Absolute polarity on the sphere; conics; loxodrome, tractrix*†

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Abstract. *Elements in the bundle and on the sphere, distance, angle and absolute polarity are explained. The spherical conic has an absolute polar conic, they are equidistant to each other; they have the same evolute curve. The well-known focus-properties are absolutely polarized.*

The loxodrome and the spherical evolute curve are presented. Polarizing the loxodrome there results a spherical tractrix. Kinematic generation of these curves is shown.

Key words: *bundle, sphere, absolute polarity; spherical conic, loxodrome, spherical tractrix; spherical evolute curve, equidistant curves; kinematic generation*

AMS subject classifications: 51N05, 51N20, 53A04, 53A17, 53A35

Received July 2, 1999 Accepted September 19, 1999

1. Geometry in the bundle

In the EUKLIDean $\mathbb{R}^3$ we fix one point $O(0/0/0)$ and so we get the nonEUKLIDean elliptic geometry in the bundle with the elements:

straight lines $\ni O$,	named "points"
planes $\ni O$,	named "straight lines".

Further we have:

cones of rotation, spit O,	named "circles"
cones of 2^{nd} deg., spit O,	named "curves of 2^{nd} degree"
cones, spit O,	named "curves"
angles, formed by two straight lines,	named "distance between points"
angles, formed by two planes,	named "angles between straight lines"

*The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society - Division Osijek, April 23, 1999.

†The author wants to express his best thanks to the referee and to the editorial staff.

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a straight line $\ni O$ and a plane $\ni O$ which are orthogonal ($\perp$) are absolutely polar	This situation is named	"a point and a straight line which are orthogonal ($\perp$) are absolutely polar"

Often we calculate with CARTESian coordinates and vectors. All elements and shapes are extended in both directions from O. We write $\perp = 90^o = 100^g = \pi/2$.

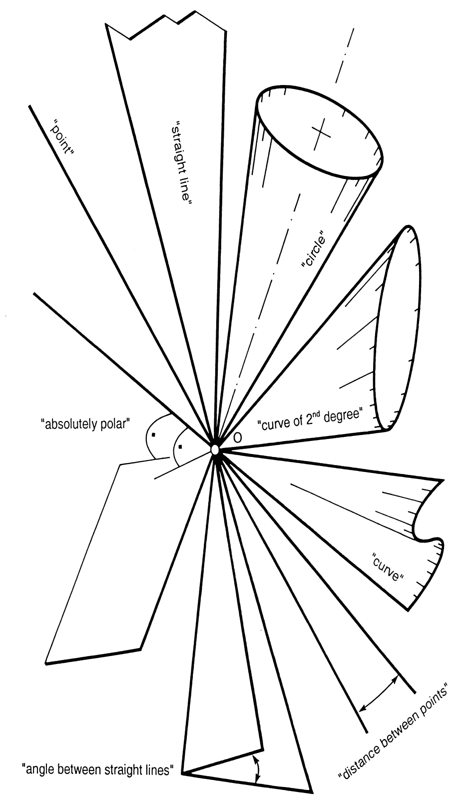


Figure 1.1. *The bundle. All elements and shapes are extended in both directions from O!*

2. Geometry on the sphere

We introduce the unit sphere $\mathcal{K}$ with center O

$$x^2 + y^2 + z^2 = 1; \tag{2.1}$$

all elements and shapes of 1. are intersected with this sphere. Now we get:

pairs of points, named "points"
great circles, named "straight lines"

pairs of small circles, named "circles"
"curves of 2^{nd} degree"
"curves"
"distance between points"
"angles between straight lines"

If $\widehat{\text{point}_1\text{point}_2} = \frac{\pi}{2}$, we are allowed to say "orthogonal points".

If $\widehat{\text{straight line}_1\text{straight line}_2} = \frac{\pi}{2}$, we say "orthogonal straight lines".

If a point and a straight line which have the shortest distance $= \frac{\pi}{2}$, are absolutely polar, we are allowed to say that "they are orthogonal".

This well-known transformation $\mathbf{P}$ which transforms points into straight lines, so that distance $= \frac{\pi}{2}$, is the ABSOLUTE POLARITY $\mathbf{P}$, in this paper short POLARITY $\mathbf{P}$. We remark that $\mathbf{P}$ transforms distances into angles. $\mathbf{P}$ is an involutoric transformation, so that $\mathbf{P} \circ \mathbf{P} = \mathbf{I} =$ identity.

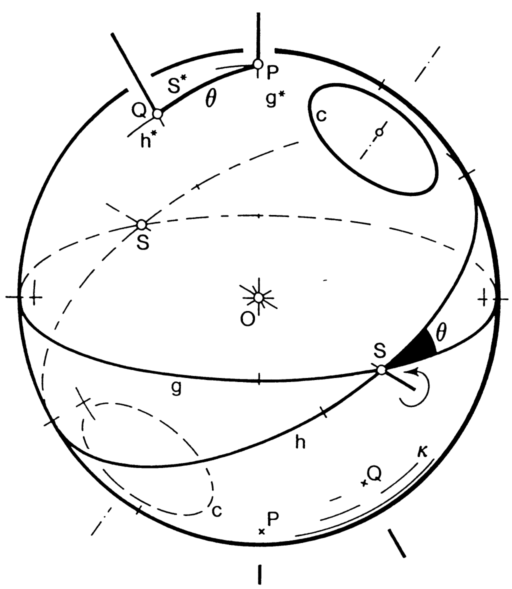
The geometries in the bundle 1. and on the sphere 2. are isomorphic; there remain only questions of convenience, visualization, didactics, calculation, computing, constructive and descriptive geometry to choose the method of work. Often we use spherical trigonometry. It is allowed to use the vocabulary of geography.

A line element is a point with direction. $\mathbf{P}$ transforms such a line element into a line element.

Let "*" be the symbol for absolutely polarized elements.

