Some relations and properties concerning tangential polygons

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Abstract. *The k-tangential polygon is defined, and some of its properties are proved.*

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1. Preliminaries

A polygon with the vertices $A_1, ..., A_n$ (in this order) will be denoted by $A_1...A_n$. The lengths of the sides of the polygon $A_1...A_n$ will be denoted by $|A_1A_2|, ..., |A_nA_1|$ or $a_1, ..., a_n$. The interior angle at the vertex A_i will be denoted by α_i or $\angle A_i$, i.e.

$$\angle A_i = \angle A_{n-1+i}A_iA_{i+1}, \quad i = 1, ..., n \ (0 < \alpha_i < \pi). \tag{1}$$

Of course, indices are calculated modulo n.

A polygon $\mathbf{A} = A_1...A_n$ is a tangential polygon if there exists a circle $\mathcal{C}$ such that each side of $\mathbf{A}$ is on a tangent line of $\mathcal{C}$.

Definition 1. *Let $\mathbf{A} = A_1...A_n$ be a tangential polygon, and let k be a positive integer such that $k \leq \lfloor \frac{n-1}{2} \rfloor$, that is, $k \leq \frac{n-1}{2}$ if n is odd and $k \leq \frac{n-2}{2}$ if n is even. Then the polygon $\mathbf{A}$ will be called a k-tangential polygon if any two of its consecutive sides have only one point in common, and if there holds*

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}, \tag{2}$$

where $2\beta_i = \angle A_i$, $i = 1, ..., n$.

Consequently, a tangential polygon $\mathbf{A}$ is k-tangential if

$$\sum_{i=1}^{n} \varphi_i = 2k\pi,$$

where $\varphi_i = \angle A_iCA_{i+1}$ and C is the centre of the circle inscribed into the polygon $\mathbf{A}$. Namely, from this and the relation $2\beta_i = \angle A_i$ it follows that

$$\beta_i + \beta_{i+1} = \pi - \varphi_i, \quad i = 1, ..., n$$

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$$\sum_{i=1}^{n}(\beta_i + \beta_{i+1}) = n\pi - 2k\pi,$$

$$\sum_{i=1}^{n}\alpha_i = (n-2k)\pi.$$

For example, the tangential star-like pentagon is a 2-tangential polygon (*Figure 1*). Let us remark that the integer k can be most $\frac{n-1}{2}$ if n is odd and $\frac{n-2}{2}$ if n is even. If $\mathbf{A} = A_1 ... A_n$ is a k-tangential polygon and if $t_1, ..., t_n$ are the lengths of its tangents, then

$$t_i = r \cdot \operatorname{ctg}\beta_i, \quad i = 1, ..., n \tag{3}$$

where r is the radius of the circle inscribed to $\mathbf{A}$.

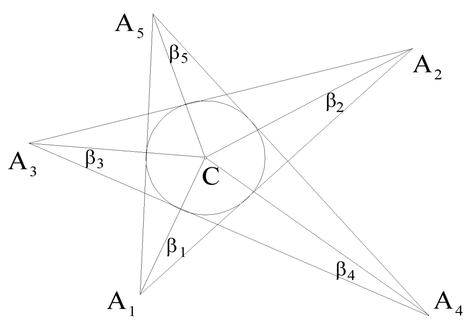

Figure 1.

2. Some properties and relations concerning k-tangential polygons

First we consider the existence of a k-tangential polygon.

Theorem 1. *Let $a_1, ..., a_n$, $t_1, ..., t_n$ be any given lengths (in fact positive numbers) such that*

$$t_i + t_{i+1} = a_i, \quad i = 1, ..., n$$

and let k be a positive integer such that $k \leq \lfloor\frac{n-1}{2}\rfloor$. Then there exists a k-tangential polygon whose sides have the lengths $a_1, ..., a_n$.

