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THE ARAKAWA–KANEKO ZETA FUNCTION AND POLY-BERNOULLI POLYNOMIALS

Yoshinori Hamahata
Ritsumeikan University, Japan

Abstract. The purpose of this paper is to introduce a generalization of the Arakawa–Kaneko zeta function and investigate their special values at negative integers. The special values are written as the sums of products of Bernoulli and poly-Bernoulli polynomials. We establish the basic properties for this zeta function and their special values.

1. Introduction

Let $\mathrm{Li}_k(x)$ $(k \in \mathbb{Z})$ be the formal series defined by

$$\mathrm{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}. \tag{1.1}$$

The formal power series $\mathrm{Li}_k(x)$ is the k-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. When $k = 1$, we see easily that

$$\mathrm{Li}_1(x) = -\log(1-x). \tag{1.2}$$

The *Arakawa–Kaneko zeta function* $\xi_k(s, x)$, for $s \in \mathbb{C}$, $x > 0$, $k \in \mathbb{Z}$, is defined by

$$\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1} dt. \tag{1.3}$$

It is defined for $\mathrm{Re}(s) > 0$, $x > 0$ if $k \geq 1$, and for $\mathrm{Re}(s) > 0$, $x > |k|+1$ if $k < 0$. The function $\xi_k(s, x)$ is a generalization of the Hurwitz zeta function $\zeta(s, x)$ in that $\xi_1(s, x) = s\zeta(s, x)$. Especially, $\xi_k(s) := \xi_k(s, 1)$, which was defined by Arakawa and Kaneko [1], is a generalization of the Riemann zeta function

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$\zeta(s)$ in that $\xi_k(s) = s\zeta(s+1)$. In this paper, we introduce a generalization of $\xi_k(s,x)$ and investigate their special values at negative integers.

To consider the special values of $\xi_k(s,x)$, we recall Bernoulli and poly-Bernoulli polynomials. Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

These polynomials are related to special values of the Hurwitz zeta function. There exist some relations among Bernoulli polynomials. For instance, the following identity is known:

$$\sum_{n=0}^{n}\binom{n}{i}B_i(x)B_{n-i}(y) = n(x+y-1)B_{n-1}(x+y) - (n-1)B_n(x+y) \tag{1.4}$$

(see [6, (3.2)]). Dilcher [6] and Chen [4] gave a generalization of this identity for sums of products of Bernoulli polynomials given by

$$\sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\ldots,i_m\geq 0}}\binom{n}{i_1,\ldots,i_m}B_{i_1}(x_1)\cdots B_{i_{m-1}}(x_{m-1})B_{i_m}(x_m), \tag{1.5}$$

where

$$\binom{n}{i_1,\ldots,i_m} = \frac{n!}{i_1!\cdots i_m!}$$

are multinomial coefficients.

We next recall poly-Bernoulli polynomials introduced in [3, 5]. For every integer k, we define polynomials $B_n^{(k)}(x)$, which we call *poly-Bernoulli polynomials*, by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.6}$$

We remark that $B_n^{(k)}(x)$ are defined in [5] by replacing e^{xt} by e^{-xt} in the left-hand side of (1.5). By definition, it is easy to see that for any $n \geq 0$

$$B_n(x) = (-1)^n B_n^{(1)}(-x), \tag{1.7}$$

or equivalently

$$B_n^{(1)}(x) = B_n(x+1). \tag{1.8}$$

The numbers $B_n^{(k)} := B_n^{(k)}(0)$ are called *poly-Bernoulli numbers*. These numbers are introduced by Kaneko [8], and then investigated in [1, 2]. Poly-Bernoulli polynomials $B_n^{(k)}(x)$ were defined in [3] to generalize the properties of Bernoulli polynomials and poly-Bernoulli numbers.

The purpose of this paper is to introduce a generalization of $\xi_k(s,x)$ and describe the special values at negative integers in terms of

$$S_m^{(k)}(n;x)=\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\ldots,i_m\geq 0}}\binom{n}{i_1,\ldots,i_m}B_{i_1}(x_1)\cdots B_{i_{m-1}}(x_{m-1})B_{i_m}^{(k)}(x_m),$$

where $x=x_1+\cdots+x_n$. Since $S_m^{(k)}(n;x)$ is a generalization of (1.5), it is of interest to investigate $S_m^{(k)}(n;x)$. We show the outline of this paper. In Section 2, after some preparations of notations needed later, we present the basic properties for $S_m^{(k)}(n;x)$. In Sections 3, 4 and 5, the proofs of these results are given. In Section 6, we introduce a generalization of $\xi_k(s,x)$ and investigate the special values at negative integers in terms of $S_m^{(k)}(n;x)$.

