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HOUSEHOLDER'S APPROXIMANTS AND CONTINUED FRACTION EXPANSION OF QUADRATIC IRRATIONALS

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Abstract. There are numerous methods for rational approximation of real numbers. Continued fraction convergent is one of them and Newton's iterative method is another one. Connections between these two approximation methods were discussed by several authors. Householder's methods are generalisation of Newton's method. In this paper, we will show that for these methods analogous connection with continued fractions hold.

1. Introduction and main results

Let α be a quadratic irrational, i.e., $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$ and d is not a square of a rational number. It is well known that continued fraction expansion of α is periodic, i.e., has the form

$$\alpha = [\, a_0, a_1, \ldots, a_h, \overline{a_{h+1}, a_{h+2}, \ldots, a_{h+\ell}}\,].$$

Here $\ell = \ell(\alpha)$ denotes the length of the shortest period in the expansion of α. We will observe quadratic irrationals whose period begins with a_1. We will say that period is palindromic if it holds $a_1 = a_{\ell-1}$, $a_2 = a_{\ell-2}$, $\ldots$, i.e., the period without the last term is symmetric.

Continued fraction convergents $\frac{p_n}{q_n} = [\, a_0, a_1, \ldots, a_n\,]$ give good rational approximations of α. Another approximation method we obtain using the Householder's iterative method of order p. This method is a numerical algorithm for solving the nonlinear equation $f(x) = 0$, where $f(x)$ is a $p+1$ times continuously differentiable function and α is a zero of f but not of its

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derivative. Householder's method of order p consists of a sequence of iterations

$$x_{i+1} = x_i + p \cdot \frac{(1/f)^{(p-1)}(x_i)}{(1/f)^{(p)}(x_i)}$$

beginning with an initial guess x_0. Householder's method of order 1 is just Newton's method and for Householder's method of order 2 one gets Halley's method.

In this paper we study connections between continued fraction convergents of quadratic irrational $\alpha = c + \sqrt{d}$ and Householder's iterative method of order $m-1$, $m \in \mathbb{N}$, $m \geq 2$ (with rate of convergence m) for the equation $f(x) = (x-\alpha)(x-\alpha') = 0$, where $\alpha' = c - \sqrt{d}$. Precisely, if the initial iteration $x_0 = \frac{p_n}{q_n}$ is the nth continued fraction convergent of α, the principal question is whether the first iteration $R_n^{(m)} = x_1$ also a convergent of α. In that case we say that $R_n^{(m)}$ is a *good approximant*.

We will show that for quadratic irrational α whose period of the length ℓ begins with a_1, there is a good approximant at the end of the period, i.e., it holds

$$R_{k\ell-1}^{(m)} = \frac{p_{mk\ell-1}}{q_{mk\ell-1}}, \text{ for all } k \in \mathbb{N}, \tag{1.1}$$

and when period is palindromic and has even length, say $\ell = 2t$, there is a good approximant in the half of the period, i.e., it holds

$$R_{kt-1}^{(m)} = \frac{p_{mkt-1}}{q_{mkt-1}}, \text{ for all } k \in \mathbb{N}. \tag{1.2}$$

In Section 3 we show

THEOREM 1.1. *To be a good approximant is a periodic property, i.e., for all $k \in \mathbb{N}$ it holds*

$$R_n^{(m)} = \frac{p_s}{q_s} \qquad \iff \qquad R_{k\ell+n}^{(m)} = \frac{p_{km\ell+s}}{q_{km\ell+s}},$$

and when period is palindromic, it is also a palindromic property, i.e., it holds

$$R_n^{(m)} = \frac{p_s}{q_s} \qquad \iff \qquad R_{\ell-n-2}^{(m)} = \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}.$$

When $\ell \leq 2$, from (1.1) and (1.2) it follows that then every approximant is good, and then it holds $R_n^{(m)} = \frac{p_{m(n+1)-1}}{q_{m(n+1)-1}}$ for all $n \geq 0$. So if $R_n^{(m)}$ is a good approximant, one might expect that it hits convergent with m-times larger index. However, this is not always true. If $R_n^{(m)} = \frac{p_s}{q_s}$, we can define numbers $j_n^{(m)} = j_n^{(m)}(\alpha)$ as half of the distance from convergent with m times larger index

$$j_n^{(m)} = \frac{s+1-m(n+1)}{2}. \tag{1.3}$$

We prove that it is unbounded by constructing an explicit family of quadratic irrationals, which involves the Fibonacci numbers.

THEOREM 1.2. *Let F_ℓ denote the ℓ-th Fibonacci number. Let $\ell > 3, \ell \equiv \pm 1 \pmod 6$. Then for $d_\ell = \left(\frac{F_{\ell-3}F_\ell+1}{2}\right)^2 + F_{\ell-3}F_{\ell-1} + 1$ and $M \in \mathbb{N}$ it holds $\ell(\sqrt{d_\ell}) = \ell$ and*

$$j_0^{(3M-1)}(\sqrt{d_\ell}) = j_0^{(3M)}(\sqrt{d_\ell}) = j_0^{(3M+1)}(\sqrt{d_\ell}) = \tfrac{\ell-3}{2} \cdot M.$$

Connection between Newton's iterative method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

for solving nonlinear equations $f(x) = 0$ and continued fractions was discussed by several authors. So, let us briefly mention what is known in the case $m = 2$.

It is well known that for $\alpha = \sqrt{d}$, $d \in \mathbb{N}$, d not a perfect square, and the corresponding Newton's approximant $R_n^{(2)} = \frac{1}{2}(\frac{p_n}{q_n} + \frac{dq_n}{p_n})$ it follows that (see e.g. [1, p. 468])

$$R_{k\ell-1}^{(2)} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \tag{1.4}$$

It was proved by Mikusiński [9] (see also Elezović [4]) that if $\ell = 2t$, then

$$R_{kt-1}^{(2)} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1. \tag{1.5}$$

These results imply that if $\ell(\sqrt{d}) \leq 2$, then all approximants $R_n^{(2)}$ are convergents of $\sqrt{d}$. In 2001, Dujella [2] proved the converse of this result. Namely, if all approximants $R_n^{(2)}$ are convergents of $\sqrt{d}$, then $\ell(\sqrt{d}) \leq 2$. Thus, if $\ell(\sqrt{d}) > 2$, we know that some of approximants $R_n^{(2)}$ are convergents and some of them are not. Using a result of Komatsu [8] from 1999, Dujella also showed that being a good approximant is a periodic and a palindromic property, so he defined the number $b(\sqrt{d})$ as the number of good approximants in the period. Formulas (1.4) and (1.5) suggest that $R_n^{(2)}$ should be convergent whose index is twice as large when it is a good approximant. However, this is not always true, and Dujella defined the number $j(\sqrt{d})$ as half of the distance from two times larger index. He also pointed out that $j(\sqrt{d})$ is unbounded. In 2005, Dujella and the author [3] proved that $b(\sqrt{d})$ is unbounded, too.

In 2011, the author [13] proved the analogous results for $\alpha = \frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, d not a perfect square and $d \equiv 1 \pmod 4$.