Proof. It is sufficient to prove that there exist acute angles $\beta_1, ..., \beta_n$ such that

$$t_1 \operatorname{tg}\beta_1 = \cdots = t_n \operatorname{tg}\beta_n \tag{4}$$

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}. \tag{5}$$

For this purpose we can use the expressions

$$\operatorname{tg}\beta_i = \frac{t_1}{t_i}\operatorname{tg}\beta_1, \quad i = 2, ..., n \tag{6}$$

For any choice of the acute angle β_1 there are completely determined acute angles $\beta_2, ..., \beta_n$ and there holds one of the relations

$$\beta_1 + \cdots + \beta_n < (n-2k)\frac{\pi}{2} \tag{7}$$

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2} \tag{8}$$

$$\beta_1 + \cdots + \beta_n > (n-2k)\frac{\pi}{2}. \tag{9}$$

If there holds (7) or (9), let then β_1 increase or decrease until (8) holds. It is possible because $\mathrm{tg}\beta$ is a continuous function for $0 < \beta < \frac{\pi}{2}$.

When (4) and (5) hold, then, as it is easily seen, we have

$$t_1 \mathrm{tg}\beta_1 = \cdots = t_n \mathrm{tg}\beta_n = r_k, \tag{10}$$

where r_k is the radius of the circle inscribed to the corresponding a k-tangential polygon. So, *Theorem 1* is proved. □

Corollary 1. *Let $n \geq 3$ be an odd number, and let $a_1, ..., a_n$ be given lengths. Then for any $k \leq \frac{n-1}{2}$ there exists a k-tangential polygon whose sides have lengths $a_1, ..., a_n$ if and only if*

$$\sum_{i=0}^{n-1} (-1)^{2+i} a_{i+j} > 0, \quad j = 1, ..., n. \tag{11}$$

Proof. The solution of the system

$$t_i + t_{i+1} = a_i, \quad i = 1, ..., n \tag{12}$$

is unique and given by

$$2t_j = \sum_{i=0}^{n-1} (-1)^{2+i} a_{i+j}, \quad j = 1, ..., n.$$

□

Remark 1. *It can be easily proved that the condition (11) follows from the condition*

$$\sum_{i=1}^{n} a_i > 2ka_j, \quad j = 1, ..., n \tag{13}$$

where $k = \frac{n-1}{2}$.

Corollary 2. *Let $n \geq 4$ be an even number, and let $a_1, ..., a_n$ be given lengths. Then for any $k \leq \frac{n-2}{2}$ there exists a k-tangential polygon whose sides have lengths $a_1, ..., a_n$ if and only if it is fulfilled*

$$\sum_{i=1}^{n} (-1)^i a_i = 0, \tag{14}$$

$$\min\{b_1, b_3, ..., b_{n-1}\} > \max\{b_2, b_4, ..., b_n\} \qquad (15)$$

where

$$b_j = \sum_{i=1}^{j} (-1)^{i+1} a_i, \quad j = 1, ..., n.$$

Proof. If (14) is not valid, then the system

$$t_i + t_{i+1} = a_i, \quad i = 1, ..., n \qquad (16)$$

has no solution. In the case when (14) is valid, then the above system has infinitely many solutions. But it can happen that none of these is positive. To see that the system (16) has infinitely many positive solutions if (14) and (15) are valid, it can be concluded as follows:

From (16) it follows

$$\begin{array}{rcl} t_2 & = & a_1 - t_1, \\ t_3 & = & a_2 - a_1 + t_1, \\ t_4 & = & a_3 - a_2 + a_1 - t_1, \\ & \vdots & \end{array}$$

In order to have $t_1, ..., t_n$ positive, it must be

$$\begin{array}{rcl} t_1 & < & a_1 \\ a_1 - a_2 & < & t_1 \\ t_1 & < & a_1 - a_2 + a_3 \\ & \vdots & \end{array}$$

□

Before making a statement of the following theorem we introduce some notations.