We write $P^* = \mathbf{P} \circ P$ or $P^* \perp P$, where P is a point, but P^* is a straight line, we speak: P^* is the polar line of P.

We write $g^* = \mathbf{P} \circ g$ or $g^* \perp g$, where g is a straight line, but g^* is a point, we speak: g^* is the pole of g.



$P =$point
$g =$ straight line
$c =$circle
$\widehat{PQ} =$distance
$\widehat{gh} =$angle
$P \perp g$
$Q \perp h$
$\widehat{PQ} = \widehat{gh}$

Figure 2.1. *The sphere*

3. Spherical conics c

A cone Φ of 2^{nd} degree has 3 midpoints, which we put into:

$$\left.\begin{array}{l} U = x - \text{axis} \cap \mathcal{K} \\ V = y - \text{axis} \cap \mathcal{K} \end{array}\right\} \text{ are on the "equator" ä}$$

$$W = z - \text{axis} \cap \mathcal{K} = \text{ the "north pole" and the "south pole".}$$

Now Φ can be written as

$$\Phi \ \cdots \ \frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} - z^2 = 0 \tag{3.1}$$

$$0 < b \leq a < \frac{\pi}{2}, \tag{3.2}$$

where a and b are the half lengths of the axes of Φ in the bundle.
Now the spherical conic c is

$$c = \mathcal{K} \cap \Phi, \ \ c = (2.1) \cap (3.1), \ \ \text{let } (3.2) \tag{3.3}$$

a and b are the half spherical lengths of the axes of c.

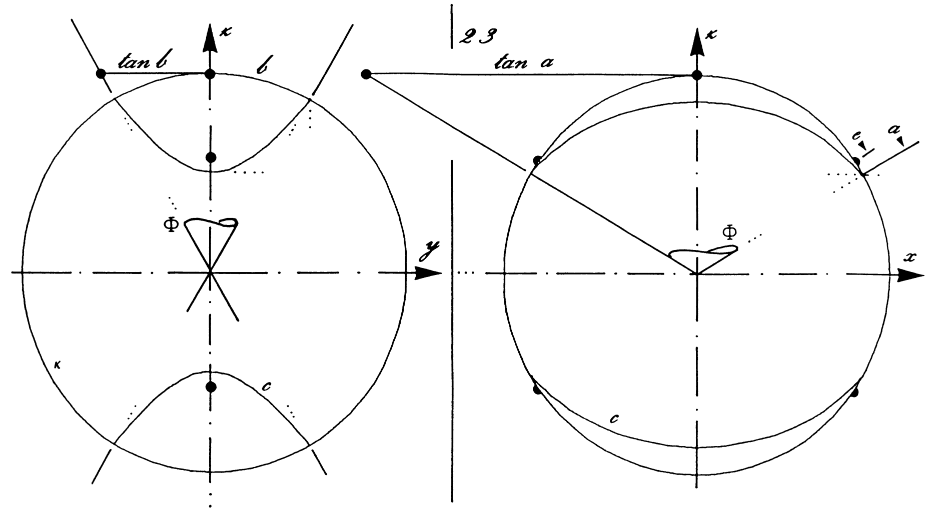
Eliminating x or y or z from (3.3) you can get the front view, the side view, the top view, the formulas for *Figure 3.1.* Sometimes it is clever to work in the planes $z = 1$ or $z = -1$:

$$\left.\begin{array}{lll} c_z = \Phi \cap (z = \pm 1) & \cdots & \frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} = 1 \\ & & \text{AND} \qquad z = \pm 1 \end{array}\right\} \tag{3.4}$$

$$\left.\begin{array}{lll} c_z & \cdots & x = \cos T \tan a \\ & & y = \sin T \tan b \\ & & z = \pm 1 \end{array}\right\} \tag{3.5}$$

Now we have

$$\left.\begin{array}{lll} c & \cdots & R = \sqrt{(\cos T \tan a)^2 + (\sin T \tan b)^2 + 1} \\ & & x = \frac{1}{R} \cos T \tan a \\ & & y = \frac{1}{R} \sin T \tan b \\ & & z = \pm \frac{1}{R} \end{array}\right\} \tag{3.6}$$



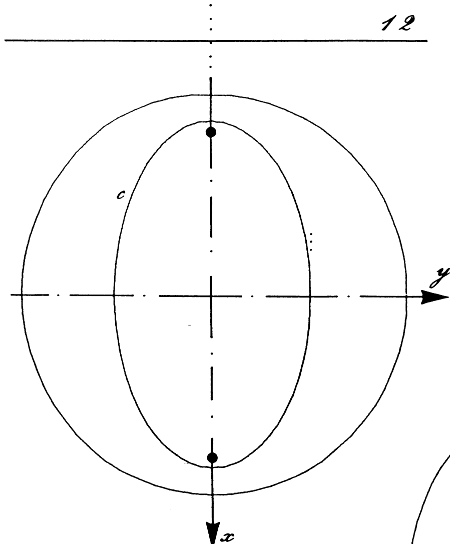


Fig. 3.1 :