2. Sums of products of Bernoulli and poly-Bernoulli polynomials

2.1. *Preliminaries.* Let $x_1,\ldots,x_m$ be variables and set $x=x_1+\cdots+x_m$. For $n\geq 0$, set

$$S_m^{(k)}(n;x)=\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\ldots,i_m\geq 0}}\binom{n}{i_1,\ldots,i_m}B_{i_1}(x_1)\cdots B_{i_{m-1}}(x_{m-1})B_{i_m}^{(k)}(x_m),$$

where $\binom{n}{i_1,\ldots,i_m}$ are multinomial coefficients defined by

$$\binom{n}{i_1,\ldots,i_m}=\frac{n!}{i_1!\cdots i_m!}.$$

When $m=1$, $S_m^{(k)}(n)$ is nothing other than $B_n^{(k)}(x)$. By definition, we have

$$\left(\frac{t}{e^t-1}\right)^{m-1}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}S_m^{(k)}(n;x)\frac{t^n}{n!}.$$

For $m\geq 1$, the *Stirling numbers of the first kind* $\begin{bmatrix}m\\l\end{bmatrix}$ are defined by

$$x(x+1)\cdots(x+m-1)=\sum_{l=0}^{m}\begin{bmatrix}m\\l\end{bmatrix}x^l$$

and $\begin{bmatrix}m\\l\end{bmatrix}=0$ for $l\geq m+1$ and $l\leq -1$. These numbers satisfy

$$\begin{bmatrix}m\\0\end{bmatrix}=0,\quad \begin{bmatrix}m\\m\end{bmatrix}=1\quad (m\geq 1), \tag{2.1}$$

$$\begin{bmatrix}m+1\\l\end{bmatrix}=\begin{bmatrix}m\\l-1\end{bmatrix}+m\begin{bmatrix}m\\l\end{bmatrix}\qquad (m\geq 1, l\in\mathbb{Z}). \tag{2.2}$$

Let us introduce a differential operator by

$$D(x)=\frac{d}{dt}-x.$$

For the exponential generating function $\sum_{n=0}^{\infty}a_nt^n/n!$ of a sequence $\{a_n\}$, it holds that

$$D(x)\left(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}(a_{n+1}-xa_n)\frac{t^n}{n!}.$$

Fix $k\in\mathbb{Z}$. We denote by P_k the set of $x^lB_n^{(k)}(x)$ $(l,n\geq 0)$. Let U and V be maps of P_k to itself with the conditions

$$U(x^lB_n^{(k)}(x))=x^lB_{n+1}^{(k)}(x),\quad V(x^lB_n^{(k)}(x))=x^{l+1}B_n^{(k)}(x).$$

2.2. *Results.* We are now ready to state our results.

THEOREM 2.1. *We have*

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix}m+1\\ l+1\end{bmatrix}S_{m+1}^{(k-l)}(n;x)$$
$$=\begin{cases}\dfrac{n!}{(n-m)!}\displaystyle\sum_{l=1}^{m}\begin{bmatrix}m+1\\ l+1\end{bmatrix}(U-V)^lB_{n-m}^{(k)}(x), & n\geq m\\ 0, & 0\leq n\leq m-1\end{cases}.$$

The proof is given in Section 3.