Symbol S_j^n. Let $t_1, ..., t_n$ be any given lengths of some line segments, and let j be any given integer such that $1 \le j \le n$. Then S_j^n is the sum of all $\binom{n}{j}$ products of the form $t_{i_1}...t_{i_j}$, where $i_1, ..., i_j$ are different elements of the set $\{1, ..., n\}$, that is

$$S_j^n = \sum_{1 \le i_1 < \cdots < i_j \le n} t_{i_1} \cdots t_{i_j}.$$

For example, if $n = 3$, then

$$S_1^3 = t_1 + t_2 + t_3, \quad S_2^3 = t_1 t_2 + t_2 t_3 + t_3 t_1, \quad S_3^3 = t_1 t_2 t_3.$$

Symbol C_j^n. Let $\beta_1, ..., \beta_n$ be any given angles such that

$$0 < \beta_i < \frac{\pi}{2}, \quad i = 1, ..., n$$

and let j be a given integer such that $1 \le j \le n$. Then C_j^n is the sum of all $\binom{n}{j}$ products of the form $\mathrm{ctg}\beta_{i_1}...\mathrm{ctg}\beta_{i_j}$, where $i_1, ..., i_j$ are different elements of the set $\{1, ..., n\}$.

Now we can express the connection between the lengths $t_1, ..., t_n$ of the tangents of the given k-tangential polygon and the radius r of its circle. For the reason of simplicity we shall first consider a 1-tangential polygon.

Theorem 2. *Let $n \geq 3$ be any given odd number. Then*

$$S_1^n r^{n-1} - S_3^n r^{n-3} + S_5^n r^{n-5} - \cdots + (-1)^s S_n^n = 0, \tag{17}$$

$$S_1^{n+1} r^{n-1} - S_3^{n+1} r^{n-3} + S_5^{n+1} r^{n-5} - \cdots + (-1)^s S_n^{n+1} = 0, \tag{18}$$

where $s = (1 + 3 + 5 + \cdots + n) + 1$.

Proof. First we state some special cases:

$$\begin{array}{ll} S_1^3 r^2 - S_3^3 = 0 & \text{(triangle)} \\ S_1^4 r^2 - S_3^4 = 0 & \text{(quadrangle)} \\ S_1^5 r^4 - S_3^5 r^2 + S_5^5 = 0 & \text{(pentagon)} \\ S_1^6 r^4 - S_3^6 r^2 + S_5^6 = 0 & \text{(hexagon)} \\ S_1^7 r^6 - S_3^7 r^4 + S_5^7 r^2 - S_7^7 = 0 & \text{(heptagon)} \\ S_1^8 r^6 - S_3^8 r^4 + S_5^8 r^2 - S_7^8 = 0 & \text{(octagon)} \end{array}$$

In proving this theorem the following lemma will be used.

Lemma 1. *Let $n \geq 3$ be a given odd number. If $\beta_1, ..., \beta_n$ are acute angles such that $\beta_1 + \cdots + \beta_n = (n-2)\frac{\pi}{2}$, then*

$$C_1^n - C_3^n + C_5^n - ... + (-1)^s C_n^n = 0, \tag{19}$$

and if $\beta_1, ..., \beta_{n+1}$ are acute angles such that $\beta_1 + \cdots + \beta_{n+1} = (n-1)\frac{\pi}{2}$, then

$$C_1^{n+1} - C_3^{n+1} + C_5^{n+1} - \cdots + (-1)^s C_n^{n+1} = 0, \tag{20}$$

where in both $s = (1 + 3 + 5 + \cdots + n) + 1$.

Proof of Lemma 1. If $\varphi_1, \varphi_2, \varphi_3, ...$ are acute angles, it can be easily found that

$$\operatorname{ctg}(\varphi_1 + \varphi_2) = \frac{C_2^2 - 1}{C_1^2}$$

$$\operatorname{ctg}(\varphi_1 + \varphi_2 + \varphi_3) = \frac{C_3^3 - C_1^3}{C_2^3 - 1}$$

$$\operatorname{ctg}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = \frac{C_4^4 - C_2^4 + 1}{C_3^4 - C_1^4}$$

$$\operatorname{ctg}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5) = \frac{C_5^5 - C_3^5 + C_1^5}{C_4^5 - C_2^5 + 1}, \ \cdots$$

