Tensor products of C*-algebras, operator spaces and Hilbert C*-modules*

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Abstract. *This article is a review of the basic results on tensor products of C*-algebras, operator spaces and Hilbert C*-modules.*

Key words: *tensor products, C*-algebras, operator spaces, Hilbert C*-modules*

1. Preliminaries

Let us recall a few basic facts about tensor products, C*-algebras, operator spaces and Hilbert C*-modules which shall be used in this short overview. Algebraic tensor products shall be denoted by $\otimes$, while tensor products completed with respect to a norm β shall be denoted by $\otimes_\beta$. Bilinear maps on the Cartesian product of vector spaces (algebras, ...) are canonically identified by linear maps on the corresponding tensor products: if $\Phi : X \times Y \to Z$ is bilinear, the corresponding (unique) linear operator $\phi : X\otimes Y \to Z$ is given on elementary tensors by $\phi(x\otimes y) = \Phi(x, y)$.

The *standard tensor product of Hilbert spaces* $\mathcal{H}$ and $\mathcal{K}$ (i.e. the tensor product $\mathcal{H}\otimes\mathcal{K}$ completed with respect to the norm induced by the inner product given on elementary tensors by $(\xi\otimes\eta \,|\, \xi'\otimes\eta') = (\xi \,|\, \xi')_{\mathcal{H}}(\eta \,|\, \eta')_{\mathcal{K}})$ shall be denoted by $\mathcal{H}\bar{\otimes}\mathcal{K}$. Let it be reminded that for the standard tensor product of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ there is a natural identification of tensor products of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$ as bounded linear operators on $\mathcal{H}\bar{\otimes}\mathcal{K}$, i.e. $\mathbf{B}(\mathcal{H})\otimes\mathbf{B}(\mathcal{K}) \subseteq \mathbf{B}(\mathcal{H}\bar{\otimes}\mathcal{K})$ via $(T\otimes S)(\xi\otimes\eta) = T(\xi)\otimes S(\eta)$ for $T \in \mathbf{B}(\mathcal{H})$, $S \in \mathbf{B}(\mathcal{K})$.

A *C*-algebra* is a norm-closed selfadjoint subalgebra $\mathcal{A}$ of $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Equivalently, it is a Banach $*$-algebra whose norm satisfies the *C*-property*

$$\|a\|^2 = \|a^*a\|$$

for all $a \in \mathcal{A}$. C*-algebras shall be denoted by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... The dual space of a C*-algebra $\mathcal{A}$ (i.e. the space of all continuous linear functionals on $\mathcal{A}$) shall be denoted by $\mathcal{A}^*$. The finite-dimensional C*-algebras $M_n(\mathbf{C})$ shall be denoted by M_n.

A $*$-*morphism* between $*$-algebras (i.e. algebras with an involution) is a linear operator that is multiplicative and preserves the involution. For a $*$-morphism

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between C*-algebras the continuity (even contractivity) is automatic, so the $*$-morphisms between C*-algebras shall be referred to as *C*-morphisms*. A *representation of a C*-algebra* is a C*-morphism $\pi : \mathcal{A}\to\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Every C*-algebra can be faithfully (i.e. isometrically) represented as a (norm-closed selfadjoint) subalgebra of $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Every representation π of $\mathcal{A}$ on $\mathbf{B}(\mathcal{H})$ and vector $\xi \in \mathcal{H}$ define a continuous linear functional f on $\mathcal{A}$ by $f(a) = (\pi(a)\xi \mid \xi)$. Such a functional is positive, i.e. takes positive values on positive elements in $\mathcal{A}$ (those are elements of the form x^*x for $x \in \mathcal{A}$). What is more important is that for every positive functional on $\mathcal{A}$ (positive linear functionals are automatically continuous) there exist a Hilbert space $\mathcal{H}_f$, a vector $\xi_f \in \mathcal{H}_f$ and a representation π_f of $\mathcal{A}$ on $\mathcal{H}_f$ such that $f(a) = (\pi_f(a)\xi_f \mid \xi_f)$ for all $a \in \mathcal{A}$ (the construction is known as the Gelfand-Naimark-Segal construction). For more details on the general theory of C*-algebras, see e.g. [8].

An *operator space* is any subspace X of some C*-algebra (i.e. of $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$). It is usually (but not always) required that an operator space is norm-closed. The point is that one considers not only the vector-space structure, but also the matricial structure. Namely, for any $n\in\mathbf{N}$ one may identify the space $M_n(\mathbf{B}(\mathcal{H}))$ of $n \times n$ matrices over $\mathbf{B}(\mathcal{H})$ with the space $\mathbf{B}(\mathcal{H}^n)$, where $\mathcal{H}^n$ denotes the orthogonal sum of n copies of $\mathcal{H}$. This way, for any subspace X of $\mathbf{B}(\mathcal{H})$ one may consider $M_n(X)$ as a subspace of $M_n(\mathbf{B}(\mathcal{H})) = \mathbf{B}(\mathcal{H}^n)$ and thus one has a norm $\|\cdot\|_n$ on $M_n(X)$ for every n. Note that for any vector space V one can identify the space $M_n(V)$ with the tensor product $V\otimes M_n$. When considering maps between operator spaces, one is naturally led to consider their n-th amplifications i.e. for a linear map $T : X\to Y$ its n-th amplification is the linear operator $T_n : M_n(X)\to M_n(Y)$ defined by

$$T_n([a_{ij}]) = [T(a_{ij})]$$

for $[a_{ij}] \in M_n(X)$. Since $M_n(X)$ and $M_n(Y)$ are normed for every n, one may consider the operator norm $\|T_n\|$. If $\sup_{n\in\mathbf{N}} \|T_n\| < \infty$, then the operator T is called *completely bounded* and $\|T\|_{cb} = \sup_{n\in\mathbf{N}} \|T_n\|$ is the *completely bounded norm* (cb-norm) of T. The set of all completely bounded maps from X to Y (with the cb-norm) is denoted $CB(X, Y)$. A completely bounded map T is a *complete isometry* if all T_n are isometric and it is a *completely isometric isomorphism* if T is invertible and $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$. Operator spaces are identified up to completely isometric isomorphisms. For more information on completely bounded operators see [7] and [15].