Sharma [15] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, d is not a square of a rational number, whose continued fraction period of the length ℓ begins with a_1. He showed that for every such α and the corresponding Newton's approximant $R_n^{(2)} = \frac{p_n^2 - \alpha\alpha' q_n^2}{q_n(2p_n - (\alpha+\alpha')q_n)}$ it holds

$$R_{k\ell-1}^{(2)} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1,$$

and when $\ell = 2t$ and the period is palindromic then it holds

$$R_{kt-1}^{(2)} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1.$$

Frank and Sharma [6] discussed generalization of Newton's formula. They showed that for every quadratic irrational α, whose period begins with a_1, it holds

$$\frac{p_{mk\ell-1}}{q_{mk\ell-1}} = \frac{\alpha(p_{k\ell-1} - \alpha' q_{k\ell-1})^m - \alpha'(p_{k\ell-1} - \alpha q_{k\ell-1})^m}{(p_{k\ell-1} - \alpha' q_{k\ell-1})^m - (p_{k\ell-1} - \alpha q_{k\ell-1})^m}, \text{ for } k, m \in \mathbb{N}, \tag{1.6}$$

and when $\ell = 2t$ and the period is palindromic then it holds

$$\frac{p_{mkt-1}}{q_{mkt-1}} = \frac{\alpha(p_{kt-1} - \alpha' q_{kt-1})^m - \alpha'(p_{kt-1} - \alpha q_{kt-1})^m}{(p_{kt-1} - \alpha' q_{kt-1})^m - (p_{kt-1} - \alpha q_{kt-1})^m}, \text{ for } k, m \in \mathbb{N}. \tag{1.7}$$

2. Householder's methods

Householder's iterative method (see [14], [7, §4.4]) of order p for rootsolving, consists of a sequence of iterations

$$x_{i+1} = H^{(p)}(x_i) = x_i + p \cdot \frac{(1/f)^{(p-1)}(x_i)}{(1/f)^{(p)}(x_i)},$$

(where $(1/f)^{(p)}$ denotes p-th derivative of $1/f$) beginning with an initial guess x_0. Let $f(x)$ be a $p+1$ times continuously differentiable function and α is a zero of f but not of its derivative, then, in a neighborhood of α, the convergence has rate $p+1$.

Analogous to Newton's method, we will start with function $f(x) = (x - \alpha)(x - \alpha')$, which satisfies the above conditions. Let us first observe p-th derivative of the function $1/f$

$$\begin{aligned}(1/f)^{(p)} &= \Big(\frac{1}{(x-\alpha)(x-\alpha')}\Big)^{(p)} = \frac{1}{\alpha - \alpha'}\Big(\frac{1}{x-\alpha} - \frac{1}{x-\alpha'}\Big)^{(p)} \\ &= \frac{(-1)^p p!}{\alpha - \alpha'}\Big(\frac{1}{(x-\alpha)^{p+1}} - \frac{1}{(x-\alpha')^{p+1}}\Big).\end{aligned}$$

So we have

$$\begin{aligned}H^{(p)}(x) &= x - \frac{(x-\alpha')^p - (x-\alpha)^p}{(x-\alpha')^{p+1} - (x-\alpha)^{p+1}}(x-\alpha)(x-\alpha') \\ &= \frac{\alpha(x-\alpha')^{p+1} - \alpha'(x-\alpha)^{p+1}}{(x-\alpha')^{p+1} - (x-\alpha)^{p+1}}.\end{aligned} \tag{2.1}$$

It is not hard to show that it holds

$$H^{(p+1)}(x)=\frac{xH^{(p)}(x)-\alpha\alpha'}{H^{(p)}(x)+x-\alpha-\alpha'},\quad \text{for } p\in\mathbb{N}. \tag{2.2}$$

Formula (1.6) shows that for an arbitrary quadratic surd, whose period begins with a_1 and $k\in\mathbb{N}$, $m=2,3,\dots$, it holds

$$H^{(m-1)}\Big(\frac{p_{k\ell-1}}{q_{k\ell-1}}\Big)=\frac{p_{mk\ell-1}}{q_{mk\ell-1}}, \tag{2.3}$$

and when period is palindromic, and has even length, say $\ell=2t$, from (1.7) it follows

$$H^{(m-1)}\Big(\frac{p_{kt-1}}{q_{kt-1}}\Big)=\frac{p_{mkt-1}}{q_{mkt-1}}. \tag{2.4}$$

Let us recall the definition

$$R_n^{(1)}=\frac{p_n}{q_n},\qquad \text{and for } m>1\quad R_n^{(m)}=H^{(m-1)}\Big(\frac{p_n}{q_n}\Big),$$

and we say that $R_n^{(m)}$ is *good approximation*, if it is a convergent of α. From (2.3) and (2.4) it follows (1.1) and (1.2). From (2.1) we have

$$R_n^{(m)}=\frac{\alpha(p_n-\alpha' q_n)^m-\alpha'(p_n-\alpha q_n)^m}{(p_n-\alpha' q_n)^m-(p_n-\alpha q_n)^m}, \tag{2.5}$$

and formula (2.2) says

$$R_n^{(m+1)}=\frac{R_n^{(1)}R_n^{(m)}-\alpha\alpha'}{R_n^{(1)}+R_n^{(m)}-\alpha-\alpha'},\quad \text{for } m\in\mathbb{N},\ n=0,1,\dots. \tag{2.6}$$

3. Good approximants are periodic and palindromic

From now on, we assume that α is quadratic irrational whose period of the length ℓ begins with a_1. From formula [15, (8)] we obtain

$$(a_\ell-a_0)p_{k\ell-1}+p_{k\ell-2}=-\alpha\alpha' q_{k\ell-1}, \tag{3.1}$$

$$(a_\ell-a_0)q_{k\ell-1}+q_{k\ell-2}=p_{k\ell-1}-(\alpha+\alpha')q_{k\ell-1}, \tag{3.2}$$

for all $k\in\mathbb{N}$.

LEMMA 3.1. *For $m,k\in\mathbb{N}$ and $i=1,2,\dots,\ell$, it holds*

$$R_{k\ell+i-1}^{(m)}=\frac{R_{k\ell-1}^{(m)}R_{i-1}^{(m)}-\alpha\alpha'}{R_{k\ell-1}^{(m)}+R_{i-1}^{(m)}-\alpha-\alpha'}. \tag{3.3}$$

PROOF. For $m=1$, statement of the lemma is proven in [5, Thm. 2.1]. Suppose that (3.3) holds for some $m\in\mathbb{N}$, and let us show that it holds for

