \leq h(x) \leq M_2$ for all $x \in [a,b]$, then we have the estimation:

$$\left|\tfrac{1}{b-a}\int_a^b g(x)h(x)\,dx - \tfrac{1}{b-a}\int_a^b g(x)\,dx \cdot \tfrac{1}{b-a}\int_a^b h(x)\,dx\right| \leq \tfrac{1}{4}(M_1-m_1)(M_2-m_2). \tag{10}$$

Now, if we are choosing in (10)

$$g(x) = \begin{cases} a_1, x \in [0,1) \\ a_2, x \in [1,2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n, x \in [n-1,n] \end{cases}$$

and

$$h(x) = \begin{cases} b_1, x \in [0,1) \\ b_2, x \in [1,2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_n, x \in [n-1,n] \end{cases},$$

then $m_1 = a, M_1 = A, m_2 = b$ and $M_2 = B$ and obviously

$$\int_0^n g(x)\,dx = \sum_{i=1}^n a_i, \int_0^n h(x)\,dx = \sum_{i=1}^n b_i$$

and

$$\int_0^n g(x)\,h(x)\,dx = \sum_{i=1}^n a_i b_i$$

and the lemma is proved. □

We are able now to point out some different estimations for the modulus of the difference $A_n(p,x) - A_n(q,x)$.

Theorem 5. *With the above assumptions for p, q and X, we have the inequality:*

$$\left| A_n(p,x) - A_n(q,x) - \frac{1}{n-1}(x_n - x_1)\sum_{i=1}^{n-1}(n-i)(p_i - q_i) \right| \tag{11}$$

$$\leq \frac{n-1}{4}(\Gamma - \gamma)(\Delta - \delta)$$

provided that:

$$\delta \leq x_i \leq \Delta \text{ for all } i = 1, ..., n-1$$

and

$$\gamma \leq P_i - Q_i \leq \Gamma \text{ for all } i = 1, ..., n-1.$$

Proof. Choose in *Lemma 2*

$$a_i := P_i - Q_i, b_i = \Delta x_i, i = 1, ..., n-1.$$

Then we have:

$$\left| \frac{1}{n-1}\sum_{i=1}^{n-1}(P_i - Q_i)\Delta x_i - \frac{1}{n-1}\sum_{i=1}^{n-1}(P_i - Q_i) \cdot \frac{1}{n-1}\sum_{i=1}^{n-1}\Delta x_i \right|$$

$$\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma).$$

As we have:

$$\sum_{i=1}^{n-1}(P_i - Q_i) = \sum_{i=1}^{n-1}(n-i)(p_i - q_i),$$

$$\sum_{i=1}^{n-1} \Delta x_i = x_n - x_1$$

and

$$\sum_{i=1}^{n-1} (p_i - q_i)\, x_i = \sum_{i=1}^{n-1} (P_i - Q_i)\, \Delta x_i$$

then (see *Theorem 3*), we get that

$$\begin{aligned} & \left| A_n(p,x) - A_n(q,x) - \tfrac{1}{n-1}(x_n - x_1) \sum_{i=1}^{n-1} (n-i)(p_i - q_i) \right| \\ = & \left| \sum_{i=1}^{n-1} (P_i - Q_i)\, \Delta x_i - \tfrac{1}{n-1} \sum_{i=1}^{n-1} (P_i - Q_i) \cdot \tfrac{1}{n-1} \sum_{i=1}^{n-1} \Delta x_i \right| \\ \leq & \tfrac{n-1}{4} (\Gamma - \gamma)(\Delta - \delta) \end{aligned}$$

and the estimation is proved. □

Corollary 5. *With the above assumptions for X and if*

$$\tilde{\gamma} \leq P_i - \frac{i}{n} \leq \tilde{\Gamma}$$

then

$$\begin{aligned} & \left| A_n(p,x) - A_n(q,x) - \tfrac{1}{n-1}(x_n - x_1) \sum_{i=1}^{n-1} (n-i)\left(p_i - \tfrac{1}{n}\right) \right| \\ \leq & \tfrac{n-1}{4} \left(\tilde{\Gamma} - \tilde{\gamma}\right)(\Delta - \delta). \end{aligned} \tag{12}$$

The following theorem also holds.

