Necessary and sufficient condition for L^1-convergence of cosine trigonometric series with δ-quasimonotone coefficients

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Abstract. *For a cosine trigonometric series with coefficients in the class $S_p(\delta)$, $1 < p \le 2$, the necessary and sufficient condition for L^1-convergence is obtained.*

Key words: *δ–quasi–monotone sequence, cosine trigonometric series, Fourier series, Dirichlet kernel, Abel's transformation, Holder inequality, Hausdorff-Young inequality, L^1-convergence of Fourier series*

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1. Introduction

Let f be a 2π-periodic and even function in $L^1(0,\pi)$, and let $\{a_k\}$ be the sequence of its Fourier coefficients. Denote by $\mathcal{J}$ the class of sequences of Fourier coefficients of all such functions. It is well known that, in general, it does not follow from $\{a_n\} \in \mathcal{J}$ that

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \tag{1}$$

converges to f in the L^1-norm, i.e. it does not follow that $\|S_n - f\| = o(1),\ n \to \infty$. There is a classical example for which $\|S_n - f\| = o(1),\ n \to \infty$ is equivalent with $a_n \log n = o(1),\ n \to \infty$.

Telyakovskii [8] introduced the following class S. A sequence $\{a_k\}$ belongs to the class S if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k. A sequence $\{a_k\}$ of positive numbers is said to be quasi-monotone if $a_k \to \infty$ as $k \to 0$ and $\Delta a_k \ge -\beta \frac{a_k}{k}$, for some $\beta > 0$.

A sequence $\{a_k\}$ is said to be δ–quasi–monotone if $a_k \to 0,\ a_k > 0$, ultimately, and $\Delta a_k \ge -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

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A sequence $\{a_k\}$ is said to satisfy condition S', if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi–monotone, $\sum_{k=1}^{\infty} A_k < \infty$, $|\Delta a_k| \le A_k$, for all k.

On the other hand, a sequence $\{a_k\}$ is said to satisfy condition $S(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is δ–quasi-monotone, $\sum_{k=1}^{\infty} A_k < \infty$, $\sum_{k=1}^{\infty} k\delta_k < \infty$ and $|\Delta a_k| \le A_k$, for all k.

Now, we say that a null-sequence $\{a_k\}$ belongs to the class $S_p(\delta)$, $1 < p \le 2$ if there exists a sequence of numbers $\{A_k\}$ such that:

(a) $\{A_k\}$ is δ–quasi–monotone and $\sum_{k=1}^{\infty} k\,\delta_k < \infty$.

(b) $\sum_{k=1}^{\infty} A_k < \infty$.

(c) $\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$.

Thus, in view of the above definitions it is obvious that $S' \subset S(\delta) \subset S_p(\delta)$.

2. Lemmas

For the proof of our theorem we require the following lemmas.

Lemma 1. (Hausdorff–Young, see [3]) *Let the sequence of complex numbers* $\{c_n\} \in l^p$, $1 < p \le 2$. *Then* $\{c_n\}$ *is the sequence of Fourier coefficients of some* $\varphi \in L^q \left(\frac{1}{p} + \frac{1}{q} = 1\right)$, *and*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(x)|^q \, dx\right)^{1/q} \le \left(\sum_{n=-\infty}^{\infty} |c_n|^p\right)^{1/p}.$$

Lemma 2. (see [1],[11] case $v=1$) *If* $\{a_n\}$ *is a* δ*–quasi-monotone sequence with* $\sum_{n=1}^{\infty} n\delta_n < \infty$, *then the convergence of* $\sum_{n=1}^{\infty} a_n$ *implies that* $n a_n = o(1)$, $n \to \infty$.

Lemma 3. (see [11]) *Let* $\{a_n\}$ *be a* δ*-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n\delta_n < \infty$. *If* $\sum_{n=1}^{\infty} a_n < \infty$, *then* $\sum_{n=1}^{\infty} (n+1)\,|\Delta a_n| < \infty$.

