Numerical methods for solving stochastic differential equations*

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Abstract. *This paper provides an introduction to stochastic calculus and stochastic differential equations, in both theory and applications, emphasizing the numerical methods needed to solve such equations.*

Key words: *stochastic differential equations, strong solutions, numerical schemes*

1. Introduction

The aim of this paper is to present efficient numerical methods to compute certain quantities depending on the unknown process $(X(t))$, with algoritms based on simulations on a computer of the other processes.

Before going on, it must be said that the numerical analysis of stochastic differential equations is at its very beginning. Nevertheless, it already appears that this field is not at all direct continuation of what has been done for the numerical solving of ordinary differential equations. It is often unuseful to try to approximate the stochastic differential equation on the space of trajectories, when one wants to compute a quantity which depends on law.

We refer here to the books of Kloeden and Platen ([4]), Kloeden, Platen and Schurz ([5]) and Schurz ([7],[8]).

Consider the simple population growth model

$$\frac{dX}{dt} = a(t)X(t),\ X(0) = A, \tag{1}$$

where $X(t)$ is the size of the population at time t, and $a(t)$ is the relative rate growth at time t. It might happen that $a(t)$ is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + \text{" noise "}, \tag{2}$$

where $r(t)$ is non-random function. So the equation (1) becomes

$$\frac{dX}{dt} = r(t)X(t) + X(t) \cdot \text{" noise "} \tag{3}$$

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or more generally equation of the form

$$\frac{dX_t}{dt}(\omega) \quad = \quad a(t, X_t(\omega))dt + b(t, X_t(\omega))\xi_t(\omega) \tag{4}$$

where a and b are some given functions, and (ξ_t) were standard Gaussian random variables for each t and $b(t,x)$ a (generally) time-space dependent intensity factor. This symbolic differential was interpreted as an integral equation

$$X_t(\omega) \quad = \quad X_{t_0}(\omega) + \int_{t_0}^{t} a(s, X_s(\omega))ds + \int_{t_0}^{t} b(s, X_s(\omega))\xi_s(\omega)ds \tag{5}$$

for each sample path. For the special case of (5) with $a \equiv 0, b \equiv 1$ we see that ξ_t should be derivative of pure Brownian motion, that is the derivative of a Wiener process W_t, thus we suggest that we could write (5) alternatively as

$$X_t(\omega) \quad = \quad X_{t_0}(\omega) + \int_{t_0}^{t} a(s, X_s(\omega))ds + \int_{t_0}^{t} b(s, X_s(\omega))dW_s(\omega). \tag{6}$$

The problem with this is that a Wiener process W_t is nowhere differentiable, so strictly speaking the white noise process ξ_t does not exist as a conventional function of t. Thus the second integral in (6) cannot be an ordinary Riemann or Lebesque integral. Worse still, the continuous sample paths of a Wiener process are not of bounded variation on any bounded time interval, so the second integral cannot even be interpreted as a Riemann-Stieltjes integral for each sample path.

2. The main results

Definition 1. *A process* $\mathbf{W}$=(W_t) *defined on a complete probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *is called a standard Brownian motion if it has the following properties:*

1.) $W_0 = 0$ *(a.s),*

2.) *For a.a.ω, the trajectories* $t \to W_t(\omega)$ *are continuous* $t \in [0, T]$,

3.) *For all* $0 \le t_1 \le t_2 \le t_3 \le t_4, t_i \in [0, T]$ *the random variables* $W_{t_4} - W_{t_3}$ *and* $W_{t_2} - W_{t_1}$ *are independent,*

4.) *For every s and t (s* $\le$ *t)* $W_t - W_s$ *has Gaussian distribution with zero mean and variance equal to t-s,* $t, s, \in [0, T]$.

We suppose that we have a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, a Brownian motion $\mathbf{W} = (W_t, t \ge 0)$ and a filtration $\{\mathcal{F}_t, t \ge 0\}$.

Definition 2. *A process* $\mathbf{W}$ =(W_t) *is called a* $(\mathcal{F}_t)$*-Brownian motion if it has the following properties:*

1.) (W_t) *is* $(\mathcal{F}_t)$*-adapted,*

2.) (W_t) *is a standard Brownian motion,*

3.) *For every* $t \ge 0$ *and* $s > 0$ $W_{t+s} - W_t$ *is independent with* $\mathcal{F}_t$.