By using induction on n, it can be proved that it generally holds

$$\operatorname{ctg}(\varphi_1+\cdots+\varphi_n) = \frac{\sum\limits_{i=0}^{k}(-1)^i C^n_{n-2i}}{\sum\limits_{i=0}^{k}(-1)^i C^n_{n-1-2i}}, \quad n=2k+1 \tag{21}$$

$$\operatorname{ctg}(\varphi_1+\cdots+\varphi_n) = \frac{\sum\limits_{i=0}^{k}(-1)^i C^n_{n-2i}}{\sum\limits_{i=0}^{k-1}(-1)^i C^n_{n-1-2i}}, \quad n=2k. \tag{22}$$

From this, it is easy to see that (19) and (20) hold, because if $\beta_1+\cdots+\beta_n=(n-2)\frac{\pi}{2}$, and n is odd, then

$$\operatorname{ctg}(\beta_1+\cdots+\beta_{n-1}) = \operatorname{ctg}\left[(n-2)\frac{\pi}{2}-\beta_n\right] = \frac{1}{\operatorname{ctg}\beta_n}$$

and this can be written as (19). Similar holds for the even number $n+1$. □

Now, let us return to the relations (17) and (18). We obtain these relations if in (19) and (20) we put $\frac{t_i}{r}$ instead $\operatorname{ctg}\beta_i$. □

Having proved *Theorem 2* we are going to consider some of its corollaries.

Corollary 3. *Let m be a positive integer such that $m=\frac{n-1}{2}$ if $n\geq 3$ is odd, and let r_k, $k=1,\ldots,m$ be the radius of a k-tangential polygon whose lengths of tangents are $t_1,\ldots,t_n$. Then every r_k is a solution of the equation (17), i.e.*

$$S_1^n r_k^{n-1} - S_3^n r_k^{n-3} + S_5^n r_k^{n-5} - \cdots + (-1)^s S_n^n = 0. \tag{23}$$

Similar holds for the equation (18) where $n+1$ is an even number.

Proof. If instead of $\beta_1+\cdots+\beta_n=(n-2)\frac{\pi}{2}$ we put $\beta_1+\cdots+\beta_n=(n-2k)\frac{\pi}{2}$, then all essential remains unchanged because

$$\operatorname{ctg}\left[(n-2k)\frac{\pi}{2}-\beta_n\right] = \frac{1}{\operatorname{ctg}\beta_n}, \quad k=1,\ldots,\frac{n-1}{2}.$$

Similar holds if instead of $\beta_1+\cdots+\beta_{n+1}=(n+1-2)\frac{\pi}{2}$ we take $\beta_1+\cdots+\beta_{n+1}=(n+1-2k)\frac{\pi}{2}$. □

Here are some examples which may be interesting.