Lemma 4. *Let the coefficients of the series (1) satisfy the condition* $S_p(\delta)$, $1 < p \le 2$. *Then the following relations hold* *a)* $\displaystyle\int_0^\pi \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x)\right| dx = O_p(k)$, *where* O_p *depends on* p.

b) $\displaystyle A_n \int_0^\pi \left|\sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x)\right| dx = o(1),\ n \to \infty.$

Proof. a) We have

$$\int_0^\pi \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x)\right| dx = \int_0^{\pi/k} + \int_{\pi/k}^{\pi} = I_k + J_k\,.$$

Recalling the uniform estimate of the Dirichlet kernel we have:

$$I_k \le A \sum_{j=0}^{k} \frac{|\Delta a_j|}{A_j} \le Ak \left(\frac{1}{k} \sum_{j=0}^{k} \frac{|\Delta a_k|^p}{A_j^p}\right)^{1/p}$$

where A is an absolute constant.

To estimate the second integral:

$$J_k = \int_{\pi/k}^{\pi} \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x)\right| dx = \int_{\pi/k}^{\pi} \frac{1}{2\sin\frac{x}{2}} \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin\left(j+\frac{1}{2}\right)x\right| dx\,.$$

We shall first apply the Holder inequality, where $\frac{1}{p}+\frac{1}{q}=1$,

$$J_k \le \left[\int_{\pi/k}^{\pi} \left(\frac{1}{2\sin\frac{x}{2}}\right)^p dx\right]^{1/p} \left[\int_0^\pi \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin\left(j+\frac{1}{2}\right)x\right|^q dx\right]^{1/q}.$$

Since

$$\int_{\pi/k}^{\pi} \frac{dx}{\left(\sin\frac{x}{2}\right)^p} \le \pi^p \int_{\pi/k}^{\pi} \frac{dx}{x^p} \le \frac{\pi}{p-1} k^{p-1}\,,$$

it follows that

$$J_k \le \frac{1}{2}\left(\frac{\pi}{p-1}\right)^{1/p} k^{(p-1)/p} \left[\int_0^\pi \sum_{j=0}^{k} \left|\frac{\Delta a_j}{A_j} \sin\left(j+\frac{1}{2}\right)x\right|^q dx\right]^{1/q}.$$

Then using the Hausdorff-Young inequality we get:

$$\left[\int_0^\pi \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} \sin\left(j+\frac{1}{2}\right)x\right|^q dx\right]^{1/q} \le \left[\int_0^\pi \sum_{j=0}^{k} \left|\frac{\Delta a_j}{A_j} e^{ijx}\right|^q dx\right]^{1/q} \le \left(\sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p}.$$

Finally,

$$J_k \leq B_p \left(\frac{1}{k} \sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p},$$

where B_p is an absolute constant dependent on p. Thus

$$\int_0^\pi \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p(k).$$

b) Applying first the relation a) of this lemma, then *Lemma 2* yields

$$\int_0^\pi \left| \sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p(nA_n) = o(1), \ \ n \to \infty .$$

□

3. Main result

Theorem 1. *Let $\{a_k\} \in S_p(\delta)$, $1 < p \leq 2$. Then (11) is a Fourier series of some $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$.*

Proof. By summation by parts, we have:

$$\begin{aligned} \sum_{k=1}^{n} |\Delta a_k| &= \sum_{k=1}^{n} A_k \tfrac{|\Delta a_k|}{A_k} \leq \sum_{k=1}^{n-1} |\Delta A_k| \sum_{j=1}^{k} \tfrac{|\Delta a_j|}{A_j} + A_n \sum_{j=1}^{n} \tfrac{|\Delta a_j|}{A_j} \\ &\leq \sum_{k=1}^{n-1} k|\Delta A_k| \left(\tfrac{1}{k} \sum_{j=1}^{k} \tfrac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + nA_n \left(\tfrac{1}{n} \sum_{j=1}^{n} \tfrac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \\ &= O(1) \left[\sum_{k=1}^{n-1} k|\Delta A_k| + nA_n \right] \leq O(1) \left[\sum_{k=1}^{n-1} (k+1)|\Delta A_k| + nA_n \right]. \end{aligned}$$

Application of *Lemma 2* and *Lemma 3* yields, $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, i.e. $S_n(x)$ converges to $f(x)$, for $x \neq 0$.