$m+1$ too. Using the notation $s = R^{(1)}_{k\ell-1}$, $S = R^{(m)}_{k\ell-1}$, $t = R^{(1)}_{i-1}$ and $T = R^{(m)}_{i-1}$, we have

$$R^{(m+1)}_{k\ell+i-1} \overset{(2.6)}{=} \frac{R^{(1)}_{k\ell+i-1}R^{(m)}_{k\ell+i-1} - \alpha\alpha'}{R^{(1)}_{k\ell+i-1} + R^{(m)}_{k\ell+i-1} - \alpha - \alpha'} = \frac{\frac{st-\alpha\alpha'}{s+t-\alpha-\alpha'} \cdot \frac{ST-\alpha\alpha'}{S+T-\alpha-\alpha'} - \alpha\alpha'}{\frac{st-\alpha\alpha'}{s+t-\alpha-\alpha'} + \frac{ST-\alpha\alpha'}{S+T-\alpha-\alpha'} - \alpha - \alpha'}$$
$$= \frac{(st-\alpha\alpha')(ST-\alpha\alpha') - \alpha\alpha'(s+t-\alpha-\alpha')(S+T-\alpha-\alpha')}{(st-\alpha\alpha')(S+T-\alpha-\alpha') + (ST-\alpha\alpha')(s+t-\alpha-\alpha') - (\alpha+\alpha')(s+t-\alpha-\alpha')(S+T-\alpha-\alpha')}$$
$$= \frac{(sS-\alpha\alpha')(tT-\alpha\alpha') - \alpha\alpha'(s+S-\alpha-\alpha')(t+T-\alpha-\alpha')}{(sS-\alpha\alpha')(t+T-\alpha-\alpha') + (tT-\alpha\alpha')(s+S-\alpha-\alpha') - (\alpha+\alpha')(s+S-\alpha-\alpha')(t+T-\alpha-\alpha')}$$
$$= \frac{\frac{sS-\alpha\alpha'}{s+S-\alpha-\alpha'} \cdot \frac{tT-\alpha\alpha'}{t+T-\alpha-\alpha'} - \alpha\alpha'}{\frac{sS-\alpha\alpha'}{s+S-\alpha-\alpha'} + \frac{tT-\alpha\alpha'}{t+T-\alpha-\alpha'} - \alpha - \alpha'} \overset{(2.6)}{=} \frac{R^{(m+1)}_{k\ell-1}R^{(m+1)}_{i-1} - \alpha\alpha'}{R^{(m+1)}_{k\ell-1} + R^{(m+1)}_{i-1} - \alpha - \alpha'}.$$

□

We have (see e.g. [12, §23]) $-(\alpha - a_0)' = [\,\overline{a_\ell, a_{\ell-1}, \ldots, a_2, a_1}\,]$, and so

$$\frac{1}{a_0 - a_\ell - \alpha'} = [\,\overline{a_{\ell-1}, \ldots, a_2, a_1, a_\ell}\,].$$

So when period is palindromic, we have $\alpha + \alpha' = 2a_0 - a_\ell$, thus formulas (3.1) and (3.2) in palindromic case become

$$a_0 p_{k\ell-1} + p_{k\ell-2} = (\alpha + \alpha')p_{k\ell-1} - \alpha\alpha' q_{k\ell-1}, \tag{3.4}$$
$$a_0 q_{k\ell-1} + q_{k\ell-2} = p_{k\ell-1}. \tag{3.5}$$

Lemma 3.2. *For $m, k \in \mathbb{N}$ and $i = 1, 2, \ldots, \ell - 1$, when period is palindromic, it holds*

$$R^{(m)}_{k\ell-i-1} = \frac{R^{(m)}_{k\ell-1}(R^{(m)}_{i-1} - \alpha - \alpha') + \alpha\alpha'}{R^{(m)}_{i-1} - R^{(m)}_{k\ell-1}}. \tag{3.6}$$

Proof. For $m = 1$ we have

$$R^{(1)}_{k\ell-i-1} = \frac{p_{k\ell-i-1}}{q_{k\ell-i-1}} = \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [\,a_0, a_1, \ldots, a_{k\ell-i-1}, a_{k\ell-i}, 0\,]$$
$$= [\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i+1}, \ldots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \ldots, -a_{i-1}\,]$$
$$= [\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i+1}, \ldots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}}\,]$$
$$= \frac{p_{k\ell-1}(a_0 - R^{(1)}_{i-1}) + p_{k\ell-2}}{q_{k\ell-1}(a_0 - R^{(1)}_{i-1}) + q_{k\ell-2}} \underset{(3.5)}{\overset{(3.4)}{=}} \frac{R^{(1)}_{k\ell-1}(R^{(1)}_{i-1} - \alpha - \alpha') + \alpha\alpha'}{R^{(1)}_{i-1} - R^{(1)}_{k\ell-1}}.$$

Suppose that (3.6) holds for some $m \in \mathbb{N}$, and let us show that it holds for $m+1$ too. With the same notation as in the proof of Lemma 3.1, we have

$$\begin{aligned}
R_{k\ell-i-1}^{(m+1)} &\overset{(2.6)}{=} \frac{R_{k\ell-i-1}^{(1)} R_{k\ell-i-1}^{(m)} - \alpha\alpha'}{R_{k\ell-i-1}^{(1)} + R_{k\ell-i-1}^{(m)} - \alpha - \alpha'} \\
&= \frac{\frac{s(t-\alpha-\alpha')+\alpha\alpha'}{t-s} \cdot \frac{S(T-\alpha-\alpha')+\alpha\alpha'}{T-S} - \alpha\alpha'}{\frac{s(t-\alpha-\alpha')+\alpha\alpha'}{t-s} + \frac{S(T-\alpha-\alpha')+\alpha\alpha'}{T-S} - \alpha - \alpha'} \\
&= \frac{(s(t-\alpha-\alpha')+\alpha\alpha')(S(T-\alpha-\alpha')+\alpha\alpha')-\alpha\alpha'(t-s)(T-S)}{(s(t-\alpha-\alpha')+\alpha\alpha')(T-S)+(S(T-\alpha-\alpha')+\alpha\alpha')(t-s)-(\alpha+\alpha')(t-s)(T-S)} \\
&= \frac{(sS-\alpha\alpha')(tT-\alpha\alpha'-(\alpha+\alpha')(t+T-\alpha-\alpha'))+\alpha\alpha'(s+S-\alpha-\alpha')(t+T-\alpha-\alpha')}{(tT-\alpha\alpha')(s+S-\alpha-\alpha')-(sS-\alpha\alpha')(t+T-\alpha-\alpha')} \\
&= \frac{\frac{sS-\alpha\alpha'}{s+S-\alpha-\alpha'}\left(\frac{tT-\alpha\alpha'}{t+T-\alpha-\alpha'} - \alpha - \alpha'\right) + \alpha\alpha'}{\frac{tT-\alpha\alpha'}{t+T-\alpha-\alpha'} - \frac{sS-\alpha\alpha'}{s+S-\alpha-\alpha'}} \\
&\overset{(2.6)}{=} \frac{R_{k\ell-1}^{(m+1)}(R_{i-1}^{(m+1)} - \alpha - \alpha') + \alpha\alpha'}{R_{i-1}^{(m+1)} - R_{k\ell-1}^{(m+1)}}.
\end{aligned}$$

∎

Let us show that each approximant can be expressed as the combination of convergent with m times larger index and carefully selected numbers $\beta_i^{(m)}$, which are periodic (so we take them for $i = 1, \ldots \ell - 1$).