Theorem 6. *Suppose that p, q, x satisfy the condition:*

$$x \leq X_i \leq X, i = 1, ..., n-1$$

and

$$\varphi \leq \Delta(p_i - q_i) \leq \Phi, i = 1, ..., n-1.$$

Then we have:

$$\begin{aligned} & \left| A_n(p,x) - A_n(q,x) - X_n(p_n - q_n) + \tfrac{1}{n-1}[p_n - p_1 - (q_n - q_1)] \sum_{i=1}^{n-1} (n-i)\, x_i \right| \\ & \leq \tfrac{n-1}{4} (X - x)(\Phi - \varphi). \end{aligned} \tag{13}$$

Proof. Let us apply the discrete Grüss' inequality (2.7) for $a_i := X_i$, $b_i := \Delta(p_i - q_i)$, $i = 1, ..., n-1$ to get:

$$\left|\frac{1}{n-1}\sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) - \frac{1}{n-1}\sum_{i=1}^{n-1} X_i \cdot \frac{1}{n-1}\sum_{i=1}^{n-1} \Delta(p_i - q_i)\right|$$

$$\leq \frac{1}{4}(X - x)(\Phi - \varphi).$$

But

$$\sum_{i=1}^{n-1} X_i = \sum_{i=1}^{n-1} (n-i) x_i,$$

$$\sum_{i=1}^{n-1} \Delta(p_i - q_i) = p_n - q_n - p_1 + q_1 = p_n - p_1 - (q_n - q_1),$$

and then we get:

$$\left|-\sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) + \frac{1}{n-1}(p_n - p_1 - (q_n - q_1))\sum_{i=1}^{n-1}(n-i) x_i\right| \tag{14}$$

$$\leq \frac{n-1}{4}(X - x)(\Phi - \varphi).$$

But,

$$-\sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) = A_n(p, x) - A_n(q, x) - X_n(p_n - q_n)$$

(see the proof of *Theorem 4*) and then by (14) we obtain:

$$\left|A_n(p, x) - A_n(q, x) - X_n(p_n - q_n) + \frac{1}{n-1}(p_n - p_1 - (q_n - q_1))\sum_{i=1}^{n-1}(n-i) x_i\right|$$

$$\leq \frac{n-1}{4}(X - x)(\Phi - \varphi)$$

and the theorem is thus proved. □

The following corollary also holds.

Corollary 6. *If x and p satisfy the condition:*

$$x \leq X_i \leq X, i = 1, ..., n-1$$

and

$$\tilde{\varphi} \leq \Delta p_i \leq \tilde{\Phi}, i = 1, ..., n-1$$

then

$$\left| A_n(p, X) - A_n(X) - X_n\left(p_n - \frac{1}{n}\right) + \frac{1}{n-1}(p_n - p_1)\sum_{i=1}^{n-1}(n-i)\, x_i \right| \tag{15}$$

$$\leq \frac{n-1}{4}(X - x)\left(\tilde{\Phi} - \tilde{\varphi}\right).$$

3. Applications to the moments of guessing mapping

To simplify the notation further, we assume that the x_i are numbered such that x_k is always the k^{th} guess with respect to G. This yields

$$E(G^p) = \sum_{i=1}^{n} i^p p_i, (p \geq 0).$$

Now, if we consider another guessing mapping L, we can write

$$E(L^p) = \sum_{i=1}^{n} i^p p_{\sigma(i)}$$

where σ is a permutation of the indices $\{1, ..., n\}$ associated with L.

Using the results from Section 2 we can give the following theorems.

Theorem 7. *Let $G(X)$ and $L(X)$ be two guessing mappings associated with the random variable X and $E(G(X)^p)$, $E(L(X)^p)$ $(p \geq 1)$ their p-moments. Then we have the estimation:*

$$|E(G(X)^p) - E(L(X)^p)|$$

$$\leq \begin{cases} [n^p - (n-1)^p] \sum\limits_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|, \\ \left(\sum\limits_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|^s\right)^{\frac{1}{s}} \left(\sum\limits_{i=1}^{n-1} \left[(i+1)^p - i^p\right]^l\right)^{\frac{1}{l}} \ \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty, \\ \max\limits_{i=\overline{1,n-1}} \left|P_i - P_{\sigma(i)}\right| (n^p - 1), \end{cases} \tag{16}$$

where $P_i = \sum\limits_{k=1}^{i} p_k$ and $P_{\sigma(i)} = \sum\limits_{k=1}^{i} p_{\sigma(k)}$.

Proof. Let us choose in *Theorem 3*, $q_i = p_{\sigma(i)}$ and $x_i = i^p$. As for $p \geq 1$ the sequence x_i is convex, we have that $(x_{i+1} - x_i)_{i=\overline{1,n-1}}$ is increasing, and then

$$\max_{i=\overline{1,n-1}} |x_{i+1} - x_i| = n^p - (n-1)^p .$$

Also, as $x_{i+1} \geq x_i$, we have $\sum_{i=1}^{n-1} |x_{i+1} - x_i| = n^p - 1$.