(i) If $n=5$ and $t_j=j$, $j=1,2,3,4,5$, then

$$15r_k^4 - 195r_k^2 + 120 = 0, \quad k=1,2$$

$$r_1 \approx 3.51459, \qquad r_2 \approx 0.80477.$$

(ii) If $n=6$ and $t_j=j$, $j=1,2,\ldots,6$, then

$$21r_k^4 - 735r_k^2 + 1764 = 0, \quad k=1,2$$

$$r_1 \approx 5.69280, \qquad r_2 \approx 1.60995.$$

(iii) If $n = 7$ and $t_j = j$, $j = 1, 2, ..., 7$, then

$$28r_k^6 - 1960r_k^4 + 13132r_k^2 - 5040 = 0, \quad k = 1, 2, 3$$

$$r_1 \approx 7.90871, \quad r_2 \approx 2.65399, \quad r_3 \approx 0.63919.$$

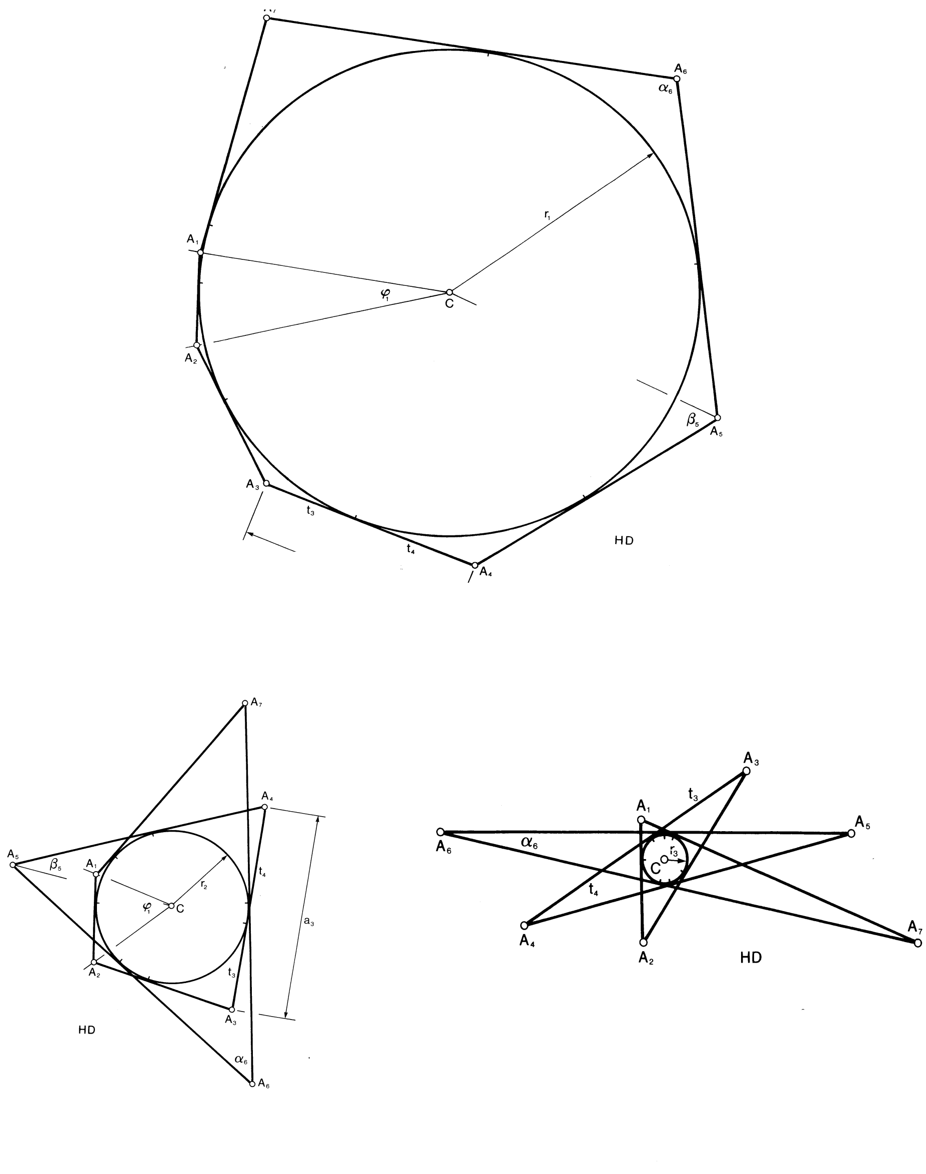

Figure 2.

Corollary 4. *The equations*

$$S_1^n x^{\frac{n-1}{2}} - S_3^n x^{\frac{n-3}{2}} + S_5^n x^{\frac{n-5}{2}} - \cdots + (-1)^s S_n^n = 0$$

$$S_1^{n+1} x^{\frac{n-1}{2}} - S_3^{n+1} x^{\frac{n-3}{2}} + S_5^{n+1} x^{\frac{n-5}{2}} - \cdots + (-1)^s S_n^{n+1} = 0$$

have all positive solutions.