front view .... side view

top view of a spherical conic c

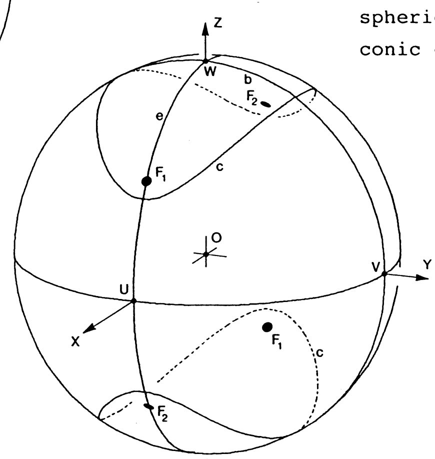


Fig. 3.2 : Orthogonal axonometric view of a spherical conic c

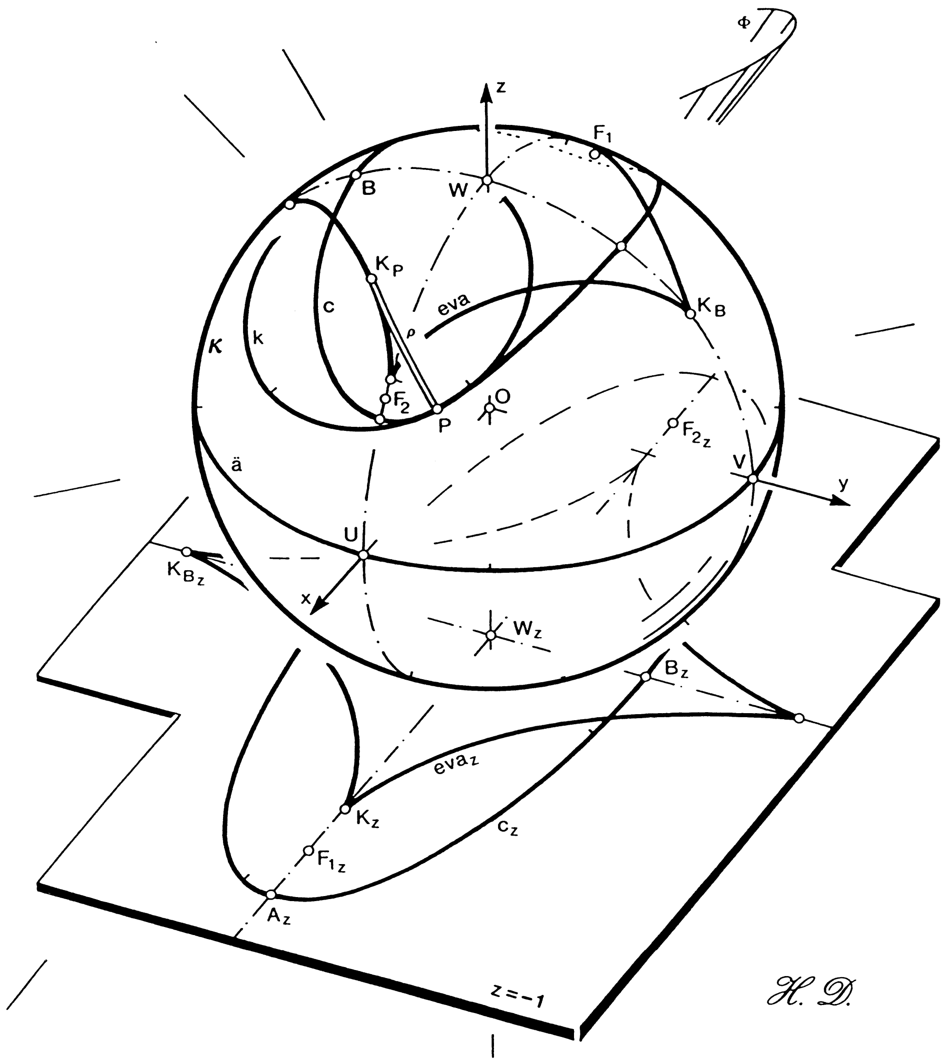


Figure 3.3. *c and* c_z, $a = \frac{\pi}{3}$, $b = \frac{\pi}{6}$

The figure also shows for the next chapter: k = circle of curvature, eva = evolute curve of c, $eva_z = [eva\ O] \cap (z = -1)$.

4. Spherical evolute curve *eva*

Let

$$\left.\begin{aligned} m &= \tfrac{\sin^2 a - \sin^2 b}{\sin a \cos a} \\ n &= \tfrac{\sin^2 b - \sin^2 a}{\sin b \cos b} \end{aligned}\right\} \tag{4.1}$$

Then we get the evolutic cone of c as

$$[O\ eva]\ \ldots\ \left(\frac{x}{m}\right)^{\frac{2}{3}} + \left(\frac{y}{n}\right)^{\frac{2}{3}} = z^{\frac{2}{3}} \tag{4.2}$$

$$\left.\begin{aligned} eva_z\ \ldots\ \left(\tfrac{x}{m}\right)^{\frac{2}{3}} + \left(\tfrac{y}{n}\right)^{\frac{2}{3}} &= 1 \\ \text{AND} \qquad z &= \pm 1 \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} eva_z \quad \cdots \quad & x = m\cos^3 T \\ & y = n \sin^3 T \\ & z = \pm 1 \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} eva \quad \cdots \quad & R = \sqrt{m^2\cos^6 T + n^2 \sin^6 T + 1} \\ & x = \tfrac{m}{R}\cos^3 T \\ & y = \tfrac{n}{R}\sin^3 T \\ & z = \pm\tfrac{1}{R} \end{aligned}\right\} \tag{4.5}$$

(4.1) shows that we have the chance to find a nontrivial equilateral spherical evolute curve, the condition is

$$a + b = \frac{\pi}{2}. \tag{4.6}$$

An equilateral evolute curve of an ellipsis cannot appear in EUKLIDean geometry! The case (4.6) and all facts of *Section 4* are shown in *Figure 3.3*.