Now, by the inequality (5) we get (16). □

Remark 1. *If we choose $p = 1$, $s = l = 2$, then we have the estimation:*

$$|E(G(X)) - E(L(X))| \leq \begin{cases} \sum_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|, \\ (n-1)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|^2\right)^{\frac{1}{2}}, \\ (n-1) \max_{i=\overline{1,n-1}} \left|P_i - P_{\sigma(i)}\right|. \end{cases} \tag{17}$$

Remark 2. *If we choose $p = 2$, $s = l = 2$, then we have the estimation:*

$$\left|E\left(G(X)^2\right) - E\left(L(X)^2\right)\right| \leq \begin{cases} (2n-1) \sum_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|, \\ \left[\frac{(n-1)\left(4n^2+4n+3\right)}{3}\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \left|P_i - P_{\sigma(i)}\right|^2\right)^{\frac{1}{2}}, \\ \left(n^2-1\right) \max_{i=\overline{1,n-1}} \left|P_i - P_{\sigma(i)}\right|. \end{cases} \tag{18}$$

Indeed, as a simple calculation shows us that

$$\begin{aligned} \sum_{i=1}^{n-1} \left[(i+1)^2 - i^2\right]^2 &= \sum_{i=1}^{n-1} (2i+1)^2 = 4\sum_{i=1}^{n-1} i^2 + 4\sum_{i=1}^{n-1} i + n - 1 \\ &= 4\tfrac{(n-1)n(2n-1)}{6} + \tfrac{4(n-1)n}{2} + n - 1 \\ &= (n-1)\left[\tfrac{2n(2n-1)}{3} + 2n + 1\right] \\ &= \tfrac{(n-1)\left(4n^2-2n+6n+3\right)}{3} = \tfrac{(n-1)\left(4n^2+4n+3\right)}{3}. \end{aligned}$$

Another result is embodied in the following theorem.

Theorem 8. *With the assumptions of* Theorem 7*, we have the estimation:*

$$\left|E(G(X)^p) - E(L(X)^p) - S_p(n)\left(p_n - p_{\sigma(n)}\right)\right| \tag{19}$$

$$\leq \begin{cases} \max_{i=\overline{1,n-1}} \left|\Delta p_i - \Delta p_{\sigma(i)}\right| \sum_{i=1}^{n-1} S_p(i); \\ \left(\sum_{i=1}^{n-1} \left|\Delta p_i - \Delta p_{\sigma(i)}\right|^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} S_p^l(i)\right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty; \\ S_p(n-1) \sum_{i=1}^{n-1} \left|\Delta p_i - \Delta p_{\sigma(i)}\right|; \end{cases}$$

where $S_p(k) = \sum_{i=1}^{k} i^p, k = 1, ..., n.$

The proof is obvious by *Theorem 4* choosing $x_i = i^p, q_i = p_{\sigma(i)}, i = 1, ..., n$. We shall omit the details.

Remark 3. *If we choose in* (19) $p = 1$, *we get :*

$$\left| E(G(X)) - E(L(X)) - \frac{n(n+1)}{2} \left(p_n - p_{\sigma(n)}\right) \right| \tag{20}$$

$$\leq \begin{cases} \frac{(n-1)n(n+1)}{6} \max_{i=\overline{1,n-1}} \left|\Delta p_i - \Delta p_{\sigma(i)}\right| \\ \frac{(n-1)n}{2} \sum_{i=1}^{n-1} \left|\Delta p_i - \Delta p_{\sigma(i)}\right| \end{cases}.$$

Indeed, as a simple calculation shows us that:

$$\begin{aligned} \sum_{i=1}^{n-1} S_1(i) &= \sum_{i=1}^{n-1} \frac{i(i+1)}{2} = \frac{1}{2} \left(\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i \right) \\ &= \frac{1}{2} \left[\frac{(n-1)n(2n+1)}{6} + \frac{(n-1)n}{2} \right] = \frac{n(n-1)}{4} \left(\frac{2n-1}{3} + 1 \right) \\ &= \frac{2n(n-1)(n+1)}{12} = \frac{(n-1)n(n+1)}{6}. \end{aligned}$$

A similar inequality can be pointed out if we choose $p = l = 2$ in the second inequality in (19). We shall omit the details.

Now, if we use *Theorem 5*, we can state the following theorem:

Theorem 9. *With the assumptions from* Theorem 7*, we have:*

$$\begin{aligned} &\left| E(G(X)^p) - E(L(X)^p) - \frac{n^p-1}{n-1} \sum_{i=1}^{n-1} (n-i) \left(p_i - p_{\sigma(i)}\right) \right| \\ \leq\ & \frac{(n-1)(n^p-1)}{4} \left(M_{P,\sigma} - m_{p,\sigma}\right), p \geq 1, \end{aligned} \tag{21}$$

where

$$M_{p,\sigma} := \max_{i=\overline{1,n-1}} \left|P_i - P_{\sigma(i)}\right|$$

and

$$m_{p,\sigma} := \min_{i=\overline{1,n-1}} \left|P_i - P_{\sigma(i)}\right|.$$

Finally, by the use of *Theorem 6*, we can state another estimation result.