Of course, $t_1, ..., t_n, t_{n+1}$ may be arbitrary positive numbers because to the length of a line segment there corresponds a positive number. For example, if $t_1, ..., t_6$ are arbitrary positive numbers, then

$$S_1^5 x^2 - S_3^5 x + S_5^5 = 0,$$

$$S_1^6 x^2 - S_3^6 x + S_5^6 = 0,$$

where

$$(S_3^5)^2 - 4S_1^5 S_5^5 > 0,$$
$$(S_3^6)^2 - 4S_1^6 S_5^6 > 0.$$

Especially, if $t_1 = \cdots = t_n = \frac{1}{2}$, and n is odd, then

$$\frac{1}{2}\binom{n}{1} r^{n-1} - \left(\frac{1}{2}\right)^3 \binom{n}{3} r^{n-3} + \cdots + (-1)^s \left(\frac{1}{2}\right)^n = 0,$$

$$\frac{1}{2}\binom{n+1}{1} r^{n-1} - \left(\frac{1}{2}\right)^3 \binom{n+1}{3} r^{n-3} + \cdots + (-1)^s \left(\frac{1}{2}\right)^n \binom{n+1}{n} = 0.$$

For example, if $n = 5$, then (*Figure 3*)

$$r^4 - \frac{1}{2} r^2 + \frac{1}{80} = 0$$

$$r_1 = \sqrt{\frac{1}{4} + \frac{\sqrt{5}}{10}} \approx 0.68819, \qquad r_2 = \sqrt{\frac{1}{4} - \frac{\sqrt{5}}{10}} \approx 0.16249.$$

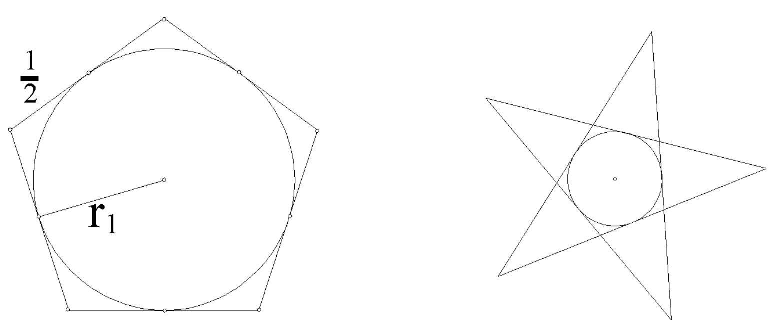

Figure 3.

Generally, if $m = \frac{n-1}{2}$ when n is odd and $m = \frac{n-2}{2}$ when n is even, then there are m tangential n-gons with equal sides and equal interior angles. For example, if $n = 9$ the situation is shown in *Figure 4.* (where the regular 1-tangential 9-gon is not drawn).

The radii of the corresponding circles are solutions of the equation

$$\frac{1}{2}\binom{9}{1}r^8 - \left(\frac{1}{2}\right)^3\binom{9}{3}r^6 + \left(\frac{1}{2}\right)^5\binom{9}{5}r^4 - \left(\frac{1}{2}\right)^7\binom{9}{7}r^2 + \left(\frac{1}{2}\right)^9 = 0.$$

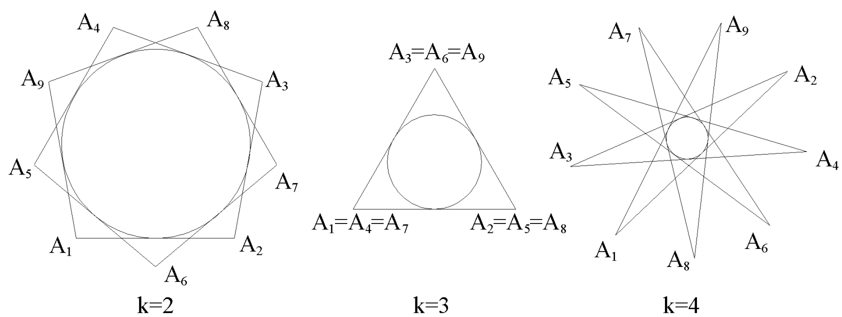

Figure 4.