The most important result on operator spaces is Ruan's theorem ([17]) which gives an abstract characterization of operator spaces:

Theorem 1. (Ruan) *A vector space X with a sequence of norms $\|\cdot\|_n$ on $M_n(X)$ ($n \in \mathbb{N}$) is an operator space if and only if the following two conditions are satisfied (for all $n, m \in \mathbf{N}$):*

(i) $\|\Gamma A\Lambda\|_n \le \|\Gamma\|\|A\|_n\|\Lambda\|$ for all $\Gamma, \Lambda\in M_n$, $A\in M_n(X)$ (the matrix multiplication being the natural one);

(ii) $\|A \oplus B\|_{n+m} = max\{\|A\|_n,\ \|B\|_m\}$ for all $A \in M_n(X)$, $B \in M_m(X)$ (where $A \oplus B$ denotes the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{n+m}(X)$).

A (right) pre-Hilbert C*-module is a vector space E which is a (right) module over a C*-algebra $\mathcal{A}$ with an *inner product* $< . \mid . >: E \times E \to \mathcal{A}$ with the following properties:

$$\begin{array}{ll} \text{(H1)} & < x \mid y+z > = < x \mid y > + < x \mid z > \\ \text{(H2)} & < x \mid ya > = < x \mid y > a \\ \text{(H3)} & < x \mid y >^* = < y \mid x > \\ \text{(H4)} & < x \mid x > \geq 0 \ (\text{in } \mathcal{A}) \\ \text{(H5)} & < x \mid x > = 0 \Leftrightarrow x = 0 \end{array}$$

for all $x, y, z \in E$, $a \in \mathcal{A}$. These properties obviously generalize the ones required for the inner product of a Hilbert space.

If E is complete in the norm defined by

$$\|x\| = \sqrt{\| < x \mid x > \|}$$

then E is called a *Hilbert $\mathcal{A}$-module*. Every Hilbert C*-module is an operator space (see e.g. [3]). The only classes of operators on Hilbert C*-modules which shall be considered in this paper are $B_{\mathcal{A}}(E)$ and $\mathbf{B}_{\mathcal{A}}(E)$. $B_{\mathcal{A}}(E)$ is the Banach algebra of all bounded operators on E that are $\mathcal{A}$-linear, i.e. T such that $T(xa) = T(x)a$ for all $x \in E$, $a \in \mathcal{A}$. $\mathbf{B}_{\mathcal{A}}(E)$ is the C*-algebra (subalgebra of $B_{\mathcal{A}}(E)$) of all operators $T : E \to E$ for which there exists a map $T^* : E \to E$ (called the *adjoint* of T) such that $< Tx \mid y > = < x \mid T^*y >$ for all $x, y \in E$. An overview of the basic theory of Hilbert C*-modules can be found in [19].

2. Tensor products of C*-algebras

The norm of a C*-algebra is unique in the sense: on a given $*$-algebra $\mathcal{A}$ there is at most one norm which makes $\mathcal{A}$ into a C*-algebra. Still, on a $*$-algebra $\mathcal{A}$ there may exist different norms satisfying the C*-property. The completion with respect to any of such norms results in a C*-algebra which contains $\mathcal{A}$ as a dense subalgebra. This is precisely what happens when the tensor product of C*-algebras is considered: in the general case there are many different norms on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ (which is a $*$-algebra) with the C*-property (these shall be referred to as C*-(tensor) norms and tensor products of C*-algebras completed with respect to a C*-norm shall be referred to as C*-tensor products).

One such norm is the *spatial norm* $\| \cdot \|_\alpha$ defined by the inclusion $\mathcal{A} \otimes \mathcal{B} \subseteq \mathbf{B}(\mathcal{H}) \otimes \mathbf{B}(\mathcal{K}) \subseteq \mathbf{B}(\mathcal{H} \bar{\otimes} \mathcal{K})$, assuming that $\mathcal{A}$ and $\mathcal{B}$ are faithfully represented on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. This norm was introduced by T.Turumaru in 1953. The definition does not depend on particular representations of $\mathcal{A}$ and $\mathcal{B}$, that is for all $t \in \mathcal{A} \otimes \mathcal{B}$

$$\|t\|_\alpha = \|(\pi \otimes \rho)(t)\|_{\mathbf{B}(\mathcal{H} \bar{\otimes} \mathcal{K})}$$

for any two faithful representations π of $\mathcal{A}$ on $\mathcal{H}$ and ρ of $\mathcal{B}$ on $\mathcal{K}$.

The second natural norm on $\mathcal{A} \otimes \mathcal{B}$ was introduced in 1965 by A. Guichardet. It is the *maximal C*-norm* $\| \cdot \|_\nu$ defined as

$$\|t\|_\nu = \sup\{\|t\|_\beta : \| \cdot \|_\beta \text{ C*-seminorm on } \mathcal{A} \otimes \mathcal{B}\}.$$

The maximal C*-norm is obviously adequatly named. As for the spatial norm, it is shown that it is the minimal C*-norm on the tensor product of C*-algebras (and the norm is often referred to as "the minimal C*-norm"). Being minimal resp. maximal means

$$\|t\|_\alpha \leq \|t\|_\beta \leq \|t\|_\nu$$

for every other C*-norm $\|\cdot\|_\beta$, for all C*-algebras $\mathcal{A}$ and $\mathcal{B}$ and all $t \in \mathcal{A}\otimes\mathcal{B}$. The set of all C*-norms on $\mathcal{A}\otimes\mathcal{B}$ is a complete lattice (with respect to the ordering $\|\cdot\|_\beta \leq \|\cdot\|_\gamma$ if $\|t\|_\beta \leq \|t\|_\gamma$ for all $t \in \mathcal{A}\otimes\mathcal{B}$) and all C*-norms $\|\cdot\|_\beta$ on tensor products of C*-algebras are *cross-norms*, i.e.