EXAMPLE 2.2.

$$-S_2^{(k)}(n;x)+S_2^{(k-1)}(n;x)=nB_n^{(k)}(x)-nxB_{n-1}^{(k)}(x)\quad(n\geq 1),$$
$$2S_3^{(k)}(n;x)-3S_3^{(k-1)}(n;x)+S_3^{(k-2)}(n;x)$$
$$=n(n-1)\left[B_n^{(k)}(x)+(3-2x)B_{n-1}^{(k)}(x)+(x^2-3x)B_{n-2}^{(k)}(x)\right]\quad(n\geq 2),$$
$$-6S_4^{(k)}(n;x)+11S_4^{(k-1)}(n;x)-6S_4^{(k-2)}(n;x)+S_4^{(k-3)}(n;x)$$
$$=n(n-1)(n-2)\Big[B_n^{(k)}(x)+(6-3x)B_{n-1}^{(k)}(x)$$
$$+(11-12x+3x^2)B_{n-2}^{(k)}(x)-(11x-6x^2+x^3)B_{n-3}^{(k)}(x)\Big]\quad(n\geq 3).$$

THEOREM 2.3. *For $k\geq 1$ and $n\geq 1$, we have*

$$S_2^{(0)}(n;x)=B_n^{(1)}(x),\tag{2.3}$$

$$S_2^{(k)}(n;x)=B_n(x+1)-n\sum_{j=1}^{k}B_n^{(j)}(x)+nx\sum_{j=1}^{k}B_{n-1}^{(j)}(x),\tag{2.4}$$

$$S_2^{(-k)}(n;x)=B_n(x+1)+n\sum_{j=0}^{k-1}B_n^{(-j)}(x)+nx\sum_{j=0}^{k-1}B_{n-1}^{(-j)}(x).\tag{2.5}$$

The proof is given in Section 4.

THEOREM 2.4. *For $k \geq 1$ and $n \geq 2$, we have*

$$S_3^{(0)}(n;x) = -(n-1)B_n^{(1)}(x) + nxB_{n-1}^{(1)}(x), \tag{2.6}$$

$$\begin{aligned} S_3^{(k)}(n;x) = {} & (2-2^{-k})[nxB_{n-1}(x) - (n-1)B_n(x)] \\ & - (1-2^{-k})[n(x+1)B_{n-1}(x+1) \\ & - (n-1)B_n(x+1)] \\ & + n(n-1)\sum_{j=1}^{k}(1-2^{j-k-1})\Big[B_n^{(j)}(x) \\ & + (1-2x)B_{n-1}^{(j)}(x) + (x^2-x)B_{n-2}^{(j)}(x)\Big], \end{aligned} \tag{2.7}$$

$$\begin{aligned} S_3^{(-k)}(n;x) = {} & (2-2^{k})\,[nxB_{n-1}(x) - (n-1)B_n(x)] \\ & - (1-2^{k})[n(x+1)B_{n-1}(x+1) \\ & - (n-1)B_n(x+1)] \\ & + n(n-1)\sum_{j=1}^{k-2}(2^{k-j-1}-1)\Big[B_n^{(-j)}(x) \\ & + (1-2x)B_{n-1}^{(-j)}(x) + (x^2-x)B_{n-2}^{(-j)}(x)\Big], \end{aligned} \tag{2.8}$$

where $S_3^{(-1)}(n;x) = n(x+1)B_{n-1}(x+1) - (n-1)B_n(x+1)$.

The proof is given in Section 5.

REMARK 2.5. It is difficult to find a formula for $S_m^{(k)}(n;x)$ when $m \geq 4$.

2.3. *Corollaries.* We present some results derived from Theorems 2.1, 2.3 and 2.4.

2.3.1. *Sums of products of Bernoulli and poly-Bernoulli numbers.* Let k be an integer. For $m \geq 1$ and $n \geq 0$, set

$$S_m^{(k)}(n) = \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\ldots,i_m \geq 0}} \binom{n}{i_1,\ldots,i_m} B_{i_1}\cdots B_{i_{m-1}}B_{i_m}^{(k)}.$$

When $m = 1$, $S_m^{(k)}(n)$ becomes $B_n^{(k)}$. By definition, we have

$$\left(\frac{t}{e^t-1}\right)^{m-1}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} S_m^{(k)}(n)\frac{t^n}{n!}.$$

Putting $x = 0$ in Theorem 2.1, we have the following theorem.

Theorem 2.6 (Kamano [7]). *For $k \in \mathbb{Z}$ and $m \geq 1$, we have*

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) = \begin{cases} n(n-1)\cdots(n-m+1)\displaystyle\sum_{l=1}^{m}\begin{bmatrix} m \\ l \end{bmatrix} B_{n-m+l}^{(k)}, & n \geq m \\ 0, & 0 \leq n \leq m-1 \end{cases}. \tag{2.9}$$

Putting $x = 0$ in Theorem 2.3, we obtain the following theorem.