PROPOSITION 3.3. *Let $m \in \mathbb{N}$. For $i = 1, 2, \ldots, \ell - 1$ let*

$$\beta_i^{(m)} = -\frac{p_{mi-1} - R_{i-1}^{(m)} q_{mi-1}}{p_{mi} - R_{i-1}^{(m)} q_{mi}}.$$

Then it holds

$$R_{k\ell+i-1}^{(m)} = \frac{\beta_i^{(m)} p_{m(k\ell+i)} + p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)} + q_{m(k\ell+i)-1}}, \text{ for all } k \geq 0, \tag{3.7}$$

and when period is palindromic, then

$$R_{k\ell-i-1}^{(m)} = \frac{p_{m(k\ell-i)-1} - \beta_i^{(m)} p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)} q_{m(k\ell-i)-2}}, \text{ for all } k \geq 1. \tag{3.8}$$

PROOF. Let us first consider the continued fraction expansion of $\beta_i^{(m)}$.

$$\begin{aligned}
\beta_i^{(m)} &= -\left[0, \frac{p_{mi} - R_{i-1}^{(m)} q_{mi}}{p_{mi-1} - R_{i-1}^{(m)} q_{mi-1}}\right] \\
&\overset{[13,\text{ Lm. } 3]}{=} -\left[0, a_{mi}, a_{mi-1}, \ldots, a_1, a_0 - R_{i-1}^{(m)}\right] \\
&= \left[0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)}\right].
\end{aligned}$$

If $k=0$ we have

$$\frac{\beta_i^{(m)} p_{mi}+p_{mi-1}}{\beta_i^{(m)} q_{mi}+q_{mi-1}} = \left[\, a_0, a_1, \ldots, a_{mi-1}, a_{mi}, \beta_i^{(m)} \,\right]$$
$$= \left[\, a_0, a_1, \ldots, a_{mi-1}, a_{mi}, 0, -a_{mi}, -a_{mi-1}, \ldots, -a_1, -a_0 + R_{i-1}^{(m)} \,\right] = R_{i-1}^{(m)},$$

and if $k>0$ we have

$$\begin{aligned}
&\frac{\beta_i^{(m)} p_{m(k\ell+i)}+p_{m(k\ell+i)-1}}{\beta_i^{(m)} q_{m(k\ell+i)}+q_{m(k\ell+i)-1}} \\
&\quad= \left[\, a_0, a_1, \ldots, a_{mk\ell-1}, a_{mk\ell}, a_{mk\ell+1}, \ldots, a_{m(k\ell+i)}, \beta_i^{(m)} \,\right] \\
&\quad= \left[\, a_0, a_1, \ldots, a_{mk\ell-1}, a_{mk\ell} - a_0 + R_{i-1}^{(m)} \,\right] \\
&\quad= \frac{p_{mk\ell-1}(a_{mk\ell} - a_0 + R_{i-1}^{(m)}) + p_{mk\ell-2}}{q_{mk\ell-1}(a_{mk\ell} - a_0 + R_{i-1}^{(m)}) + q_{mk\ell-2}} \\
&\quad\overset{(3.1)}{\underset{(3.2)}{=}} \frac{p_{mk\ell-1}R_{i-1}^{(m)} - \alpha\alpha' q_{mk\ell-1}}{p_{mk\ell-1} + q_{mk\ell-1}(R_{i-1}^{(m)} - \alpha - \alpha')} \\
&\quad\overset{(2.3)}{=} \frac{R_{k\ell-1}^{(m)} R_{i-1}^{(m)} - \alpha\alpha'}{R_{k\ell-1}^{(m)} + R_{i-1}^{(m)} - \alpha - \alpha'} \overset{\text{Lm. 3.1}}{=} R_{k\ell+i-1}^{(m)}.
\end{aligned}$$

When period is palindromic we have

$$\begin{aligned}
&\frac{p_{m(k\ell-i)-1} - \beta_i^{(m)} p_{m(k\ell-i)-2}}{q_{m(k\ell-i)-1} - \beta_i^{(m)} q_{m(k\ell-i)-2}} = \left[a_0, a_1, \ldots, a_{m(k\ell-i)-1}, -\frac{1}{\beta_i^{(m)}} \right] \\
&\quad= \left[\, a_0, a_1, \ldots, a_{m(k\ell-i)-1}, 0, 0, a_{mi}, a_{mi-1}, \ldots, a_1, a_0 - R_{i-1}^{(m)} \,\right] \\
&\quad= \left[\, a_0, a_1, \ldots, a_{m(k\ell-i)-1}, a_{m(k\ell-i)}, a_{m(k\ell-i)+1}, \ldots, a_{mk\ell-1}, a_0 - R_{i-1}^{(m)} \,\right] \\
&\quad= \frac{p_{mk\ell-1}(a_0 - R_{i-1}^{(m)}) + p_{mk\ell-2}}{q_{mk\ell-1}(a_0 - R_{i-1}^{(m)}) + q_{mk\ell-2}} \overset{(3.4)}{\underset{(3.5)}{=}} \frac{p_{mk\ell-1}(R_{i-1}^{(m)} - \alpha - \alpha') + \alpha\alpha' q_{mk\ell-1}}{q_{mk\ell-1}R_{i-1}^{(m)} - p_{mk\ell-1}} \\
&\quad\overset{(2.3)}{=} \frac{R_{k\ell-1}^{(m)}(R_{i-1}^{(m)} - \alpha - \alpha') + \alpha\alpha'}{R_{i-1}^{(m)} - R_{k\ell-1}^{(m)}} \overset{\text{Lm. 3.2}}{=} R_{k\ell-i-1}^{(m)}.
\end{aligned}$$

□

Remark 3.4. [8, Thm. 1] and [13, Thm. 2] are special cases of the last proposition for $m=2$ and $\alpha=\sqrt{d}$ and $\alpha=\frac{1+\sqrt{d}}{2}$, respectively.

Proof of Thereom 1.1. The first part for $n=\ell-1$ is (1.1) and for $n=0,1,\ldots,\ell-2$ follows from (3.7). The second part follows similarly as in [2, Lm. 3], but we have three cases. Let $R_n^{(m)} = \frac{p_s}{q_s} = [\, a_0, a_1, \ldots, a_s \,]$.

If $s = m(n+1) - 1$, from (3.7) we have $\beta_{n+1}^{(m)} = 0$, so from (3.8) we have $R_{\ell-n-2}^{(m)} = \frac{p_{m(\ell-n-1)-1}}{q_{m(\ell-n-1)-1}} = \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}$.