$$\|a\otimes b\|_\beta = \|a\|\|b\|$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

The spatial C*-norm has several good properties, the most important of them being its injectivity, i.e. the restriction of the spatial norm on $\mathcal{A}\otimes\mathcal{B}$ to the tensor product $\mathcal{A}_1\otimes\mathcal{B}_1$ of subalgebras $\mathcal{A}_1$ of $\mathcal{A}$ and $\mathcal{B}_1$ of $\mathcal{B}$ is the spatial norm on $\mathcal{A}_1\otimes\mathcal{B}_1$. This may not be the case for other C*-tensor norms: although the restriction of any C*-norm $\|\cdot\|_\beta$ from $\mathcal{A}\otimes\mathcal{B}$ to $\mathcal{A}_1\otimes\mathcal{B}_1$ is always a C*-norm, it is not necessarily the β-norm on $\mathcal{A}_1\otimes\mathcal{B}_1$.

Other good properties of the spatial norm include the following: the C*-tensor product of two C*-algebras is a simple C*-algebra if and only if both C*-algebras are simple and the norm is the spatial one; the tensor product of continuous linear functionals on C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is always continuous on $\mathcal{A}\otimes\mathcal{B}$ with respect to the spatial C*-norm (and consequently with respect to every C*-norm):

$$|\ (f\otimes g)(t)\ | \leq \|f\|\|g\|\|t\|_\alpha$$

for $f \in \mathcal{A}^*$, $g \in \mathcal{B}^*$, $t \in \mathcal{A}\otimes\mathcal{B}$; the tensor product of C*-morphisms is always continuous if the range is equipped with the spatial C*-norm (and the domain is equipped with any C*-norm).

The maximal C*-norm has its good properties, too. The most important is that the representation defined by

$$\sum_{i=1}^{n} a_i\otimes b_i \mapsto \sum_{i=1}^{n} \pi(a_i)\rho(b_i)$$

can be continuously extended to a representation of the C*-algebra $\mathcal{A}\otimes_\nu\mathcal{B}$ for any pair of commuting representations π and ρ of $\mathcal{A}$ and $\mathcal{B}$ on the same Hilbert space. A pair (π, ρ) of representations is called commuting if $\pi(a)\rho(b) = \rho(b)\pi(a)$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. An algebraic representation $\pi : \mathcal{A}\otimes\mathcal{B}\to\mathbf{B}(\mathcal{H})$ which satisfies $\|\pi(a\otimes b)\| \leq \|a\|\|b\|$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ is called a *subtensor representation*. Since for every subtensor representation σ of $\mathcal{A}\otimes\mathcal{B}$ there exists a pair of commuting representations π and ρ of $\mathcal{A}$ and $\mathcal{B}$ such that $\sigma(a\otimes b) = \pi(a)\rho(b) = \rho(b)\pi(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and since every representation of $\mathcal{A}\otimes_\beta\mathcal{B}$ is subtensor (for every C*-norm $\|\cdot\|_\beta$), one gets

$$\|t\|_\nu = \sup\{\|\pi(t)\| : \pi \text{ subtensor representation of } \mathcal{A}\otimes\mathcal{B}\}$$

for $t \in \mathcal{A}\otimes\mathcal{B}$. This is the original Guichardet's definition of the maximal C*-norm for the tensor product of C*-algebras.

As for other C*-norms, they can all be obtained in the following way:

Let $S(\mathcal{A}\otimes\mathcal{B})$ be the set of all linear functionals f on $\mathcal{A}\otimes\mathcal{B}$ such that $f(t^*t)\geq 0$ for all $t\in\mathcal{A}\otimes\mathcal{B}$ and $\sup\{|f(a\otimes b)| : \|a\|\leq 1,\ \|b\|\leq 1\}=1$. To each $f\in\mathcal{A}\otimes\mathcal{B}$ one can associate an algebraic representation $\pi_f:\mathcal{A}\otimes\mathcal{B}\to\mathbf{B}(\mathcal{H})$ (and a vector $\xi_f\in\mathcal{H}_f$) such that $f(t)=(\pi_f(t)\xi_f \mid \xi_f)$ for all $t\in\mathcal{A}\otimes\mathcal{B}$ (the construction imitates the Gelfand-Naimark-Segal construction and then proves the continuity of the resulting $\pi_f(t)$ for all t). If we define $\|t\|_f=\|\pi_f(t)\|$, $\|\cdot\|_f$ is obviously a C*-seminorm. For a subset S of $S(\mathcal{A}\otimes\mathcal{B})$ let

$$\|t\|_S=\sup_{f\in S}\|t\|_f.$$

Then the collection of all $S\subseteq S(\mathcal{A}\otimes\mathcal{B})$ such that $\|\cdot\|_S$ is a C*-norm is in 1-1-correspondence to the set of all C*-norms on $\mathcal{A}\otimes\mathcal{B}$, i.e. every C*-norm is uniquely determined by such set S (in particular, the maximal norm corresponds to the whole $S(\mathcal{A}\otimes\mathcal{B})$ and the spatial one to $S(\mathcal{A}\otimes\mathcal{B})\cap(\mathcal{A}^*\otimes\mathcal{B}^*)$). The correspondence was proven by Lance and Effros in 1977 ([10]).