Definition 3. *Let $M_{[0,T]}$ be the class of all processes* $\mathbf{f}$ =(f_t) *satisfying the conditions:*

1.) (f_t) *is* $\mathcal{B}(R)/\mathcal{B}([0,T]) \times \mathcal{F}$*-measurable,*

2.) (f_t) *is* $(\mathcal{F}_t)$*-adapted,*

3.) $E\int_0^T \| f_t \|^2 dt < +\infty$.

Definition 4. *The Ito integral* $\int_0^T f(t,\omega)dW_t(\omega)$*, sign by I(f), for an integrand* $f \in M_{[0,T]}$ *is equal to the mean-square limit of the sums*

$$S_n(\omega) = \sum_{j=1}^{n} f(\xi_j^{(n)}, \omega)(W_{t_{j+1}^{(n)}}(\omega) - W_{t_j^{(n)}}(\omega))$$

with evalution points $\xi_j^{(n)} = t_j^{(n)}$ *for partitions* $0 = t_1^{(n)} < t_2^{(n)} < \ldots < t_{n+1}^{(n)} = T$ *for which*

$$\delta^n = \max_{1\le j\le n}(t_{j+1}^{(n)} - t_j^{(n)}) \to 0 \text{ as } n \to +\infty.$$

Other choices of evalution points $t_j^{(n)} \le \xi_j^{(n)} \le t_{j+1}^{(n)}$ are possible, but generally lead to different random variables in the limit. While arbitrarily chosen evalution points

$$\xi_j^{(n)} = (1-\lambda)t_j^{(n)} + \lambda t_{j+1}^{(n)}$$

for the same fixed $0 \le \lambda \le 1$ lead to limits, which we shall denote here by

$$(\lambda)\int_0^T f(t,\omega)dW_t(\omega).$$

As an indication of how the value of these integrals vary with λ we observe that

$$(\lambda)\int_0^T W_t(\omega)dW_t(\omega) = \frac{1}{2}W_T^2(\omega) + (\lambda - \frac{1}{2})T.$$

Lemma 1. *For any* $f, g \in M_{[0,T]}$ *and* $\alpha, \beta \in R$ *the Ito stochastic integral satisfies*

1.) $I(f)$ *is* $\mathcal{F}_T$*-measurable,*

2.) $E[I(f)] = 0$,

3.) $E[I(f)]^2 = \int_0^T E(f(t,\cdot))^2 dt$.

The problem considered here is that of approximating strong solutions of the following type of the Ito stochastic differential equation:

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t, \text{ for } 0 \le t \le T. \tag{7}$$

We suppose that $E\| X_0 \|^2 < +\infty$ and X_0 is independent of $\mathcal{F}_t = \sigma\{W_s, 0 \le s \le t\}$, the σ-algebra generated by the underlying process.

Also, suppose that the coefficients $a(t,x)$ and $b(t,x)$ satisfy conditions which guarantee the existence of the unique, strong solution of the stochastic differential equation.

Theorem 1. *Let $U : [0,T]\times R \to R$ have continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial x^2}$ and define the process $(Y_t, 0 \le t \le T)$ by $Y_t = U(t, X_t)$ w.p.1, where X_t satisfies the differential (7). Then the stochastic differential equation for Y_t is given by*

$$\begin{aligned} dY_t &= [\frac{\partial U}{\partial t}(t,X_t) + a(t,X_t)\frac{\partial U}{\partial x}(t,X_t) + \frac{1}{2}b^2(t,X_t)\frac{\partial^2 U}{\partial x^2}(t,X_t)]dt \\ &+ b(t,X_t)\frac{\partial U}{\partial x}(t,X_t)dW_t. \end{aligned}$$

Details about this stochastic object and the corresponding calculus can be found in Karatzas and Shreve([3])

Let us consider the 1-dimensional stochastic differential equation in an integral form

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s,$$

for $t \in [0,T]$, where the second integral is an Ito stochastic integral and the coefficients a and b are sufficiently smooth real-valued functions satisfying a linear growth bound. Then, for any twice continuously differentiable function $f : R \to R$ the Ito formula gives

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t (a(X_s)\frac{\partial}{\partial x}f(X_s) + \frac{1}{2}b^2(X_s)\frac{\partial^2}{\partial x^2}f(X_s))ds \\ &+ \int_0^t b(X_s)\frac{\partial}{\partial x}f(X_s)dW_s \\ &= f(X_0) + \int_0^t L^0 f(X_s)ds + \int_0^t L^1 f(X_s)dW_s. \end{aligned}$$