If $s > m(n+1) - 1$, then from (3.7) we have

$$\begin{aligned}\beta_{n+1}^{(m)} &= [\, a_{m(n+1)+1}, a_{m(n+1)+2}, \dots, a_s \,] \\ &= [\, a_{m(\ell-n-1)-1}, a_{m(\ell-n-1)-2}, \dots, a_{m\ell-s} \,].\end{aligned}$$

From (3.8) we have

$$\begin{aligned}R_{\ell-n-2}^{(m)} &= \frac{p_{m(\ell-n-1)-1} - \beta_{n+1}^{(m)} p_{m(\ell-n-1)-2}}{q_{m(\ell-n-1)-1} - \beta_{n+1}^{(m)} q_{m(\ell-n-1)-2}} \\ &= \left[a_0, a_1, \dots, a_{m(\ell-n-1)-1}, -\frac{1}{\beta_{n+1}^{(m)}} \right] \\ &= [\, a_0, a_1, \dots, a_{m(\ell-n-1)-1}, 0, -a_{m(\ell-n-1)-1}, -a_{m(\ell-n-1)-2}, \dots, -a_{m\ell-s} \,] \\ &= [\, a_0, a_1, \dots, a_{m\ell-s-1}, 0 \,] = \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}.\end{aligned}$$

If $s < m(n+1) - 1$, then from (3.8) we have

$$\begin{aligned}\beta_{\ell-n-1}^{(m)} &= [\, a_{m(n+1)-1}, a_{m(n+1)-2}, \dots, a_{s+2} \,] \\ &= [\, a_{m(\ell-n-1)+1}, a_{m(\ell-n-1)+2}, \dots, a_{m\ell-s-2} \,].\end{aligned}$$

From (3.7) we have

$$R_{\ell-n-2}^{(m)} = \frac{\beta_{\ell-n-1}^{(m)} p_{m(\ell-n-1)} + p_{m(\ell-n-1)-1}}{\beta_{\ell-n-1}^{(m)} q_{m(\ell-n-1)} + q_{m(\ell-n-1)-1}} = \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}.$$

□

Let us show how Theorem 1.1 can be applied. The first example shows palindromic situation, the second is not palindromic (but we accidentally get good approximation in the half of the period), and the third shows that good approximants do depend on m.

Example 3.5. Let us observe $\sqrt{44} = [\, 6, \overline{1, 1, 1, 2, 1, 1, 1, 12} \,]$. The period is palindromic and we have $\ell = 8$. Let us consider e.g. the case $m = 5$. From (2.5) we have

$$R_n^{(5)} = \frac{p_n^5 + 440 p_n^3 q_n^2 + 9680 p_n q_n^4}{5 p_n^4 q_n + 440 p_n^2 q_n^3 + 1936 q_n^5}.$$

From (1.2) we have

$$\begin{aligned}&R_3^{(5)} = \frac{p_{19}}{q_{19}} = \frac{3\,160\,100}{476\,403}, R_7^{(5)} = \frac{p_{39}}{q_{39}} = \frac{4\,993\,116\,004\,999}{752\,740\,560\,150}, \\ &R_{11}^{(5)} = \frac{p_{59}}{q_{59}}, R_{15}^{(5)} = \frac{p_{79}}{q_{79}}, \dots, R_{4k-1}^{(5)} = \frac{p_{20k-1}}{q_{20k-1}}.\end{aligned}$$

Since, $R_0^{(5)}=\frac{p_8}{q_8}=\frac{2514}{379}$. From Theorem 1.1 we have $R_6^{(5)}=\frac{p_{30}}{q_{30}}=\frac{7944493914}{1197677521}$, and also $R_{8k}^{(5)}=\frac{p_{40k+8}}{q_{40k+8}}$ and $R_{8k-2}^{(5)}=\frac{p_{40k-10}}{q_{40k-10}}$.

$R_1^{(5)}=\frac{235\,487}{35\,501}$ is not a convergent of $\sqrt{44}$, so neither $R_{8k+1}^{(5)}$ nor $R_{8k-3}^{(5)}$ will be.

$R_2^{(5)}=\frac{6\,251\,453}{942\,442}$ is not a convergent of $\sqrt{44}$, so neither $R_{8k+2}^{(5)}$ nor $R_{8k-4}^{(5)}$ will be.

Example 3.6. Let us observe $\alpha=\frac{5+\sqrt{21}}{3}=[9,\overline{5,6,1,2}]$ and $m=3$. From 2.5 we have $R_m^{(3)}=\frac{37p_n^3-4572p_nq_n^2+23368q_n^3}{81p_n^2q_n-1242p_nq_n^2+4824q_n^3}$.

We have $R_3^{(3)}=\frac{p_{11}}{q_{11}}=\frac{4\,4004\,659}{435\,564}$, and so $R_{4k-1}^{(3)}=\frac{p_{12k-1}}{q_{12k-1}}$. The period is not palindromic, and accidentally we have: $R_1^{(3)}=\frac{p_7}{q_7}=\frac{36\,409}{3960}$ (in palindromic case it would be $\frac{p_5}{q_5}$), and so $R_{4k+1}^{(3)}=\frac{p_{12k+7}}{q_{12k+7}}$.

Example 3.7. Let us observe $\alpha=\frac{7+\sqrt{11}}{5}=[2,\overline{15,1,3,1,3,1}]$. For $m=3$ we have $R_{6k-1}^{(3)}=\frac{p_{18k-1}}{q_{18k-1}}$; $R_1^{(3)}=\frac{p_7}{q_7}$ and $R_{6k+1}^{(3)}=\frac{p_{18k+7}}{q_{18k+7}}$.

For $m=4$ we have: $R_{6k-1}^{(4)}=\frac{p_{24k-1}}{q_{24k-1}}$; $R_0^{(4)}=\frac{p_5}{q_5}$ and $R_{6k}^{(4)}=\frac{p_{24k+5}}{q_{24k+5}}$; $R_1^{(4)}=\frac{p_{11}}{q_{11}}$ and $R_{6k+1}^{(4)}=\frac{p_{24k+11}}{q_{24k+11}}$; $R_3^{(4)}=\frac{p_{17}}{q_{17}}$ and $R_{6k+3}^{(4)}=\frac{p_{24k+17}}{q_{24k+17}}$.

4. Which convergents may appear?

From now on, let us observe only quadratic irrationals of the form $\alpha=\sqrt{d}$, $d\in\mathbb{N}$, d not a perfect square. It is well known that period of such α begins with a_1 and is palindromic.

Lemma 4.1. a) *$R_n^{(m)}<\sqrt{d}$ if and only if n is even and m is odd. Therefore, $R_n^{(m)}$ can be an even convergent only if n is even and m is odd.*

b)

$$\left|R_{n+1}^{(m)}-\sqrt{d}\right|<\left|R_n^{(m)}-\sqrt{d}\right|. \tag{4.1}$$

Proof. *a)* From (2.5) we have

$$R_n^{(m)}-\sqrt{d}=\frac{2\sqrt{d}(p_n-q_n\sqrt{d})^m}{(p_n+q_n\sqrt{d})^m-(p_n-q_n\sqrt{d})^m}. \tag{4.2}$$

On the right side of (4.2), denominator is always greater than 0, and nominator is less than 0 if and only if n is even (then we have $p_n-\sqrt{d}q_n<0$) and m is odd.

b) Let us observe (4.2). From $1\le p_0<p_1<p_2<\dots$ and $1\le q_0<q_1<q_2<\dots$ we have $2<p_0+\sqrt{d}q_0<p_1+\sqrt{d}q_1<\dots$. On the other hand, we have $1>|p_0-\sqrt{d}q_0|>|p_1-\sqrt{d}q_1|>\dots$ (see e.g. [12, §15]), so it holds (4.1). ∎

Let us observe the definition (1.3). The number $j_n^{(m)}$ is an integer, by Lemma 4.1 a). Using Theorem 1.1 we have

$$j_n^{(m)} = j_{k\ell+n}^{(m)}, \text{ and in palindromic case } j_n^{(m)} = -j_{\ell-n-2}^{(m)}. \tag{4.3}$$

PROPOSITION 4.2. *For $n \geq 0$ and $m \in \mathbb{N}$ we have*

$$\left|j_n^{(m)}(\sqrt{d})\right| < \frac{m(\ell/2-1)}{2}.$$

PROOF. Let $R_n^{(m)} = \frac{p_{m(n+1)+2j-1}}{q_{m(n+1)+2j-1}}$. According to (4.3), it suffices to consider the case $j > 0$ and $n < \ell$.