There are C*-algebras $\mathcal{A}$ for which the minimal and the maximal norm on $\mathcal{A}\otimes\mathcal{B}$ coincide for all C*-algebras $\mathcal{B}$ (and consequently the C*-norm on $\mathcal{A}\otimes\mathcal{B}$ is unique). Such C*-algebras are called *nuclear* and their C*-tensor products obviously combine the good properties of the minimal and maximal C*-product. Among the nuclear C*-algebras are all finite-dimensional C*-algebras (in particular, the algebras M_n for $n\in\mathbf{N}$), all commutative ones, all GCR-algebras (in particular the C*-algebra of compact operators on a Hilbert space), inductive limits of nuclear C*-algebras (in particular AF-algebras), type I C*-algebras,... But not all C*-algebras are nuclear. The first example of a non-nuclear algebra is due to Takesaki ([18]): the C*-algebra generated by the left regular representation on $l^2(G)$ of a free group G with two generators.

Example 1. *Since M_n is a nuclear C*-algebra, for any C*-algebra $\mathcal{A}$ the C*-norm on $\mathcal{A}\otimes M_n$ is unique (and thus equal to the spatial norm). Moreover, the algebraic tensor product $\mathcal{A}\otimes M_n$ is already complete in this norm[1]. In particular, the algebra $M_n(\mathcal{A})$ is a C*-algebra.*

There are different characterisations of nuclear C*-algebras proven by Choi, Effros, Kirchberg, Lance and others in the 70es. They are all complicated in proof and mostly also in formulation. One of the simpler ones is: $\mathcal{A}$ is nuclear if and only if the von Neumann algebra[2] $\mathcal{A}^{**}$ is injective (i.e. for all C*-algebras $\mathcal{B}\supseteq\mathcal{A}^{**}$ there is a continuous retraction[3] $\mathcal{B}\to\mathcal{A}^{**}$). S. Wassermann showed that for infinite-dimensional separable $\mathcal{H}$ $\mathbf{B}(\mathcal{H})^{**}$ is not injective (*On tensor products of certain group C*-algebras*, J. Funct. Anal. 23 (1976) 239-254), so one gets another example of a non-nuclear C*-algebra, namely $\mathbf{B}(\mathcal{H})$ for infinite-dimensional separable $\mathcal{H}$.

Most of the characterisations of nuclear C*-algebras are of the following form: the identity operator ι on $\mathcal{A}$ (or on $\mathcal{A}^*$)

[1] In fact, the proof that M_n is nuclear is the proof of the completeness of $\mathcal{A}\otimes M_n$ in the spatial norm.

[2] A von Neumann algebra is a weakly closed selfadjoint subalgebra of $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The weak topology on $\mathbf{B}(\mathcal{H})$ is defined by the family of seminorms $\{p_{\xi,\eta}:\xi,\eta\in\mathcal{H}\}$, where $p_{\xi,\eta}(a)=(a\xi\mid\eta)$ for $a\in\mathbf{B}(\mathcal{H})$. The enveloping von Neumann algebra of a C*-algebra $\mathcal{A}$ is isometrically isomorphic to its bidual $\mathcal{A}^{**}$.

[3] A retraction is a left inverse of the inclusion morphism.

a) can be approximated (in a suitable topology) by a certain class of (finite rank) operators, or
b) there is an approximate factorization for ι, where under approximate factorizations one understands that for any neigborhood of ι (in a suitable topology) there exist an operator ι' and operators τ (into M_n for some n) and σ (with domain M_n) such that $\iota' = \sigma\tau$.

More detailed overviews of the theory of tensor products of C*-algebras can be found in [13] and [19].

3. Tensor products of operator spaces

Let X and Y be two operator spaces and $X \otimes Y$ their algebraic tensor product. We consider norms on the vector space $X \otimes Y$ which make it an operator space, i.e. sequences of norms $\|\cdot\|_n$ on $M_n(X \otimes Y)$ which satisfy the conditions of Ruan's theorem. We also pose an additional condition on these norms - that of being *cross-norms*:

$$\|A \otimes B\|_{nm} = \|A\|_n \|B\|_m$$

for all $n, m \in \mathbb{N}$, $A = [a_{ij}] \in M_n(X)$, $B = [b_{kl}] \in M_m(Y)$, where $A \otimes B$ denotes the matrix $[a_{ij} \otimes b_{kl}] \in M_{nm}(X \otimes Y)$. Such norms shall be called *operator space cross-norms*.

Similarly as in the C*-algebra case, it turns out there is a minimal (the spatial norm) and a maximal (the projective norm) among operator space cross norms, but only if one more property is required: that the dual norm of the norms in question is also an operator space cross-norm. The *dual norm* of a norm $\|\cdot\|_\beta$ is the norm induced by the natural inclusion $X' \otimes Y'$ into $(X \otimes_\beta Y)'$ where X' denotes the *standard dual of an operator space* X, $X' = CB(X, \mathbf{C})$. Besides the minimal and maximal norm, there is another very important operator space cross-norm: the Haagerup norm.

The Haagerup norm was the first one considered. The motivation was the consideration of operators of the form $\phi(a) = \sum_{i=1}^n u_i a v_i$ for $a \in \mathcal{A}$ where $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ are some fixed elements in $\mathcal{A}$. These operators result from the action of $\sum u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}^{op}$ on $\mathcal{A}$ (where $\mathcal{A}^{op}$ is the C*-algebra $\mathcal{A}$ with the reversed product). If $\mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$ then for $\xi, \eta \in \mathcal{H}$ the Cauchy-Schwarz inequality implies

$$|(\phi(a)\xi \mid \eta)| = \left|\sum (av_i\xi \mid u_i^*\eta)\right| \le \left(\sum \|av_i\xi\|^2\right)^{1/2} \left(\sum \|u_i^*\eta\|^2\right)^{1/2}.$$