5. Absolute polarization of the conic c

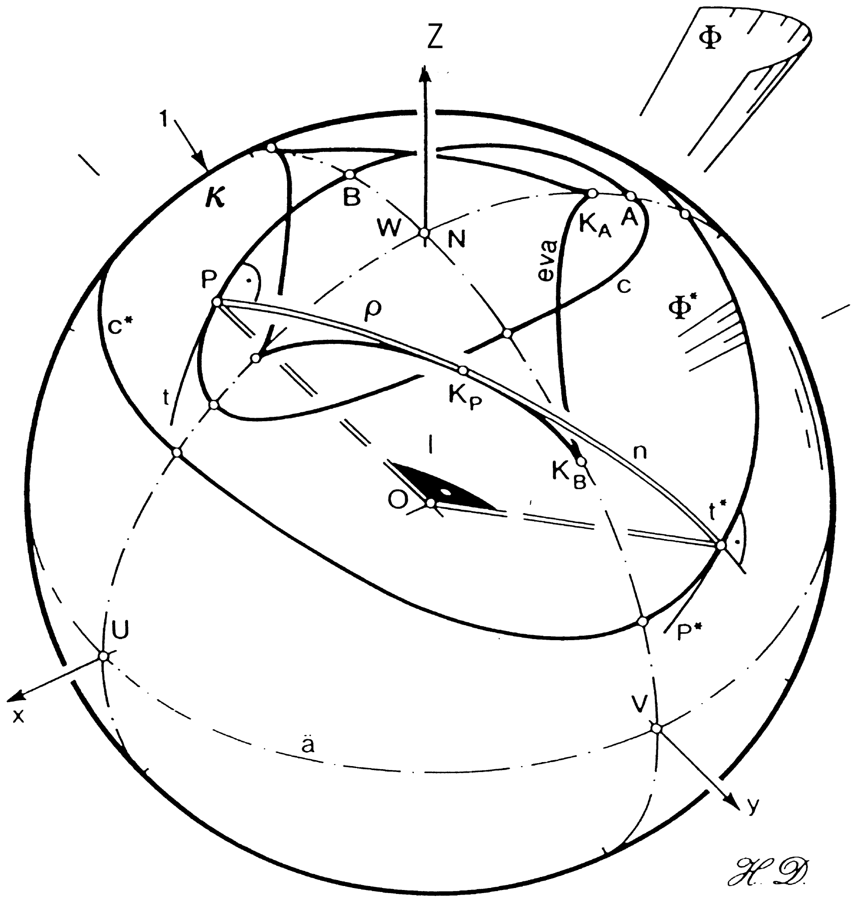


Figure 5.1. *Conic c and the absolute polar conic c^**

c is to be polarized. By $\mathbf{P}$ the line element P with t is transformed into the line element P^* with t^*, c is transformed into c^*.

c has a and b due to 3.,

$$c^* \text{ has } a^* = \frac{\pi}{2} - a, \quad b^* = \frac{\pi}{2} - b \tag{5.1}$$

as new spherical half lengths of the axes.

The distance $\widehat{Pt^*} = \frac{\pi}{2}$ and $n = [P\ t^*]$ rolls on *eva*, while P and t^* draw the paths c and c^*, while t and P^* envelop c and c^*; c and c^* are equidistant curves. c and c^* have the same evolute curve *eva*.

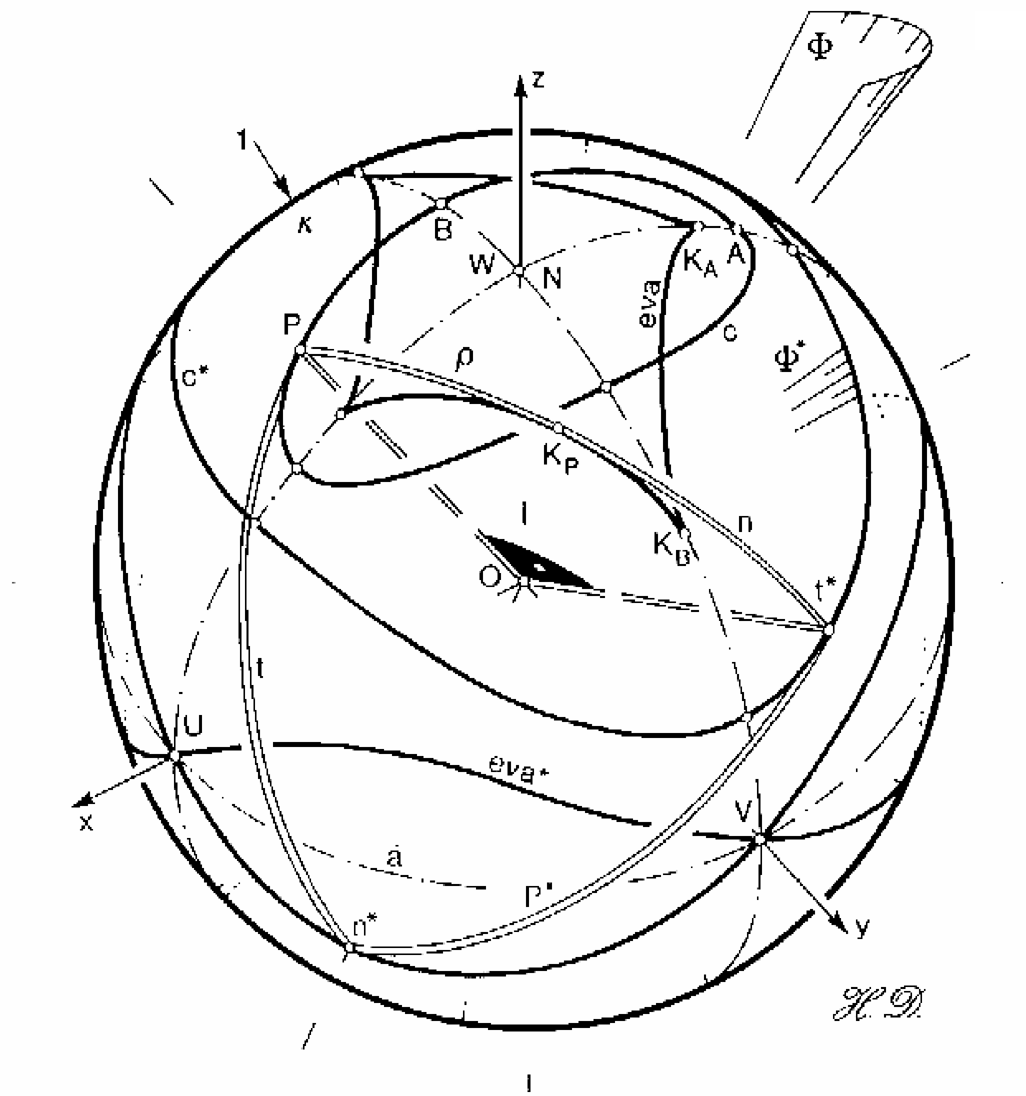


Figure 5.2. *As in* Figure 5.1*, we can additionally see the octant* $P\ t^*\ n^*$ *and the path* eva^**.*

We can absolutely polarize *eva* and we get eva^*. The points $P\ t^*\ n^*$ form a spherical octant: the three angles have $\frac{\pi}{2}$, the three edges have $\frac{\pi}{2}$.