Let us remark that in the case when $k|n$, the vertices $A_{i+n/k}$, $i = 1, ..., n/k$ are identical.

Before stating the following corollary we shall introduce one symbol. By $T(k, n)$ we shall denote a k-tangential n-gon, $k = 1, ..., \lfloor\frac{n-1}{2}\rfloor$, whose tangents have a property that $t_1 = t_2 = \cdots = t_n = 1$.

Corollary 5. *If r_k is the radius of the circle inscribed to $T(k, n)$, then by the expression*

$$x_k = r_k^2 = \mathrm{tg}^2\left((n-2k)\frac{\pi}{2n}\right), \quad k = 1, ..., \lfloor\frac{n-1}{2}\rfloor \tag{24}$$

are given all the solutions of the equation

$$\sum_{i=1}^{\frac{n+1}{2}}(-1)^{i+1}\binom{n}{2i-1}x^{\frac{n-2i+1}{2}} = 0, \quad n \text{ is odd} \tag{25}$$

or of the equation

$$\sum_{i=1}^{\frac{n}{2}}(-1)^{i+1}\binom{n}{2i-1}x^{\frac{n-2i}{2}} = 0, \quad n \text{ is even.} \tag{26}$$

Proof. If we put x instead of r_k^2 in the equations

$$\binom{n}{1}r_k^{n-1} - \binom{n}{3}r_k^{n-3} + \binom{n}{5}r_k^{n-5} - \cdots + (-1)^s\binom{n}{n} = 0, \quad n \text{ odd}$$

$$\binom{n}{1}r_k^{n-2} - \binom{n}{3}r_k^{n-4} + \binom{n}{5}r_k^{n-6} - \cdots + (-1)^s\binom{n}{n-1} = 0, \quad n \text{ even}$$

we obtain the equations (25) and (26).

From $\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}$, in the case when $\beta_1 = \cdots = \beta_n = \beta$, it follows that $\beta = (n-2k)\frac{\pi}{2n}$, and from $1 = r_k \mathrm{ctg}\beta$, it follows that

$$r_k = \mathrm{tg}\left((n-2k)\frac{\pi}{2n}\right), \quad k = 1, ..., \lfloor\frac{n-1}{2}\rfloor$$

□

Corollary 6. *If n is odd, then*

$$\sum_{k=1}^{\frac{n-1}{2}} \mathrm{tg}^2\left((n-2k)\frac{\pi}{2n}\right) = \binom{n}{3} : \binom{n}{1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathrm{tg}^2\left((n-2)\frac{\pi}{2n}\right) \quad \mathrm{tg}^2\left((n-4)\frac{\pi}{2n}\right) \quad \cdots \quad \mathrm{tg}^2\frac{\pi}{2n} = \binom{n}{n} : \binom{n}{1}$$

Similar holds when n is even.

Proof. The above equations are Vieta's formulas for the equation (25). □

Corollary 7. *If r_k^2, $k \in \left\{1, ..., \lfloor\frac{n-1}{2}\rfloor\right\}$ is a solution of the equation (25) or (26) and k such that $n \le 4k$, then*

$$r_k - \frac{1}{3}r_k^3 + \frac{1}{5}r_k^5 - \frac{1}{7}r_k^7 + \cdots = (n-2k)\frac{\pi}{2n}. \tag{27}$$

Proof. $r_k = \mathrm{tg}\left((n-2k)\frac{\pi}{2n}\right)$, it is

$$\mathrm{arc\,tg} r_k = (n-2k)\frac{\pi}{2n}.$$

□

In order to define the left-hand side of (27) ($r_k \le 1$), the condition $(n-2k)\frac{\pi}{2n} \le \frac{\pi}{4}$ or $n \le 4k$ must be fulfilled. For example:

if $k = 1$, then $n = 3, 4$

if $k = 2$, then $n = 5, 6, 7, 8$

if $k = 3$, then $n = 7, 8, 9, 10, 11, 12$

So, if $n = 4$, $k = 1$, then $r = 1$, and we have Leibniz series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

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