Further, $\|av_i\xi\| \le \|a\|\|v_i\xi\|$, $\sum \|v_i\xi\|^2 = \sum (v_i^* v_i \xi \mid \xi) \le \|\sum v_i^* v_i\| \|\xi\|^2$ and similarly for u_i. It follows that

$$|(\phi(a)\xi \mid \eta)| \le ||a|| \left|\left|\sum u_i u_i^*\right|\right|^{1/2} \left|\left|\sum v_i^* v_i\right|\right|^{1/2} ||\xi|| \, ||\eta|| \,,$$

so $\|\phi\| \le ||\sum u_i u_i^*||^{1/2} \, ||\sum v_i^* v_i||^{1/2}$.
One may allow also infinite (countable) sequences of u_i and v_i, provided that $\sum u_i u_i^*$ and $\sum v_i^* v_i$ are norm convergent. The natural definition following from these considerations is

$$\|t\|_h = \inf \left\{ \| \sum_{i=1}^n u_i u_i^* \|^{1/2} \| \sum_{i=1}^n v_i^* v_i \|^{1/2} \ : \ n \in \mathbf{N}, \ t = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{B} \right\}$$

for $t \in \mathcal{A}\otimes\mathcal{B}$. The proof that $\|\cdot\|_h$ (the *Haagerup norm*) is a norm is not completely trivial (the proof of the triangle inequality and the definiteness are non-trivial). These properties are proven by Effros and Kishimoto in 1987 ([9]). The norm was named after Uffe Haagerup, whose unpublished manuscript *Decomposition of completely bounded maps on operator algebras* (1980) initialized the definition and research of the properties of this norm.

The Haagerup norm is not a C*-norm, but if the definition is repeated for $n \in \mathbf{N}$ and $t \in M_n(X\otimes Y)$ for operator spaces X and Y, it turns out that the Haagerup norm is an operator space cross-norm (see e.g. [16]) with a number of good properties. First of all, the definition can be somewhat simplified: for $U \in M_n(X\otimes Y)$ one has

$$\|U\|_h = \inf\{\|A\|\|B\| \,:\, U = A \odot B, k \in \mathbf{N}, A \in M_{nk}(X), B \in M_{kn}(Y)\},$$

where $A \odot B$ denotes the matrix $\left[\sum_{r=1}^k a_{ir}\otimes b_{rj}\right]$. In particular, the formula implies that the first definition doesn't depend on the C*-algebras on which X and Y are represented (note that in the original formula adjoints of elements occur and that in the definition of operator spaces one does not require that the operator space is closed upon taking adjoints). Moreover, for $U \in M_n(X\otimes Y)$ the infimum is achieved for some particular $k \in \mathbf{N}$, $A \in M_{nk}(X)$ and $B \in M_{kn}(Y)$.

The Haagerup tensor product (i.e. $X\otimes Y$ equipped with the Haagerup norm) is associative:

$$(X\otimes_h Y)\otimes_h Z = X\otimes_h (Y\otimes_h Z)$$

(completely isometrically isomorphic). Further, the norm is (completely) injective i.e. for subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$ the restriction of the Haagerup norm from $X\otimes_h Y$ to $X_0\otimes Y_0$ is the Haagerup norm and consequently $X_0\otimes_h Y_0$ is completely isometrically embedded into $X\otimes_h Y$. The tensor product $T\otimes S$ of completely bounded operators T and S on X and Y is completely bounded on $X\otimes_h Y$, and more: $\|T\otimes S\|_{cb} \le \|T\|_{cb}\|S\|_{cb}$. The Haagerup norm is also selfdual, i.e. the dual norm of the Haagerup norm on $X\otimes Y$ is the Haagerup norm on $X'\otimes Y'$.

What is perhaps most important is the correspondence between completely bounded bilinear operators[4] $\Phi : X\times Y \to \mathbf{B}(\mathcal{H})$ and their linearisations $\phi : X\otimes Y \to \mathbf{B}(\mathcal{H})$ when the tensor product $X\otimes Y$ is equipped with the Haagerup norm (a bilinear operator $\Phi : X \times Y \to \mathbf{B}(\mathcal{H})$ is completely bounded if and only if the corresponding linear operator $\phi : X\otimes_h Y \to \mathbf{B}(\mathcal{H})$ is completely bounded and in that case their cb-norms are equal). Moreover, any completely bounded map on the Haagerup tensor product of two operator spaces is essentially operator multiplication:

Theorem 2. (Christensen-Sinclair [6]/Paulsen-Smith[16])
If $\phi : X\otimes_h Y \to \mathbf{B}(\mathcal{H})$ is completely contractive (i.e. $\|\phi\|_{cb} \le 1$) and $X \subseteq \mathcal{A}$, $Y \subseteq \mathcal{B}$, then there exist representations $\pi : \mathcal{A} \to B(\mathcal{K}_1)$ and $\rho : \mathcal{B} \to B(\mathcal{K}_2)$, isometries $v_i : \mathcal{H} \to \mathcal{K}_i$ and a contraction $t : \mathcal{K}_2 \to \mathcal{K}_1$ such that for all $x \in X$, $y \in Y$

$$\psi(x\otimes y) = v_1^* \pi(x) t \rho(y) v_2.$$

[4] A bilinear operator $\Phi : X \times Y \to Z$, where X, Y and Z are operator spaces, is completely bounded if $\sup_{n\in\mathbf{N}} \|\Phi_n\| < \infty$, where the n-th amplification of Φ is defined as $\Phi_n : M_n(X) \times M_n(Y) \to M_n(Z)$, $\Phi_n([x_{ij}],[y_{ij}]) = [\sum_{k=1}^n \Phi(x_{ik}, y_{kj})]$. The cb-norm of a completely bounded bilinear map is defined by $\|\Phi\|_{cb} = \sup_{n\in\mathbf{N}} \|\Phi_n\|$.