When n is rolling on *eva*, n^* draws the path eva^*.

In the language of spherical kinematic geometry: *eva* is the fixed polhode, n is the moving polhode of this spherical motion.

6. Focus-properties of the spherical conic c

A didactical trick: we begin with c^*. The spherical conic c^* has the well-known foci F_1^* and F_2^*. c^* has the well-known properties

$$\left.\begin{array}{lcl} \widehat{F_1^* t^*} & = & r_1^* \\ \widehat{F_2^* t^*} & = & r_2^* \\ r_1^* + r_2^* & = & 2b^* \end{array}\right\} \tag{6.1}$$

t^* is a point of c^*.

The supplements of these distances are

$$\left.\begin{array}{l} r_1^{H*} = \pi - r_1^* \\ r_2^{H*} = \pi - r_2^* \end{array}\right\} \tag{6.2}$$

(6.1) looks like an ellipsis, but if we substitute one of the terms (6.2), it looks like a hyperbola; in fact, there exists only one kind of a spherical conic. The tangent line P^* in t^* bisects the angle between r_1^* and r_2^{H*};

$$\text{the half angle is denoted as } \sigma \tag{6.3}$$

Now let us use *Figure 6.1* and let us polarize these facts step by step!

$$\mathbf{P} \circ c^* = c$$

$$\begin{array}{lcl} \mathbf{P} \circ (\text{point } \ t^* \ \text{ of } \ c^*) & = & \text{tangent line } t \text{ of } c \\ \mathbf{P} \circ (\text{tangent line } \ P^* \ \text{ of } \ c^*) & = & \text{point } P \text{ of } c \\ \mathbf{P} \circ F_1^* = F_1 & = & \text{“focus line” of } c \\ \mathbf{P} \circ F_2^* = F_2 & = & \text{“focus line” of } c \\ [F_1^* \ t^*] & = & \text{line } v_1^* \\ [F_2^* \ t^*] & = & \text{line } v_2^* \\ \mathbf{P} \circ \text{ line } v_1^* & = & \text{point } v_1 \\ \mathbf{P} \circ \text{ line } v_2^* & = & \text{point } v_2 \end{array}$$

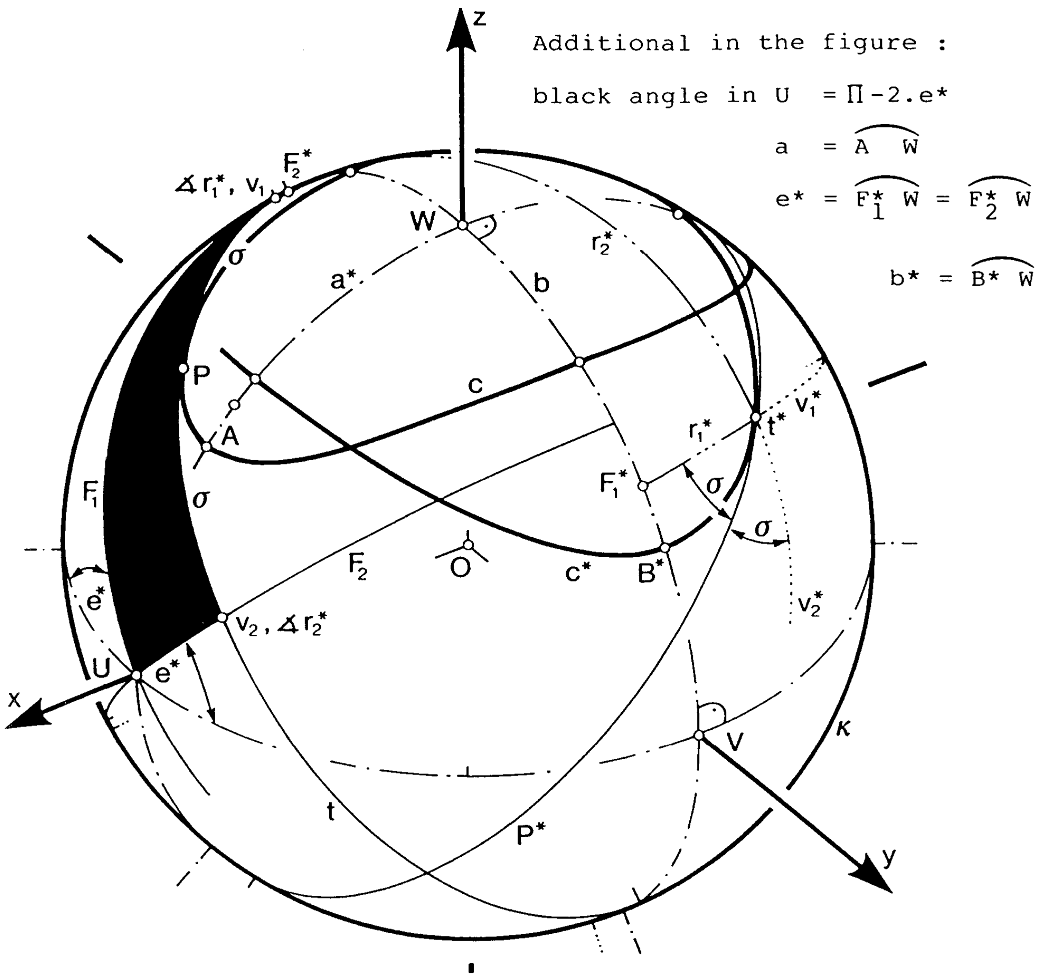


Figure 6.1. *The well-known focus-properties of c^*, triangle of constant area for c*