Theorem 10. *With the above assumptions, we have*

$$| E(G(X)^p) - E(L(X)^p) - \frac{nS_p(n) - S_{p+1}(n)}{n-1}\left[p_n - p_1 - p_{\sigma(n)} + p_{\sigma(1)}\right] \tag{22}$$

$$-S(n)\left(p_n - p_{\sigma(n)}\right) |$$

$$\leq \frac{(n-1)(S_p(n)-1)}{4}\left(\Phi_{p,\sigma} - \varphi_{p,\sigma}\right)$$

where

$$\Phi_{p,\sigma} = \max_{i=\overline{1,n-1}}\left(\Delta p_i - \Delta p_{\sigma(i)}\right)$$

and

$$\varphi_{p,\sigma} = \min_{i=\overline{1,n-1}}\left(\Delta p_i - \Delta p_{\sigma(i)}\right).$$

Remark 4. *In all previous results we have compared the moments of two guessing mappings $G(X)$ and $L(X)$ and obtain estimations in terms of the corresponding probabilities $p_i\left(p_{\sigma(i)}\right)$ $(i = 1, ..., n)$.*

In papers [4]-[6], S.S. Dragomir and J. van der Hoek obtained among others some estimations of moments of guessing mapping $E(G(X)^p)$ in terms of $S_p(n) := \sum_{i=1}^{n-1} i^p$ and the probabilities $P_M := \max\{p_i \mid i = 1, ..., n\}$ and $P_m := \min\{p_i \mid i = 1, ..., n\}$. Let us recall only one of them:

Theorem 11. *With the above assumptions we have :*

$$\left|E(G(X)^p) - \frac{1}{n}S_p(n)\right| \leq \frac{n(n^p - 1)}{4}(P_M - P_m). \tag{23}$$

Remark 5. *If we put in* (23) $p = 1$, *we get*

$$\left|E(G(X)) - \frac{n+1}{4}\right| \leq \frac{n(n-1)}{4}(P_M - P_m).$$

If we choose $p = 2$, we get

$$\left|E\left(G(X)^2\right) - \frac{(n+1)(2n+1)}{6}\right| \leq \frac{n\left(n^2-1\right)}{4}(P_M - P_m)$$

and, finally for $p = 3$, we obtain:

$$\left|E\left(G(X)^3\right) - \frac{n(n+1)^2}{4}\right| \le \frac{n\left(n^3-1\right)}{4}\left(P_M - P_m\right).$$

In this way, if we apply *Corollary 3*, we can obtain the following estimation result:

Theorem 12. *Let $G(X)$ be an arbitrary guessing mappig for the random variable X and $p \ge 1$. Then we have the estimation*

$$\begin{aligned}&\left|E\left(G(X)^p\right) - \tfrac{1}{n}S_p(n)\right| \\ &\le \begin{cases} \left[n^p - (n-1)^p\right]\sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|, \\ \left(\sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|^s\right)^{\frac{1}{s}}\left(\sum\limits_{i=1}^{n-1}\left[(i+1)^p - i^p\right]^l\right)^{\frac{1}{l}}, \frac{1}{s}+\frac{1}{l} = 1, 1 < s < \infty, \\ \max\limits_{i=\overline{1,n-1}}\left|P_i - \frac{i}{n}\right|\left(n^p - 1\right). \end{cases}\end{aligned}$$

Remark 6. *If we choose $p = 1$, $s = l = 2$, then we have the estimation*

$$\left|E(G(X)) - \frac{n+1}{2}\right| \le \begin{cases} \sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|, \\ n^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|^2\right)^{\frac{1}{2}}, \\ (n-1)\max\limits_{i=\overline{1,n-1}}\left|P_i - \frac{i}{n}\right|. \end{cases}$$

Remark 7. *If we choose $p = 2$, $s = l = 2$, then we have*

$$\left|E\left(G(X)^2\right) - \frac{(n+1)(2n+1)}{6}\right| \le \begin{cases} (2n-1)\sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|, \\ \left[\frac{(n-1)\left(4n^2+4n+3\right)}{3}\right]^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n-1}\left|P_i - \frac{i}{n}\right|^2\right)^{\frac{1}{2}}, \\ \left(n^2-1\right)\max\limits_{i=\overline{1,n-1}}\left|P_i - \frac{i}{n}\right|. \end{cases}$$

Similar results can be obtained, if we are going to apply the other results embodied in *Corollaries 4, 5* and *6*. We shall omit the details.

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