Theorem 2.7 (Kamano [7]). *For $k \geq 1$ and $n \geq 1$, we have*

$$S_2^{(0)}(n) = B_n^{(1)}, \tag{2.10}$$

$$S_2^{(k)}(n) = B_n^{(1)} - n\sum_{j=1}^{k} B_n^{(j)}, \tag{2.11}$$

$$S_2^{(-k)}(n) = B_n^{(1)} + n\sum_{j=0}^{k-1} B_n^{(-j)}. \tag{2.12}$$

In Theorem 2.4, replacing $B_{n-1}(x+1)$ by $(-1)^{n-1}B_{n-1}(-x)$ and putting $x = 0$, we obtain the following theorem.

Theorem 2.8 (Kamano [7]). *For $k \geq 1$ and $n \geq 2$, we have*

$$S_3^{(0)}(n) = -(n-1)B_n, \tag{2.13}$$

$$\begin{aligned} S_3^{(k)}(n) &= n(1-2^{-k})(-1)^n B_{n-1} - (n-1)B_n \\ &\quad + n(n-1)\sum_{j=1}^{k}(1-2^{j-k-1})(B_n^{(j)} + B_{n-1}^{(j)}), \end{aligned} \tag{2.14}$$

$$\begin{aligned} S_3^{(-k)}(n) &= n(2^k-1)(-1)^{n-1} B_{n-1} - (n-1)B_n \\ &\quad + n(n-1)\sum_{j=0}^{k-2}(2^{k-j-1}-1)(B_n^{(-j)} + B_{n-1}^{(-j)}), \end{aligned} \tag{2.15}$$

where $S_3^{(-1)}(n) = n(-1)^{n-1}B_{n-1} - (n-1)B_n$.

Remark 2.9. In the proofs of Theorems 2.6, 2.7 and 2.8, the operator $D(0) = d/dt$ was used. On the other hand, in the proofs of Theorems 2.1, 2.3 and 2.4, $D(x) = d/dt - x$ will be used.

2.3.2. *Sums of products of Bernoulli polynomials.* The identity (2.4) for $k=1$ turns into

$$\sum_{i=0}^{n}\binom{n}{i}B_i(x_1)B_{n-i}(x_2+1) = n(x_1+x_2)B_{n-1}(x_1+x_2+1)-(n-1)B_n(x_1+x_2+1).$$

Putting $x=x_1$ and $y=x_2+1$, (1.4) is gained.

Similarly, using (2.7) for $k=1$ and $n\geq 2$, we obtain the following result:

$$\begin{aligned}\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\geq 0}}&\binom{n}{i_1,i_2,i_3}B_{i_1}(x_1)B_{i_2}(x_2)B_{i_3}(x_3)\\ &= \frac{n(n-1)}{2}(x-1)(x-2)B_{n-2}(x)\\ &\quad+\frac{n}{2}\left(3n-3-2nx+x\right)B_{n-1}(x)+\frac{3}{2}n(x-1)B_{n-1}(x-1)\\ &\quad+\frac{n^2-1}{2}B_n(x)-\frac{3}{2}(n-1)B_n(x-1).\end{aligned}$$

3. Proof of Theorem 2.1

Let $G_k(t,x)$ be the generating function of poly-Bernoulli polynomials of index k given by the left-hand side of (1.6). For example, we have

$$G_{-1}(t,x)=e^{(x+2)t},\quad G_0(t,x)=e^{(x+1)t},\quad G_1(t,x)=\frac{te^{(x+1)t}}{e^t-1}.$$

The following lemma is a key result in the proofs of Theorems 2.1, 2.3 and 2.4.

Lemma 3.1. *For $k\in\mathbb{Z}$, we have*

$$D(x)G_k(t,x)=\frac{1}{e^t-1}(G_{k-1}(t,x)-G_k(t,x)). \tag{3.1}$$

Proof. For $F_k(t)=\mathrm{Li}_k(1-e^{-t})/(1-e^{-t})$, Kamano [7] proved

$$\frac{d}{dt}F_k(t)=\frac{1}{e^t-1}\left(F_{k-1}(t)-F_k(t)\right).$$

From this, we deduce

$$\frac{d}{dt}G_k(t,x)=\frac{1}{e^t-1}\left(G_{k-1}(t,x)-G_k(t,x)\right)+xG_k(t,x).$$

∎

Let us generalize the lemma just proved.