Additional in *Figure 6.1*:
$P \in c$, P bisects the distance $\widehat{v_1 v_2}$ on t,
distance $\widehat{WF_1^*} = e^*$,
distance $\widehat{WF_2^*} = e^*$,
$\mathbf{P} \circ \widehat{WF_1^*} =$ angle e^*,
$\mathbf{P} \circ \widehat{WF_2^*} =$ angle e^*,
spherical triangle $v_1 v_2 U$: the angle in $U = \pi - 2e^* = \textit{constant}$,
$\mathbf{P} \circ \widehat{F_1^* t^*} =$ angle r_1^*,
$\mathbf{P} \circ \widehat{F_2^* t^*} =$ angle r_2^*,
angle $r_1^* +$ angle $r_2^* +$ angle $\pi - 2e^* = \textit{constant}$.

The spherical triangle $v_1 v_2 U$ has a *constant area* (6.4).
Further:
In t^* we have angles $\sigma = \sigma$. (6.3),
Therefore, on t we have distances

$$\sigma = \sigma.$$

P bisects the distance $v_1 \widehat{v}_2$ (6.5).

Annotation: An EUKLIDean hyperbola $xy = 1$ has analogous properties as (6.4) and (6.5).

7. Loxodrome

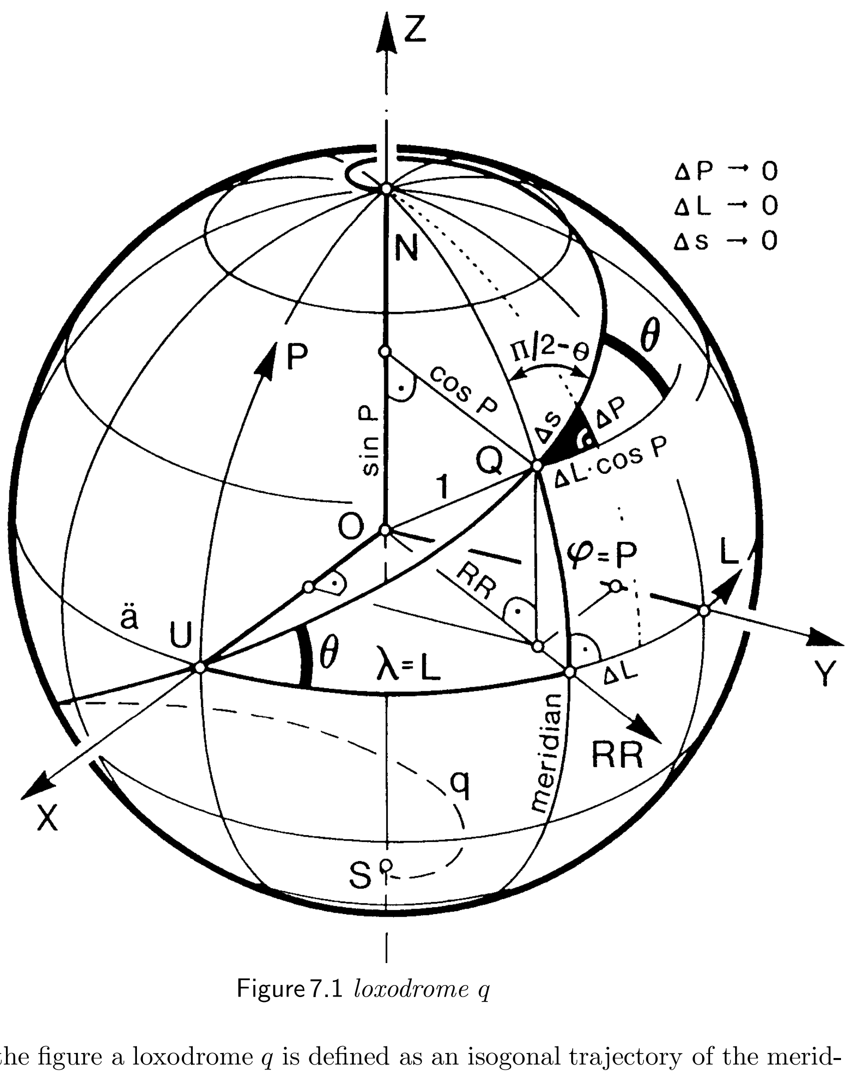


Figure 7.1 *loxodrome q*

As in the figure a loxodrome q is defined as an isogonal trajectory of the meridians of a globe; let this angle be denoted as $\frac{\pi}{2} - \theta$,

$$k = \tan\theta \tag{7.1}$$

$$k = \frac{dP}{dL\cos P} \tag{7.2}$$

we integrate and get

$$\ln\tan\left(\frac{P}{2} + \frac{\pi}{4}\right) = k(L - LD) \tag{7.3}$$

For our task we always let $LD = 0$, then we have

$$\text{the loxodrome} \quad \ni U...L = \frac{\ln\tan\left(\frac{P}{2} + \frac{\pi}{4}\right)}{k} \quad \cdots \quad q \tag{7.4}$$

$$\text{The arclength is easy:} \quad s|_0^P = \frac{1}{\sin\theta} \cdot P \tag{7.5}$$