Example 2. *The multiplication on a C^*-algebra as an operator on $\mathcal{A}\otimes\mathcal{A}$ (i.e. $m : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, $m(a\otimes b) = ab$) is not continuous with respect to any C^*-tensor norm in general, but it is continuous (even completely contractive) with respect to the Haagerup norm. This even characterizes operator algebras: if X is an operator space with an algebra multiplication which is completely bounded with respect to the Haagerup norm, then X is an operator algebra. This is proven in* D. P. Blecher, Z.-J. Ruan, A. M. Sinclair, A characterization of operator algebras*, J. Funct. Anal.* **89***(1990), 188–201.*

There are many more remarkable properties of the Haagerup norm, which resulted in a wide range of applications and identities. For example,

$$C_m(X)\otimes_h R_n(Y) = M_{mn}(X\otimes_h Y)$$

for all operator spaces X and Y and $m, n \in \mathbb{N}$, in particular

$$C_n\otimes_h R_n = M_n.$$

On the other hand, $R_n\otimes_h C_n$ can be isometrically identified with the dual of M_n equipped with the trace-class norm. Here, $C_m(X)$ denotes the subspace of $M_m(X)$ consisting of all matrices with nonzero entries only in the first column. Similarly, $R_n(Y)$ denotes the matrices in $M_n(Y)$ with nonzero entries only in the first row. C_n denotes $C_n(\mathbf{C})$ and R_n denotes $R_n(\mathbf{C})$. Further, A. Chatterjee and A. M. Sinclair showed in 1992 that if X and Y are both subspaces of $\mathbf{B}(\mathcal{H})$, then $X\otimes_h Y$ is a subspace of the space of completely bounded operators on $\mathbf{K}(\mathcal{H})$ (the compact operators), via the identification $(x\otimes y)(K) = xKy$. In fact, the only nice property the Haagerup norm does not have seems to be the commutativity: $X\otimes_h Y$ is not completely isometrically isomorphic to $Y\otimes_h X$. For more details about the Haagerup norm, see [2], [4], [5], [11] and [16].

The spatial norm $\|\cdot\|_\vee$ on the tensor product of operator spaces is defined in the same way as in the C*-algebra case, of course taking into account the matricial structure: $X\otimes_\vee Y \subseteq \mathbf{B}(\mathcal{H})\otimes\mathbf{B}(\mathcal{K}) \subseteq \mathbf{B}(\mathcal{H}\bar{\otimes}\mathcal{K})$. The corresponding formula is

$$\begin{aligned}\|U\|_\vee \;=\; \sup\{\|(S\otimes T)_n(U)\| : \;& \mathcal{H},\mathcal{K} \text{ Hilbert spaces},\\ & S : X\to\mathbf{B}(\mathcal{H}),\; T : Y\to\mathbf{B}(\mathcal{K}) \text{complete contractions}\}\end{aligned}$$

and it turns out that it is sufficient to consider only finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The spatial norm is the least between all operator space cross-norms whose dual norm is also an operator space cross-norm. It has the property that $X\otimes_\vee Y$ can be completely isometrically embedded into $CB(X', Y)$, resp. into $CB(Y', X)$.

The projective norm $\|\cdot\|_\wedge$ for the tensor product of operator spaces is in such a way constructed that $(X\otimes_\wedge Y)'$ is completely isometrically isomorphic to the space $JCB(X\times Y, \mathbf{C})$ via the usual identification of bilinear maps with linear maps on tensor products. $JCB(X\times Y, \mathbf{C})$ is the set of all bilinear maps $\Phi : X\times Y\to\mathbf{C}$ such that $\sup\{\|[\Phi(x_{ij}, y_{kl})]\| : p, q \in \mathbf{N},\ A = [x_{ij}] \in M_p(X), B = [y_{kl}] \in M_q(X), \|A\| \leq 1, \|B\| \leq 1\}$ is finite, adequately equipped with an matricial structure which makes $JCB(X\times Y, \mathbf{C})$ a operator space. Therefore,

$$\|U\|_\wedge = \sup\{\| < U, V > \| : m \in \mathbf{N}, V \in M_m(JCB(X\times Y, \mathbf{C})), \|V\|_m \leq 1\}$$

for $U \in M_n(X \otimes Y)$ and where $< U, V >= [V_{ij}(U_{kl})] \in M_{mn}$. $(X \otimes_\wedge Y)'$ is also completely isometrically isomorphic to the spaces $CB(X, Y')$ and $CB(Y, X')$. The projective norm is the largest operator space cross-norm. Its dual norm is the spatial norm (but not the other way around).

Both the spatial and the projective norm are associative and commutative and they also both give some useful identites, but they have far less good properties than the Haagerup norm. For more details about the spatial and projective operator space tensor norms (and results for operator space tensor norms in general) see [2] and [4].

4. Tensor products of Hilbert C*-modules

There are two tensor products which have been considered for Hilbert C*-modules : the interior and the exterior tensor product.

Let E be a Hilbert $\mathcal{A}$-module and F a Hilbert $\mathcal{B}$-module and let $\pi : \mathcal{A} \rightarrow \mathbf{B}_\mathcal{B}(F)$ be a C*-morphism. Then F can be considered a left $\mathcal{A}$-module, setting $ax = \pi(a)(x)$ for $x \in F$ and $a \in \mathcal{A}$. Thus, the algebraic tensor product of a right and a left $\mathcal{A}$-module can be constructed: $E \otimes_\mathcal{A} F$ is the quotient of the vector space tensor product $E \otimes F$ by the subspace N spanned by all the expresions of the form $xa \otimes y - x \otimes ay$ with $x \in E$, $y \in F$ and $a \in \mathcal{A}$. Elementary tensors in $E \otimes_\mathcal{A} F$ are denoted as $x \otimes_\mathcal{A} y$. One can regard $E \otimes_\mathcal{A} F$ as a right $\mathcal{B}$-module, setting $(x \otimes_\mathcal{A} y)b = x \otimes_\mathcal{A} yb$. Defining

$$< x \otimes_\mathcal{A} y \mid x' \otimes_\mathcal{A} y' >_\pi := < y \mid \pi(< x \mid x' >_E) y' >_F$$