Using Abel's transformation, we obtain:

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x),$$

by the fact that $\lim_{n\to\infty} a_n D_n(x) = 0$, if $x \neq 0$, where $D_n(x)$ is the Dirichlet kernel.

Then,

$$\begin{aligned}
\|S_n - f\| &= \left\| \tfrac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - f(x) \right\| \\
&= \left\| \tfrac{a_0}{2} - \tfrac{a_{n+1}}{2} + \sum_{k=1}^{n} (a_k - a_{n+1}) \cos kx - f(x) + \tfrac{a_{n+1}}{2} + \sum_{k=1}^{n} a_{n+1} \cos kx \right\| \\
&= \left\| \tfrac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx - f(x) + a_{n+1} D_n(x) \right\| \\
&= \| g_n(x) - f(x) + a_{n+1} D_n(x) \| ,
\end{aligned}$$

where $g_n(x) = \dfrac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx$ are the Rees-Stanojević sums (see [2],[6],[7]).

We have:

$$\begin{aligned}
g_n(x) &= \tfrac{\Delta a_0}{2} + \sum_{k=1}^{n} \left(\tfrac{1}{2}\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx \right) \\
&= \tfrac{\Delta a_0}{2} + \sum_{k=1}^{n} \tfrac{1}{2}\Delta a_k + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) + \tfrac{1}{2} a_{n+1} .
\end{aligned}$$

Using Abel's transformation, we obtain:

$$\begin{aligned}
g_n(x) &= \tfrac{\Delta a_0}{2} + \sum_{k=1}^{n} \tfrac{1}{2}\Delta a_k + \sum_{k=1}^{n-1} \Delta a_k \left(D_k(x) - \tfrac{1}{2} \right) + a_n \left(D_n(x) - \tfrac{1}{2} \right) \\
&\quad - a_{n+1} D_n(x) + \tfrac{1}{2} a_{n+1} \\
&= \Delta a_0 D_0(x) + \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x) \\
&= \sum_{k=0}^{n} \Delta a_k D_k(x) .
\end{aligned}$$

Since $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, the series $\sum_{k=0}^{\infty} \Delta a_k D_k(x)$ converges. Hence $\lim_{n\to\infty} g_n(x)$ exists for $x \neq 0$.

Then,

$$\| f(x) - g_n(x) \| = \left\| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right\| = \frac{1}{\pi} \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx .$$

Application of Abel's transformation and *Lemma 4.b)* yields

$$\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \le \sum_{k=n+1}^{\infty} |\Delta A_k| \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx + o(1), \quad n \to \infty .$$

Then, by *Lemma 4.a)* and *Lemma 3*, we have:

$$\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx = O_p \left(\sum_{k=n+1}^{\infty} |\Delta A_k| (k+1) \right) + o(1) = o(1), \quad n \to \infty .$$

Thus $\|f(x) - g_n(x)\| = o(1), \ \ n \to \infty$.

"If": Let $\|S_n - f\| = o(1), \ \ n \to \infty$, then by the formulae:

$$S_n(x) = g_n(x) + a_{n+1} D_n(x),$$

we get:

$$\|a_{n+1} D_n(x)\| = \|S_n - g_n\| \leq \|S_n - f\| + \|f - g_n\| = o(1) + o(1), \ \ n \to \infty\,.$$

Since $\|D_n(x)\| = O(\log n)$, we have, $a_n \log n = o(1), \ n \to \infty$.

"Only if": Let $a_n \log n = o(1), \ n \to \infty$. Then,

$$\|S_n - f\| \leq \|g_n - f\| + \|a_{n+1} D_n(x)\| = o(1) + a_{n+1} O(\log n) = o(1), \ \ n \to \infty.$$

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