In CARTESian coordinates we have

$$Q = \vec{X} = \begin{bmatrix} \frac{\cos L}{\mathrm{Ch}\,(kL)} \\ \frac{\sin L}{\mathrm{Ch}\,(kL)} \\ \mathrm{Th}\,(kL) \end{bmatrix} \quad \cdots \quad q \tag{7.6}$$

When derivating with respect to L, we write "$\cdot$",
when derivating with respect to s, we write "$'$".
Useful are the following vectors:

$$\vec{X}' = \begin{bmatrix} \frac{1}{\sqrt{1+k^2}} \frac{-\mathrm{Ch}\,(kL)\sin L - \mathrm{Sh}\,(kL)k\cos L}{\mathrm{Ch}\,(kL)} \\ \frac{1}{\sqrt{1+k^2}} \frac{\mathrm{Ch}\,(kL)\cos L - \mathrm{Sh}\,(kL)k\sin L}{\mathrm{Ch}\,(kL)} \\ \frac{1}{\sqrt{1+k^2}} \frac{k}{\mathrm{Ch}\,(kL)} \end{bmatrix} \tag{7.7}$$

$$\vec{X}'' = \begin{bmatrix} \frac{-\mathrm{Ch}^2(kL)\cos L + \mathrm{Sh}\,(kL)\mathrm{Ch}\,(kL)k\sin L - k^2\cos L}{(1+k^2)\mathrm{Ch}\,(kL)} \\ \frac{-\mathrm{Ch}^2(kL)\sin L - \mathrm{Sh}\,(kL)\mathrm{Ch}\,(kL)k\cos L - k^2\sin L}{(1+k^2)\mathrm{Ch}\,(kL)} \\ \frac{-k^2\mathrm{Sh}\,(kL)}{(1+k^2)\mathrm{Ch}\,(kL)} \end{bmatrix} \tag{7.8}$$

$$\vec{X}''_e = \text{ unit vector of (7.8)} \tag{7.9}$$

8. Spherical evolute curve $e^s_{1,2}$ of the loxodrome q

(7.7) × (7.9) is the unit vector orthogonal to (7.7) and orthogonal to (7.9), it is

$$\vec{b}_e = \vec{X}' \times \vec{X}''_e = \begin{bmatrix} \frac{k\sin L}{\sqrt{k^2+\mathrm{Ch}^2(kL)}} \\ \frac{-k\cos L}{\sqrt{k^2+\mathrm{Ch}^2(kL)}} \\ \frac{\mathrm{Ch}\,(kL)}{\sqrt{k^2+\mathrm{Ch}^2(kL)}} \end{bmatrix} \tag{8.1}$$

also it is the representation of the spherical center of curvature $M^s_{1,2}$; while L is variable, it represents the evolute curve $e^s_{1,2}$ of q. The + sign gives e^s_1, the − sign gives e^s_2 in *Figure 8.1*.
ψ = spherical radius of curvature$_1$ of q,
$\pi - \psi$ = spherical radius of curvature$_2$ of q,

$$\cos\psi = \frac{\mathrm{Sh}\,(kL)}{\sqrt{k^2 + \mathrm{Ch}^2(kL)}} \tag{8.2}$$