Assume first that ℓ is even, that is $\ell = 2t$. We have $R_{t-1}^{(m)} = \frac{p_{mt-1}}{q_{mt-1}}$ and $R_{\ell-1}^{(m)} = \frac{p_{m\ell-1}}{q_{m\ell-1}}$. For $n < t-1$, using (4.1) we have $m(n+1)+2j-1 < mt-1$, and $2j \leq m(t-1)-1$. For $n = t-1$ and $n = \ell-1$ we have $j = 0$, and for $t-1 < n < \ell-1$ we have $m(n+1)+2j-1 < m\ell-1$, or $2j < m\ell - m(n+1) \leq m(t-1)$, so again we get $j \leq \frac{m(\ell/2-1)-1}{2}$.

Let ℓ is odd, e.g. $\ell = 2t+1$. If for some $n, 0 \leq n < t$ holds $j \geq \frac{m(\ell/2-1)}{2}$, we would have $s := m(n+1)+2j-1 \geq m\ell/2-1$. By Theorem 1.1 it follows $R_{\ell-n-2}^{(m)} = \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}$, and $m\ell - s - 2 \leq m\ell/2 - 1$. Now it holds $\left|\sqrt{d} - \frac{p_s}{q_s}\right| \leq \left|\sqrt{d} - \frac{p_{m\ell-s-2}}{q_{m\ell-s-2}}\right|$, thus $\left|\sqrt{d} - R_n^{(m)}\right| \leq \left|\sqrt{d} - R_{\ell-n-2}^{(m)}\right|$. This is not possible by (4.1), since $\ell - n - 2 \geq t$. For $t-1 < n < \ell-1$, the proof is the same as in the even case. ∎

PROPOSITION 4.3. *Let $\ell \in \mathbb{N}$ and $a_1, \ldots, a_{\ell-1} \in \mathbb{N}$ such that $a_1 = a_{\ell-1}$, $a_2 = a_{\ell-2}$, ... The number $[\,a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0}\,]$ is of the form $\sqrt{d}$, $d \in \mathbb{N}$ if and only if*

$$2a_0 \equiv (-1)^{\ell-1} p'_{\ell-2} q'_{\ell-2} \pmod{p'_{\ell-1}}, \tag{4.4}$$

where $\frac{p'_n}{q'_n} = [\,a_1, a_2, \ldots, a_{n-1}, a_n\,]$. Then it holds

$$d = a_0^2 + \frac{2a_0 p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}. \tag{4.5}$$

PROOF. See [12, §26] ∎

LEMMA 4.4. *Let F_k denote the k-th Fibonacci number. Let $n \in \mathbb{N}$ and $k > 1, k \equiv 1, 2 \pmod 3$. For $d_k(n) = \left(\frac{(2n-1)F_k+1}{2}\right)^2 + (2n-1)F_{k-1} + 1$ it holds $\ell(\sqrt{d_k(n)}) = k$ and*

$$\sqrt{d_k(n)} = [\,\tfrac{(2n-1)F_k+1}{2}, \overline{\underbrace{1, 1, \ldots, 1, 1}_{k-1 \text{ times}}, (2n-1)F_k + 1}\,].$$

PROOF. From (4.4), it follows

$$2a_0 \equiv (-1)^{k-1} F_{k-1} F_{k-2} \equiv (-1)^{k-1} F_{k-1}(F_k - F_{k-1})$$

$$\equiv (-1)^{k-1}(-F_{k-1}^2) \pmod{F_k}.$$

Now from Cassini's identity $F_k F_{k-2} - F_{k-1}^2 = (-1)^{k-1}$ we have $2a_0 \equiv 1 \pmod{F_k}$. When $3 \mid k$, this congruence is not solvable, and if $3 \nmid k$, the solution is $a_0 \equiv \frac{F_k+1}{2} \pmod{F_k}$, i.e.,

$$a_0 = \frac{F_k+1}{2} + (n-1)F_k = \frac{(2n-1)F_k+1}{2}, \qquad n \in \mathbb{N}.$$

From (4.5) it follows

$$\begin{aligned} d &= \Big(\frac{(2n-1)F_k+1}{2}\Big)^2 + \frac{\big((2n-1)F_k+1\big)F_{k-1} + F_{k-2}}{F_k} \\ &= \Big(\frac{(2n-1)F_k+1}{2}\Big)^2 + (2n-1)F_{k-1} + 1. \end{aligned}$$

□

Remark 4.5. Periodic continued fractions involving Fibonacci numbers with all $a_i = 1$, $i = 1, \dots, \ell-1$ were known earlier. First example was shown in [16]. Construction of such examples, using Pellian equations was given in several papers by Mollin, see e.g. [11] and [10]. However, in all such examples, the numbers are of the form $\frac{1+\sqrt{d}}{2}$ with all $a_i = 1$ in symmetric part of the period, but for the numbers of the form $\sqrt{d}$ there is at least one $a_i \neq 1$. In [13, Lemma 5] we constructed, in a similar was as in Lemma 4.4, all numbers of the form $\frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, $d \equiv 1 \pmod 4$, with all $a_i = 1$ in symmetric part of the period. We have shown that all such numbers are of the form $\frac{1+\sqrt{d'_k(n)}}{2}$, where $d'_k(n) = 4\big((n \cdot F_k + 1)^2 + n \cdot F_{k-3}\big) + 1$, $k, n \in \mathbb{N}$ or $2n \in \mathbb{N}$ when $3 \mid k$. Some of those numbers was also given in [11, Example 5] $D_k(n) = 4F_{2k}^2 n^2 + (20F_k^2 + 8(-1)^k)n + 5$, i.e., it is not hard to show that it holds $D_k(n) = d'_{2k}(n)$. It turns out that for the numbers in Lemma 4.4 it holds $d_k(n) = \frac{1}{4}d'_k(\frac{2n-1}{2})$, and when $3 \nmid k$, continued fraction of $\sqrt{d_k(n)}$ have desired form (and there is no other number with such period).