Theorem 3.2. *For $k \in \mathbb{Z}$ and $m \geq 1$, it holds that*

$$\begin{aligned}(3.2)\qquad &\left(\begin{bmatrix} m \\ m \end{bmatrix} D(x)^m + \begin{bmatrix} m \\ m-1 \end{bmatrix} D(x)^{m-1} + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} D(x)\right) G_k(t,x) \\ &\quad = \frac{1}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x).\end{aligned}$$

Proof. We prove the theorem by induction on m. The case $m = 1$ follows from the lemma stated above. Assume that (3.2) holds for case m. By (2.2), the left-hand side of (3.2) for case $m+1$ is

$$\begin{aligned}&\left(\begin{bmatrix} m \\ m \end{bmatrix} D(x)^{m+1} + \begin{bmatrix} m \\ m-1 \end{bmatrix} D(x)^{m} + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} D(x)^2\right) G_k(t,x) \\ &\qquad + m\left(\begin{bmatrix} m \\ m \end{bmatrix} D(x)^m + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} D(x)\right) G_k(t,x).\end{aligned}$$

Thanks to the assumption for case m, this becomes

$$\begin{aligned}&D(x)\left(\frac{1}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x)\right) \\ &\qquad + \frac{m}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x).\end{aligned}$$

Applying Lemma 3.1 to the first term gives

$$\begin{aligned}&\frac{-me^t}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x) \\ &+ \frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} (G_{k-l-1}(t,x) - G_{k-l}(t,x)) \\ &+ \frac{m}{(e^t-1)^{m}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x) \\ &\quad = \frac{-m-1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x) \\ &\quad\quad + \frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l-1}(t,x) \\ &\quad = \frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m+1-l} (m+1) \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(t,x) \\ &\quad\quad + \frac{1}{(e^t-1)^{m+1}} \sum_{l=1}^{m+1} (-1)^{m+1-l} \begin{bmatrix} m+1 \\ l \end{bmatrix} G_{k-l}(t,x).\end{aligned}$$

Using $\begin{bmatrix} m+1 \\ m+2 \end{bmatrix} = \begin{bmatrix} m+1 \\ 0 \end{bmatrix} = 0$, the right-hand side turns into

$$\frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m+1} (-1)^{m+1-l} \left((m+1) \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} + \begin{bmatrix} m+1 \\ l \end{bmatrix} \right) G_{k-l}(t,x),$$

which yields the claim for case $m+1$. ∎

Let us return to the proof of Theorem 2.1. We see that

$$\begin{aligned} D(x)G_k(t,x) &= \sum_{n=1}^{\infty} B_n^{(k)}(x) \frac{t^{n-1}}{(n-1)!} - x \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (U-V) B_n^{(k)}(x) \frac{t^n}{n!}. \end{aligned}$$

Since $D(x)$ and $U-V$ are commutative, for $l \geq 0$ we obtain

$$D(x)^l G_k(t,x) = \sum_{n=0}^{\infty} (U-V)^l B_n^{(k)}(x) \frac{t^n}{n!}.$$

For $m \geq 0$, we have

$$t^m D(x)^l G_k(t,x) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} (U-V)^l B_{n-m}^{(k)}(x) \frac{t^n}{n!}.$$

Therefore the theorem follows from Theorem 3.2.

4. Proof of Theorem 2.3

Proof of (2.3): This case follows from $G_0(t,x) = e^{(x+1)t}$.

Proof of (2.4): By Lemma 3.1,

$$\frac{d}{dt} G_j(t,x) = \frac{1}{e^t-1} \left(G_{j-1}(t,x) - G_j(t,x) \right) + x G_j(t,x).$$

Summing over j from 1 to k, we have

$$\sum_{j=1}^{k} \frac{d}{dt} G_j(t,x) = \frac{1}{e^t-1} \left(G_0(t,x) - G_k(t,x) \right) + x \sum_{j=1}^{k} G_j(t,x),$$

or equivalently

$$\frac{1}{e^t-1} G_k(t,x) = \frac{e^{(x+1)t}}{e^t-1} - \sum_{j=1}^{k} \frac{d}{dt} G_j(t,x) + x \sum_{j=1}^{k} G_j(t,x).$$

Multiplying by t, we have

$$\frac{t}{e^t-1}G_k(t,x) = \sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!} - \sum_{n=1}^{\infty} n \sum_{j=1}^{k} B_n^{(k)}(x)\frac{t^n}{n!} + x\sum_{n=1}^{\infty} n \sum_{j=1}^{k} B_{n-1}^{(j)}(x)\frac{t^n}{n!},$$

which yields the result.