So if we apply the Ito formula to the functions $f = a$ and $f = b$ we obtain

$$\begin{aligned} X_t &= X_0 + \int_0^t a(X_z)dz + \int_0^t b(X_z)dW_z \\ &= X_0 + \int_0^t [a(X_0) + \int_0^z L^0 a(X_s)ds + \int_0^z L^1 a(X_s)dW_s]dz \\ &+ \int_0^t [b(X_0) + \int_0^z L^0 b(X_s)ds + \int_0^z L^1 b(X_s)dW_s]dW_z \\ &= X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s + R. \end{aligned}$$

So far a given discretization $0 = t_1 < t_2 < \ldots < t_N = T$ at the time interval $[0,T]$, an Euler scheme is given by

$$Y_{n+1} = Y_n + a(Y_n)(t_{n+1} - t_n) + b(Y_n)(W_{t_{n+1}} - W_{t_n}),$$

for $n = 0,1,2,\ldots,N-1$ with the initial value $Y_0 = X_0$.

We can continue the Ito-Taylor expansion by applying the Ito formula to $f = L^1 b$, in which case we get

$$X_t = X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s + L^1 b(X_0)\int_0^t \int_0^s dW_z dW_s + \overline{R}.$$

For a given discretization mentioned earlier a Milstein scheme is given by

$$\begin{aligned}\overline{Y_{n+1}} &= \overline{Y_n} + a(\overline{Y_n})(t_{n+1}-t_n) + b(\overline{Y_n})(W_{t_{n+1}} - W_{t_n})\\ +\frac{1}{2} \quad b & \quad (\overline{Y_n})\frac{\partial}{\partial x} b(\overline{Y_n})((W_{t_{n+1}} - W_{t_n})^2 - (t_{n+1}-t_n)),\end{aligned}$$

for $n = 0, 1, 2, \ldots, N-1$ with the initial value $Y_0 = X_0$.

Usually, we consider equidistant discretization times $t_n = n\Delta$, where $\Delta = \frac{T}{N}$.

Theorem 2. *Let us suppose that the functions a and b are of class C^2, with bounded derivatives of first and second orders. Then the Euler scheme satisfies: for any integration time T, there exists a positive constant C(T) such that, for any step size Δ of type $\frac{T}{N}, N = 1, 2, 3, \ldots$*

$$[E\|X(T) - Y_N\|^2]^{\frac{1}{2}} \le C(T)\sqrt{\Delta}.$$

For the Milstein scheme, we can substitute the following bound for the error:

$$[E\|X(T) - Y_N\|^2]^{\frac{1}{2}} \le C(T)\Delta.$$

To get a better rate of convergence in the mean-square sense that these schemes, one must involve multiple stochastic integrals of order strictly larger than 1, for example: $\int_0^t \int_0^s dW_z dW_s$, $\int_0^t \int_0^s \int_0^z dW_u dW_z dW_s$ and $\int_0^t \int_0^s \int_0^z du dW_z dW_s$.

Of course, most of those integrals have probability laws which seem difficult to simulate. There are some other two-step methods for numerical solutions of stochastic differential equations. Details about this stochastic object and the corresponding calculus can be found in ([1]) and ([2]).

References

[1] R. Horvát Bokor, *On two-step methods for SDE,* Acta Cybernetica **13**(1997), 197–207.

[2] R. Horvát Bokor, *On stability of two-step methods for stochastic differential equations*, in: Proceedings of the 7^{th} International Conference on Operational Reserch KOI'98, (I. Aganović, T. Hunjak and R. Scitovski, eds.), Osijek, 1999, 267–279.

[3] I. Karatzas, S. E. Shreve,. *Brownian Motion and Stochastic Calculus,* Springer-Verlag, New York, 1988.

[4] P. E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations,* Springer-Verlag, New York, 1992.

[5] P. E. Kloeden, E. Platen, H. Schurz, *Numerical Solution of Stochastic Differential Equations through Numerical Experiments,* Universitext, Springer, New York, 1994.

[6] W. Rümelin, *Numerical treatment of stochastic differential eqautions*, SIAM J. Numer. Anal. **19**(1982), 604–613.

[7] H. Schurz, *Mean square stability analysis of some numerical method solving bilinear SDEs,* Proceedings of Workshop on 'Stochastic and Finance', Sept. 1994, Humboldt University, Berlin, Special Press, 1994, 77-78.

[8] H. Schurz, *Construction of nonegative numerical solutions of SDEs: Explosions, life time, nonegativity and uniform boundedness*, Discussion Paper 40, SFB 373, Humboldt University, Berlin, 1994