and extending by ($\mathcal{B}$-)linearity one gets an inner product on $E \otimes_\mathcal{A} F$: all properties except for definiteness are quite easy provable, and definiteness follows from the fact (for the proof see [14]) that the subspace $\{t \in E \otimes F : < t \mid t >= 0\}$ of $E \otimes F$ coincides with N. The resulting Hilbert $\mathcal{B}$-module is called the *interior tensor product* $E \otimes_\pi F$ of E and F (using π). There is a natural C*-morphism $\Pi : \mathbf{B}_\mathcal{A}(E) \rightarrow \mathbf{B}_\mathcal{B}(E \otimes_\pi F)$ defined by $\Pi(T)(x \otimes y) = T(x) \otimes y$ (that is, $\Pi = T \otimes Id_F$). The morphism is injective if π is. It is interesting that when we consider Hilbert C*-modules as operator spaces, the interior tensor product coresponds to the (module) Haagerup tensor product:

$$E \otimes_\pi F = E \otimes_{h\mathcal{A}} F$$

completely isometrically, where $E \otimes_{h\mathcal{A}} F$ denotes $E \otimes_\mathcal{A} F$ completed with respect to the Haagerup norm.

The exterior tensor product of a Hilbert $\mathcal{A}$-module E and a Hilbert $\mathcal{B}$-module F is defined directly imitating the definition of the tensor product of Hilbert spaces: first the algebraic tensor product $E \otimes F$ is organized into a right $\mathcal{A} \otimes \mathcal{B}$-module setting $(x \otimes y)(a \otimes b) := xa \otimes yb$ and then it is defined

$$< x \otimes y \mid x' \otimes y' >=< x \mid x' > \otimes < y \mid y' >$$

(and of course the definitions are extended by ($\mathcal{A} \otimes \mathcal{B}$-)linearity to the whole algebraic tensor product). Here one encounters two problems - missing of the definiteness and the fact that $\mathcal{A} \otimes \mathcal{B}$ is not a C*-algebra, but only dense in one ($\mathcal{A} \otimes_\alpha \mathcal{B}$ for this case), so the module multiplication $(x \otimes y)t$ is not defined for all $t \in \mathcal{A} \otimes_\alpha \mathcal{B}$ and consequently the condition $< u \mid vt >=< u \mid v > t$ makes no sense for $t \in (\mathcal{A} \otimes_\alpha \mathcal{B}) \setminus (\mathcal{A} \otimes \mathcal{B})$. This means that we only have a semi-inner-product module $E\otimes F$ over a pre-C*-algebra $\mathcal{A}\otimes\mathcal{B}$. But, as it turns out, the problem is quite easily resolved: first the definiteness is achieved taking the usual quotient by the subspace $N = \{t \in E\otimes F : <t\,|\,t>= 0\}$ (as a matter of fact, the semi-inner product is actually already definite - see [14]). On the quotient $E\otimes F/_N$ one may define the norm $\|u + N\| = \|<u\,|\,u>\|_\alpha^{1/2}$. Then for $t \in \mathcal{A}\otimes_\alpha\mathcal{B}$ and $u \in E\otimes F$ one has $\|(u+N)t\| \le \|u+N\|\|t\|_\alpha$, so the structure of the right $\mathcal{A}\otimes\mathcal{B}$-module can be extended to the structure of the right $\mathcal{A}\otimes_\alpha\mathcal{B}$-module on $E\otimes F/_N$ and then it is easily shown that the condition $<u+N\,|\,(v+N)t>=<u+N\,|\,v+N>t$ holds for $u, v \in E\otimes F$ and $t \in \mathcal{A}\otimes_\alpha\mathcal{B}$. The other relations required for the inner product also extend to the situation now considered. The resulting Hilbert $\mathcal{A}\otimes_\alpha\mathcal{B}$-module $E\otimes_{ext}F$ is called the *exterior tensor product* of E and F. As to the properties of the exterior tensor product with respect to (adjointable) operators on its underlying modules, there is a natural C*-morphism (isometric embedding) $\mathbf{B}_\mathcal{A}(E)\otimes_\alpha\mathbf{B}_\mathcal{B}(F)\to\mathbf{B}_{\mathcal{A}\otimes_\alpha\mathcal{B}}(E\otimes_{ext}F)$ defined by $(S\otimes T)(x\otimes y) = S(x)\otimes T(y)$. The embedding will not in general be surjective (as is well known for the Hilbert space case). When we consider Hilbert C*-modules as operator spaces, the exterior tensor product corresponds to the spatial tensor product:

$$E\otimes_{ext}F = E\otimes_\vee F$$

completely isometrically, so $E\otimes_{ext}F$ can be considered as a subspace of $CB(E', F)$ or $CB(F', E)$. Both results on the correspondence of the operator space tensor norms and Hilbert module tensor products are due to Blecher ([3]).

Both tensor products are more completely described in [14]. The exterior and the interior tensor product of Hilbert modules coincide when one of the Hilbert modules considered is a Hilbert space (and if both are Hilbert spaces, this tensor product is equal to the standard Hilbert space tensor product).

5. Some examples

Let us give a few examples of interconnections between the theories of tensor products of C*-algebras, operator spaces and Hilbert C*-modules.