Proof of Theorem 1.2. By (1.3), we have to prove

$$R_0^{(3M-1)} = \frac{p_{M\ell-2}}{q_{M\ell-2}}, \qquad R_0^{(3M)} = \frac{p_{M\ell-1}}{q_{M\ell-1}}, \qquad R_0^{(3M+1)} = \frac{p_{M\ell}}{q_{M\ell}}.$$

We have $a_0 = \frac{F_{\ell-3}F_\ell+1}{2}$, and since $3 \nmid \ell$, $F_{\ell-3}$ is odd, thus by Lemma 4.4 it holds

$$\sqrt{d_\ell} = [\,a_0, \overline{\underbrace{1, 1, \dots, 1, 1}_{\ell-1 \text{ times}}, 2a_0}\,].$$

From Cassini's identity, since ℓ is odd ($\ell \equiv \pm 1 \pmod 6$), it follows

$$\begin{aligned} 2a_0 &= F_{\ell-3}(F_{\ell-1} + F_{\ell-2}) + 1 = F_{\ell-2}^2 + F_{\ell-3}F_{\ell-2} = F_{\ell-1}F_{\ell-2}, \\ d - a_0^2 &= F_{\ell-3}F_{\ell-1} + 1 = F_{\ell-2}^2. \end{aligned}$$

So we get

$$
\begin{aligned}
R_0^{(1)} &= \frac{p_0}{q_0} = a_0, \\
R_0^{(2)} &= \frac{p_0^2 + dq_0^2}{2p_0q_0} = \frac{a_0^2 + d}{2a_0} = a_0 + \frac{d - a_0^2}{2a_0} = a_0 + \frac{F_{\ell-2}}{F_{\ell-1}} = \frac{p_{\ell-2}}{q_{\ell-2}}, \\
R_0^{(3)} &= \frac{p_0(p_0^2 + 3dq_0^2)}{q_0(3p_0^2 + dq_0^2)} = \frac{a_0(a_0^2 + 3d)}{3a_0^2 + d} = a_0 + \frac{2a_0(d - a_0^2)}{4a_0^2 + d - a_0^2} \\
&= a_0 + \frac{F_{\ell-1}F_{\ell-2}^3}{F_{\ell-1}^2 F_{\ell-2}^2 + F_{\ell-2}^2} = a_0 + \frac{F_{\ell-1}F_{\ell-2}}{F_{\ell-1}^2 + 1} = a_0 + \frac{F_{\ell-1}F_{\ell-2}}{F_{\ell-2}F_\ell} \\
&= a_0 + \frac{F_{\ell-1}}{F_\ell} = \frac{p_{\ell-1}}{q_{\ell-1}}.
\end{aligned}
\tag{4.6}
$$

Let us prove the theorem using induction on M. For proving the inductive step, first observe that from (2.6) for $m \geq 3$ we have

$$
R_k^{(m)} = \frac{R_k^{(2)} R_k^{(m-2)} + d}{R_k^{(2)} + R_k^{(m-2)}}, \qquad R_k^{(m)} = \frac{R_k^{(3)} R_k^{(m-3)} + d}{R_k^{(3)} + R_k^{(m-3)}}.
\tag{4.7}
$$

Suppose that for some $i \in \{0, \ell-2, \ell-1\}$ it holds $\frac{p_{(M-1)\ell+i}}{q_{(M-1)\ell+i}} = R_0^{(m-3)}$. We have

$$
\begin{aligned}
\frac{p_{M\ell+i}}{q_{M\ell+i}} &= \left[a_0, \underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, a_0 + \frac{p_{(M-1)\ell+i}}{q_{(M-1)\ell+i}} \right] \\
&= \left[a_0, \underbrace{1, 1, \ldots, 1, 1}_{\ell-1 \text{ times}}, a_0 + R_0^{(m-3)} \right] \\
&= \frac{p_{\ell-1}(a_0 + R_0^{(m-3)}) + p_{\ell-2}}{q_{\ell-1}(a_0 + R_0^{(m-3)}) + q_{\ell-2}} \overset{(3.4)}{\underset{(3.5)}{=}} \frac{p_{\ell-1}R_0^{(m-3)} + dq_{\ell-1}}{q_{\ell-1}R_0^{(m-3)} + p_{\ell-1}} \\
&\overset{(4.6)}{=} \frac{R_0^{(3)} R_0^{(m-3)} + d}{R_0^{(3)} + R_0^{(m-3)}} \overset{(4.7)}{=} R_0^{(m)}.
\end{aligned}
$$

□

Corollary 4.6. *Let $\ell(\sqrt{d}) = \ell$ be the length of the shortest period of the continued fraction expansion of $\sqrt{d}$. Then for each $m \geq 2$ it holds*

$$
\sup_{d,n} \left\{ \left| j_n^{(m)}(\sqrt{d}) \right| \right\} = +\infty, \qquad \limsup_{d,n} \left\{ \frac{\left| j_n^{(m)}(\sqrt{d}) \right|}{\ell(\sqrt{d})} \right\} \geq \frac{m}{6}.
$$

5. Number of good approximants

Analogously as in [2], let us define

$$b^{(m)}(\alpha) = \big|\{n : 0 \le n \le \ell - 1, R_n^{(m)} \text{ is a convergent of } \alpha\}\big|.$$

For arbitrary m experimental results suggest that similar properties could hold as for $m = 2$. However, there are some differences, as the following example shows.

Example 5.1. We have $\ell(\sqrt{45}) = 6$ and

$$b^{(m)}(\sqrt{45}) = \begin{cases} 4, & \text{if } m \equiv 2 \pmod 4, \\ 6, & \text{if } m \not\equiv 2 \pmod 4. \end{cases}$$

Proof. We have $\sqrt{45} = [6, \overline{1, 2, 2, 2, 1, 12}] = [\overline{6, 1, 2, 2, 2, 1, 6, 0}]$. We will denote convergents of regular expansion with $\frac{p_n}{q_n}$. Using (2.3) and (2.4), $R_2^{(m)} = \frac{p_{3m-1}}{q_{3m-1}}$ and $R_5^{(m)} = \frac{p_{6m-1}}{q_{6m-1}}$ are good approximants, and by Theorem 1.1, we only have to check $R_0^{(m)}$ and $R_1^{(m)}$. The first few convergents of $\sqrt{45}$ are sequentially $\frac{6}{1}$, $\frac{7}{1}$, $\frac{30}{3}$, $\frac{47}{7}$, $\frac{114}{17}$, $\frac{161}{24}$, and let us observe how other convergents look like ($[\dots]_M$ denote matrix form of convergents: $[a_0, a_1, \dots, a_n]_M = \left(\begin{smallmatrix} a_0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{smallmatrix}\right)$), and let us write $\left(\frac{7+\sqrt{45}}{2}\right)^{3k} = \gamma$.