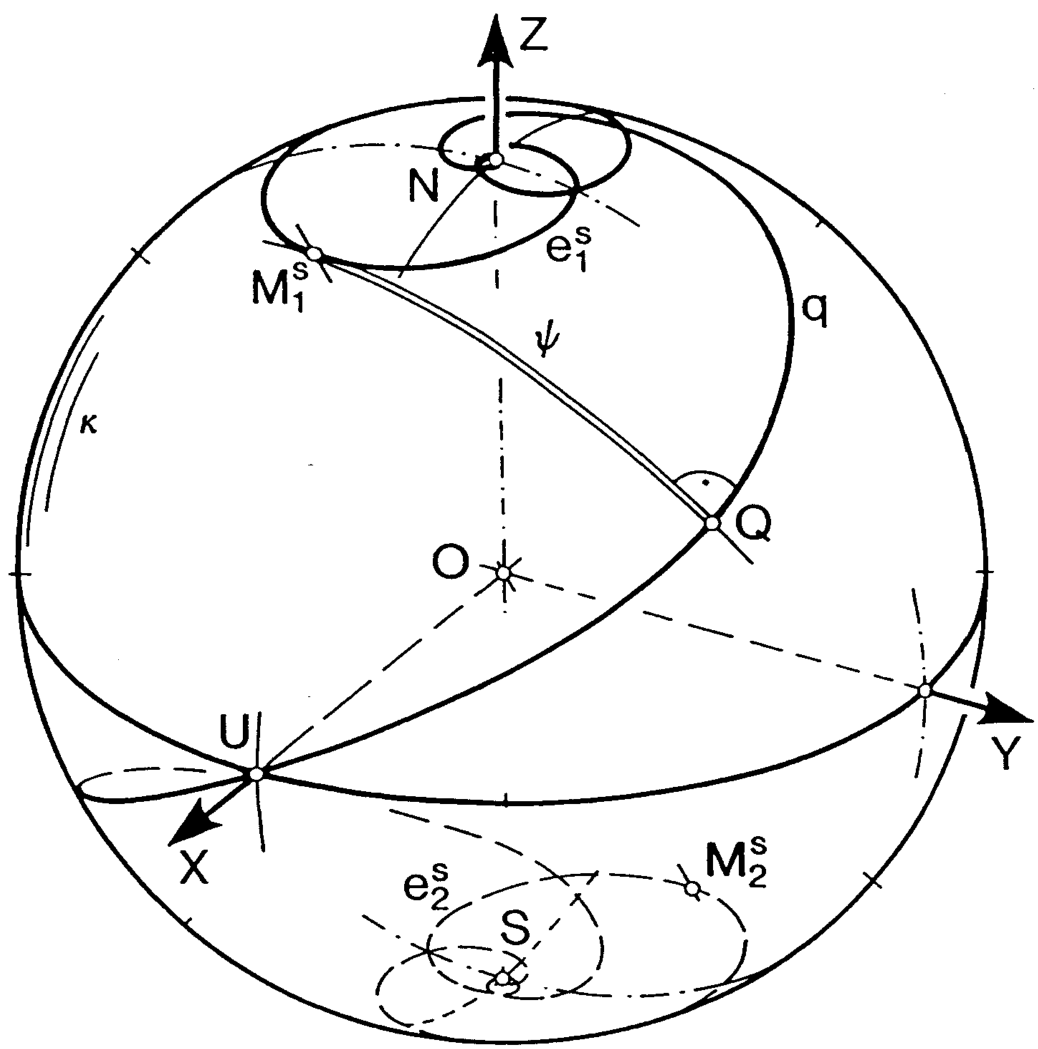


Figure 8.1. *Loxodrome q and spherical evolute curve* $e^s_{1,2}$

9. Spherical tractrix f

Now we polarize q. We will show that we get a spherical tractrix $f = \mathbf{P} \circ q$. We polarize F^*, which is the spherical tangent line of q, then we get $F \in f$:

$$F = Q \times \vec{X}' = (7.6) \times (7.7),$$

$$F = \frac{1}{\sqrt{1+k^2}} \begin{bmatrix} \frac{\mathrm{Ch}\,(kL)k \sin L - \mathrm{Sh}\,(kL)\cos L}{\mathrm{Ch}\,(kL)} \\ \frac{-\mathrm{Ch}\,(kL)k \cos L - \mathrm{Sh}\,(kL)\sin L}{\mathrm{Ch}\,(kL)} \\ \frac{1}{\mathrm{Ch}\,(kL)} \end{bmatrix} \tag{9.1}$$

Polarization of the elements and properties of q:

$$\begin{array}{lcl} \mathbf{P} \circ \text{ point } Q & = & \text{tangent line } Q^* \text{ of } f \\ \mathbf{P} \circ \text{ tangent line } F^* & = & \text{point } F \text{ of } f \\ \mathbf{P} \circ \text{ meridian in } Q & = & \ddot{A} \\ \mathbf{P} \circ (\text{angle } \widehat{\text{meridian } F^*} = \frac{\pi}{2} - \theta) & = & \text{distance } \widehat{\ddot{A}F} = \frac{\pi}{2} - \theta \end{array} \tag{9.2}$$

Therefore F draws a tractrix f, while $\ddot{A}$ is running on the equator ä; ä is an asymptotic line of f. See *Figure 9.1*, sometimes *Figure 7.1*.

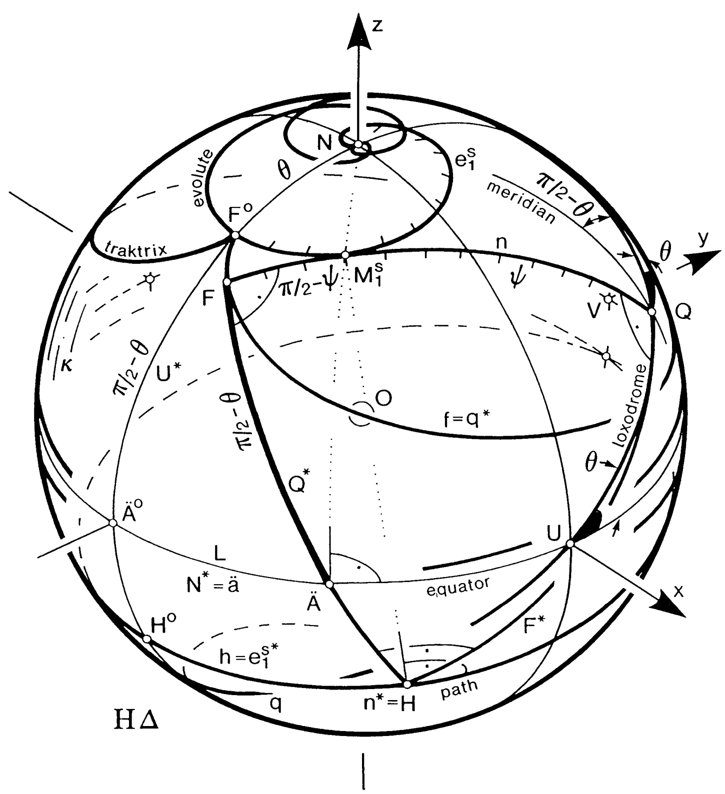


Figure 9.1. *Loxodrome q, evolute curve e_1^s, tractrix f, path h*

10. Path h

The last step: we polarize $e^s_{1,2}$. The result is the path h:
$h = \mathbf{P} \circ e^s_{1,2}$
$n =$ tangent line of $e^s_{1,2} = [Q\ F]$
$H = \mathbf{P} \circ n$
$$H = Q \times F = (7.6) \times (9.1) = -\vec{X}' \tag{7.7}$$
See *Figure 9.1.*

11. Result from 7. to 10.

$Q\ F\ H$ form a spherical octant with three angles of $\frac{\pi}{2}$ and three edges of $\frac{\pi}{2}$. $n = [Q\ F]$ rolls along the evolute curve $e^s_{1,2}$.
Q draws the loxodrome q,
F draws the tractrix f,
H draws the path h.
$e^s_{1,2}$ has two special evolvent curves q and f, they are equidistant curves from each other, the distance is $\frac{\pi}{2}$. See *Figure 9.1.*

References

[1] H. Dirnböck, *Die Syntrepenz zweier Loxodromen auf der Kugel (Syntrepence of two loxodromes on the sphere)*, paper for a lecture, Austrian Geometry Congress, April 27–May 2, 1992., Schloss Seggauberg, A-8430 Leibnitz.

[2] R. Sigl, *Ebene und sphärische Trigonometrie*, Akademische Verlagsgesellschaft, Frankfurt am Main, 1969.