PROOF OF (2.5): By Lemma 3.1,

$$\frac{d}{dt}G_{-j}(t,x) = \frac{1}{e^t-1}\left(G_{-j-1}(t,x) - G_{-j}(t,x)\right) + xG_{-j}(t,x).$$

Summing over j from 0 to $k-1$, we get

$$\sum_{j=0}^{k-1}\frac{d}{dt}G_{-j}(t,x) = \frac{1}{e^t-1}\left(G_{-k}(t,x) - G_0(t,x)\right) + x\sum_{j=0}^{k-1}G_{-j}(t,x),$$

or equivalently

$$\frac{1}{e^t-1}G_{-k}(t,x) = \frac{e^{(x+1)t}}{e^t-1} + \sum_{j=0}^{k-1}\frac{d}{dt}G_{-j}(t,x) - x\sum_{j=0}^{k-1}G_{-j}(t,x).$$

Multiplying both sides of this identity by t, we have the result.

5. PROOF OF THEOREM 2.4

PROOF OF (2.6): Using $G_0(t,x) = e^{(x+1)t}$ and $G_1(t,x) = te^{(x+1)t}/(e^t-1)$,

$$\frac{t^2}{(e^t-1)^2}G_0(t,x) = \frac{t}{e^t-1}G_1(t,x),$$

which implies

$$S_3^{(0)}(n,x) = S_2^{(1)}(n;x) = B_n(x+1) - nB_n^{(1)}(x) + nxB_{n-1}^{(1)}(x).$$

By (1.8), we obtain (2.6).

PROOF OF (2.7): By Theorem 3.2 for $m=2$, we have

$$(D(x)^2 + D(x))G_j(t,x) = \frac{1}{(e^t-1)^2}\left[(2G_j(t,x) - G_{j-1}(t,x)) - (2G_{j-1}(t,x) - G_{j-2}(t,x))\right].$$

Summing over j from 1 to l,

$$\frac{2G_l(t,x)}{(e^t-1)^2} - \frac{G_{l-1}(t,x)}{(e^t-1)^2} = \frac{2G_0(t,x) - G_{-1}(t,x)}{(e^t-1)^2} + \sum_{j=1}^{l}(D(x)^2 + D(x))G_j(t,x).$$

Multiplying both sides by 2^{l-1} and summing over l from 1 to k, we have

$$\begin{aligned}\frac{2^k G_k(t,x)}{(e^t-1)^2}-\frac{G_0(t,x)}{(e^t-1)^2} &= \left(\sum_{l=1}^{k}2^{l-1}\right)\frac{2G_0(t,x)-G_{-1}(t,x)}{(e^t-1)^2}\\ &\quad+\sum_{l=1}^{k}2^{l-1}\sum_{j=1}^{l}(D(x)^2+D(x))G_j(t,x)\\ &=\frac{(2^{k+1}-2)G_0(t,x)}{(e^t-1)^2}-\frac{(2^k-1)G_{-1}(t,x)}{(e^t-1)^2}\\ &\quad+\sum_{j=1}^{k}\sum_{l=j}^{k}2^{l-1}(D(x)^2+D(x))G_j(t,x).\end{aligned}$$

Using $G_0(t,x)=e^{(x+1)t}$ and $G_{-1}(t,x)=e^{(x+2)t}$, it holds that

$$\begin{aligned}\frac{G_k(t,x)}{(e^t-1)^2} &= (2-2^{-k})\frac{e^{(x+1)t}}{(e^t-1)^2}-(1-2^{-k})\frac{e^{(x+2)t}}{(e^t-1)^2}\\ &\quad+\sum_{j=1}^{k}(1-2^{j-k-1})(D(x)^2+D(x))G_j(t,x).\end{aligned}$$