$$\begin{pmatrix} p_{6k+1} & p_{6k} \\ q_{6k+1} & q_{6k} \end{pmatrix} = [6,1,2,2,2,1,6,0]_M^k[6,1]_M = \begin{pmatrix} 161 & 1080 \\ 24 & 161 \end{pmatrix}^k \begin{pmatrix} 7 & 6 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma\frac{7+\sqrt{45}}{2} + \gamma'\frac{7-\sqrt{45}}{2} & \gamma\frac{6+\sqrt{45}}{2} + \gamma'\frac{6-\sqrt{45}}{2} \\ \frac{\gamma\frac{7+\sqrt{45}}{2} - \gamma'\frac{7-\sqrt{45}}{2}}{\sqrt{45}} & \frac{\gamma\frac{6+\sqrt{45}}{2} - \gamma'\frac{6-\sqrt{45}}{2}}{\sqrt{45}} \end{pmatrix},$$

$$\begin{pmatrix} p_{6k+3} & p_{6k+2} \\ q_{6k+3} & q_{6k+2} \end{pmatrix} = \begin{pmatrix} \gamma\frac{47+7\sqrt{45}}{2} + \gamma'\frac{47-7\sqrt{45}}{2} & \gamma\frac{20+3\sqrt{45}}{2} + \gamma'\frac{20-3\sqrt{45}}{2} \\ \frac{\gamma\frac{47+7\sqrt{45}}{2} - \gamma'\frac{47-7\sqrt{45}}{2}}{\sqrt{45}} & \frac{\gamma\frac{20+3\sqrt{45}}{2} - \gamma'\frac{20-3\sqrt{45}}{2}}{\sqrt{45}} \end{pmatrix},$$

$$\begin{pmatrix} p_{6k+5} & p_{6k+4} \\ q_{6k+5} & q_{6k+4} \end{pmatrix} = \begin{pmatrix} \gamma\frac{161+24\sqrt{45}}{2} + \gamma'\frac{161-24\sqrt{45}}{2} & \gamma\frac{114+17\sqrt{45}}{2} + \gamma'\frac{114-17\sqrt{45}}{2} \\ \frac{\gamma\frac{161+24\sqrt{45}}{2} - \gamma'\frac{161-24\sqrt{45}}{2}}{\sqrt{45}} & \frac{\gamma\frac{114+17\sqrt{45}}{2} - \gamma'\frac{114-17\sqrt{45}}{2}}{\sqrt{45}} \end{pmatrix}.$$

From (2.5) we have $R_n^{(m)} = \frac{(p_n+q_n\sqrt{45})^m+(p_n-q_n\sqrt{45})^m}{(p_n+q_n\sqrt{45})^m-(p_n-q_n\sqrt{45})^m}\sqrt{45}$. We see now that $R_1^{(m)} = \frac{(7+\sqrt{45})^m+(7-\sqrt{45})^m}{(7+\sqrt{45})^m-(7-\sqrt{45})^m}\sqrt{45}$ is always a good approximant. Namely, from $(\frac{7\pm\sqrt{45}}{2})^2 = \frac{47\pm7\sqrt{45}}{2}$ and $(\frac{7\pm\sqrt{45}}{2})^3 = 161 \pm 24\sqrt{45}$, since $\frac{7}{1}$, $\frac{47}{7}$ and $\frac{161}{24}$ are convergents of $\sqrt{45}$, we have $R_1^{(m)} = \frac{p_{2m-1}}{q_{2m-1}}$.

Finally, let us see when $R_0^{(m)} = \frac{(6+\sqrt{45})^m+(6-\sqrt{45})^m}{(6+\sqrt{45})^m-(6-\sqrt{45})^m}\sqrt{45}$ is a convergent. First consider

$$\left(\frac{6\pm\sqrt{45}}{3}\right)^4 = 161 \pm 24\sqrt{45} = \left(\frac{7\pm\sqrt{45}}{2}\right)^3. \tag{5.1}$$

From (5.1) we see $R_0^{(4m)} = \frac{p_{6m-1}}{q_{6m-1}}$ and $R_0^{(4m+1)} = \frac{p_{6m}}{q_{6m}}$, and since $(6\pm\sqrt{45})^3 = 9(114\pm 17\sqrt{45})$ and $\frac{114}{17}$ is a convergent of $\sqrt{45}$, we have $R_0^{(4m+3)} = \frac{p_{6m+4}}{q_{6m+4}}$. From $(6\pm\sqrt{45})^2 = 3(27\pm 4\sqrt{45})$, and since $\frac{27}{4}$ is not a convergent of $\sqrt{45}$, neither $R_0^{(4m+2)}$ will be a convergent. □

Let us define

$$\ell_b^{(m)} = \min\{\ell : \text{there exists } d\in\mathbb{N} \text{ such that } \ell(\sqrt{d}) = \ell \text{ and } b^{(m)}(\sqrt{d}) = b\}.$$

In [3] Dujella and the author proved that $\sup\left\{\frac{\ell_b^{(2)}}{b} : b \geq 1\right\} \leq 2$, and in

b	$\ell_b^{(3)} \leq$	d	$\ell_b^{(3)}/b \leq$	b	$\ell_b^{(3)} \leq$	d	$\ell_b^{(3)}/b \leq$
3	5	13	1.666667	18	36	30420	2.941176
4	6	21	1.5	19	71	313157	3.736842
5	11	1625	2.2	20	44	193648	2.2
6	6	45	1.0	21	41	21125	1.952381
7	11	36125	1.571429	22	46	796500	2.0909091
8	12	558900	1.5	23	157	221425	6.826087
9	21	277	2.333333	24	66	740880	2.75
10	14	500	1.4	25	97	490625	3.88
11	37	828325	3.363636	26	50	29403	1.923077
12	20	2548	1.666667	27	113	460525	4.185185
13	45	74698	3.461538	28	78	84500	2.785714
14	28	10125	2.461538	29	171	535517	5.896552
15	41	9125	2.733333	30	80	41405	2.666667
16	28	1125	1.75	31	97	903125	3.129032
17	67	260389	3.941176	32	88	892125	2.75

TABLE 1. Upper bounds for $\ell_b^{(3)}$, $3 \leq b \leq 32$.

[13] the author showed the same inequality for $\alpha = \frac{1+\sqrt{d}}{2}$, $d \in \mathbb{N}$, $d \equiv 1 \pmod 4$ and d is not a perfect square. In Table 1 we show upper bounds for $\ell_b^{(3)}$, obtained by experiments, and corresponding d's (we tested all d's smaller then 10^6). Other experiments (we tested all m's until 20) give similar upper bounds, but $b^{(m)}(\sqrt{d})$ is not a monotonic function in m. Experimental results lead to the conclusion that for every positive integer $m \geq 3$ and every positive

integer b there exist a positive integer d such that $b^{(m)}(\sqrt{d}) = b$. Moreover, obtained upper bounds for $\frac{\ell_b^{(m)}}{b}$ suggest that

$$\sup\left\{\frac{\ell_b^{(m)}}{b} : b \geq 1\right\} \leq 2 \tag{5.2}$$

for all $m \geq 2$. In case $m = 2$ families of examples where constructed which show that for every positive integer b there exist a positive integer d such that $b^{(2)}(\sqrt{d}) = b$ and $b^{(m)}(\sqrt{d}) > \ell(\sqrt{d})/2$. To prove the inequality (5.2) for each $m \geq 3$ in a similar manner seems nearly impossible because $b^{(m)}(\sqrt{d})$ depends not only on d but also on m (see Example 5.1).

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