Example 3. *The natural contractive morphism $\mathcal{A}\otimes_h\mathcal{B}\to\mathcal{A}\otimes_\alpha\mathcal{B}$ is injective (* D. P. Blecher, Geometry of the tensor product of C*-algebras*, Math. Proc. Cam. Phil. Soc. 104 (1988) 119-127). Under this map, a closed ideal I in $\mathcal{A}\otimes_h\mathcal{B}$ induces a closed ideal I_α (the closure of the image of I) in $\mathcal{A}\otimes_\alpha\mathcal{B}$. Closed non-zero ideals in $\mathcal{A}\otimes_\alpha\mathcal{B}$ must contain a non-zero elementary tensor and every elementary tensor which lies in I_α must lie in I. Combining these results one gets that closed non-zero ideals in $\mathcal{A}\otimes_h\mathcal{B}$ must contain a non-zero elementary tensor. This fact has many consequences for investigating the ideal structure of $\mathcal{A}\otimes_h\mathcal{B}$, e.g. it immediately follows that if both $\mathcal{A}$ and $\mathcal{B}$ are simple, so is $\mathcal{A}\otimes_h\mathcal{B}$, a property belonging also to the spatial tensor product of C*-algebras.*

There are also analogies in the ideal structure of $\mathcal{A}\otimes_h\mathcal{B}$ with the ideal structure of $\mathcal{A}\otimes_\nu\mathcal{B}$, e.g. the closed ideal $\mathbf{B}(\mathcal{H})\otimes_\beta\mathbf{K}(\mathcal{H}) + \mathbf{K}(\mathcal{H})\otimes_\beta\mathbf{B}(\mathcal{H})$ is maximal in $\mathbf{B}(\mathcal{H})\otimes_\beta\mathbf{B}(\mathcal{H})$ when β is the Haagerup or the maximal C-norm (for $\mathcal{H}$ infinite dimensional separable Hilbert space).*

These results are from [1].

Example 4. ([3]) *The metric characterization of Hilbert C*-modules can be given in terms of the Haagerup tensor product:*

Theorem 3. *Let E be a Banach space (resp. operator space) which is also a (right) $\mathcal{A}$-module. Suppose that $\mathcal{A}$ is faithfully and nondegenerately[5] represented on a Hilbert space $\mathcal{H}$. Then E is a Hilbert $\mathcal{A}$-module (with its Hilbert C*-module norm coinciding with the original norm) if and only if the following three conditions hold:*
(i) The Haagerup tensor product $E\otimes_{h\mathcal{A}}\mathcal{H}^c$ is a Hilbert space[6];
(ii) The map $\phi : E\to B(\mathcal{H}, E\otimes_{h\mathcal{A}}\mathcal{H}^c)$ given by $\phi(x)(\xi) = x\otimes\xi$ is a (complete) isometry;
(iii) For all $x \in E$ we have $\phi(x)^*\phi(x) \in \mathcal{A}$.

If these conditions hold, the (unique[7]) inner product on E is given by

$$< x \mid y >= \phi(x)^*\phi(y).$$

The proof of the theorem uses a factorization theorem for Hilbert $\mathcal{A}$-modules and some (rather simple) properties of the Haagerup norm.

Moreover, the inner product $< \cdot \mid \cdot >$ on a Hilbert $\mathcal{A}$-module E and the inner product $(. \mid .)$ on the Hilbert space $E\otimes_{h\mathcal{A}}\mathcal{H}^c$ are related by the formula $(x\otimes\xi \mid y\otimes\eta) = (< y \mid x > \xi \mid \eta)_{\mathcal{H}}$, so $E\otimes_{h\mathcal{A}}\mathcal{H}^c$ coincides with the interior tensor product $E\otimes_\pi\mathcal{H}$, where π is a (nondegenerate) representation of $\mathcal{A}$ on $\mathcal{H}$ (so $\mathcal{H}$ can be thought of as a left $\mathcal{A}$-module as well as a right Hilbert $\mathbf{C}$-module). The more general correspondence between the interior tensor product of Hilbert modules and their Haagerup tensor product is already mentioned in the previous section.

Example 5. *If $\mathcal{H}$ is a (separable) Hilbert space and $\mathcal{A}$ a C*-algebra, one can form their tensor product $\mathcal{H}\otimes\mathcal{A}$, which is a pre-Hilbert-$\mathcal{A}$-module defining $< \xi\otimes a \mid \eta\otimes b >= (\xi \mid \eta)a^*b$. Its completion is a Hilbert $\mathcal{A}$-module (it is in fact the interior tensor product of the right $\mathcal{A}$-module $\mathcal{A}$, with the inner product $< a \mid b >= a^*b$, and the Hilbert space $\mathcal{H}$ considered as a right Hilbert $\mathbf{C}$-module). This Hilbert module coincides with the Hilbert $\mathcal{A}$-module of all countable sequences in $\mathcal{A}$ with natural operations and the inner product $< (a_n) \mid (b_n) >= \sum_{n\in\mathbf{N}} a_n^* b_n$. The module is usually denoted $\mathcal{H}_\mathcal{A}$ and called the standard Hilbert $\mathcal{A}$-module. It obviously generalizes the l_2 Hilbert space. The standard Hilbert $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ has great importance in the theory of Hilbert modules and their applications, e.g. in K- and KK-theory. One of its important properties is e.g. that $\mathbf{B}_\mathcal{A}(\mathcal{H}_\mathcal{A})$ is C*-isomorphic to the multiplier algebra[8] $M(\mathcal{A}\otimes_\alpha\mathbf{K}(\mathcal{H}))$.*

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[5] A representation π of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called nondegenerate if the closure of the linear span of all $\pi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{H}$) equals $\mathcal{H}$.

[6] $\mathcal{H}^c$ denotes the Hilbert column space $B(\mathbf{C}, \mathcal{H})$ with its natural operator space structure. This is one way to turn a Hilbert space into an operator space.

[7] E.C. Lance proved (*Unitary operators on Hilbert C*-modules*, Bull. London Math. Soc. 26 (1994), 363-366) that there is a $1-1$ correspondence between norm and inner product on a Hilbert C*-module.

[8] The multiplier algebra of a C*-algebra is the C*-algebra $M(\mathcal{A}) = \{a \in \mathcal{A}^{**} : ax, xa \in \mathcal{A} \text{ for all } x \in \mathcal{A}\}$ (this is just one possible realization of the more abstract definition). When considering $\mathcal{A}$ as a Hilbert $\mathcal{A}$-module, it turns out that $M(\mathcal{A})$ is C*-isomorphic to $\mathbf{B}_\mathcal{A}(\mathcal{A})$.

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