We multiply both sides by t^2, and calculate each term of the right-hand side:

$$\frac{t^2e^{(x+1)t}}{(e^t-1)^2}=1+\sum_{n=1}^{\infty}\left(nxB_{n-1}(x)-(n-1)B_n(x)\right)\frac{t^n}{n!}, \tag{5.1}$$

$$\frac{t^2e^{(x+2)t}}{(e^t-1)^2}=1+\sum_{n=1}^{\infty}\left(n(x+1)B_{n-1}(x+1)-(n-1)B_n(x+1)\right)\frac{t^n}{n!}, \tag{5.2}$$

$$\begin{aligned}&t^2D(x)^2G_j(t,x)\\ &\quad=\sum_{n=2}^{\infty}n(n-1)\left(B_n^{(j)}(x)-2xB_{n-1}^{(j)}(x)+x^2B_{n-2}^{(j)}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{5.3}$$

$$t^2D(x)G_j(t,x)=\sum_{n=2}^{\infty}n(n-1)\left(B_{n-1}^{(j)}(x)-xB_{n-2}^{(j)}(x)\right)\frac{t^n}{n!}. \tag{5.4}$$

Here (5.1) follows from

$$\frac{t^2e^{(x+1)t}}{(e^t-1)^2}=\frac{te^{xt}}{e^t-1}+xt\cdot\frac{te^{xt}}{e^t-1}-t\frac{d}{dt}\left(\frac{te^{xt}}{e^t-1}\right).$$

From these, when $n\geq 2$, we obtain (2.7).

Proof of (2.8): Using Theorem 3.2 for $m = 2$, we have

$$(D(x)^2 + D(x))G_{-j}(t,x) = \frac{1}{(e^t-1)^2}\left[(2G_{-j}(t,x) - G_{-j-1}(t,x)) - (2G_{-j-1}(t,x) - G_{-j-2}(t,x))\right].$$

Summing over j from 0 to l,

$$\frac{2G_{-l-1}(t,x)}{(e^t-1)^2} - \frac{G_{-l-2}(t,x)}{(e^t-1)^2} = \frac{2G_0(t,x) - G_{-1}(t,x)}{(e^t-1)^2} - \sum_{j=0}^{l}(D(x)^2 + D(x))G_{-j}(t,x).$$

Multiplying both sides by 2^{-l} and summing over l from 0 to $k-2$, we have

$$\begin{aligned}
&\frac{2G_{-1}(t,x)}{(e^t-1)^2} - \frac{2^{-k+2}G_{-k}(t,x)}{(e^t-1)^2}\\
&= \left(\sum_{l=0}^{k-2} 2^{-l}\right)\frac{2G_0(t,x) - G_{-1}(t,x)}{(e^t-1)^2} - \sum_{l=0}^{k-2} 2^{-l}\sum_{j=0}^{l}(D(x)^2 + D(x))G_{-j}(t,x)\\
&= \frac{2^2(1-2^{-k+1})G_0(t,x)}{(e^t-1)^2} - \frac{2(1-2^{-k+1})G_{-1}(t,x)}{(e^t-1)^2}\\
&\quad - \sum_{j=0}^{k-2}\sum_{l=j}^{k-2} 2^{-l}(D(x)^2 + D(x))G_{-j}(t,x).
\end{aligned}$$

Using $G_0(t,x) = e^{(x+1)t}$ and $G_{-1}(t,x) = e^{(x+2)t}$, we obtain

$$\frac{G_{-k}(t,x)}{(e^t-1)^2} = (2-2^k)\frac{e^{(x+1)t}}{(e^t-1)^2} - (1-2^k)\frac{e^{(x+2)t}}{(e^t-1)^2} + \sum_{j=0}^{k-2}(2^{k-j-1}-1)(D(x)^2 + D(x))G_{-j}(t,x).$$

We multiply both sides by t^2, and calculate each term of the right-hand side: above all, we see easily that

$$\begin{aligned}
&t^2 D(x)^2 G_{-j}(t,x)\\
&= \sum_{n=2}^{\infty} n(n-1)\left(B_n^{(-j)}(x) - 2xB_{n-1}^{(-j)}(x) + x^2 B_{n-2}^{(-j)}(x)\right)\frac{t^n}{n!},
\end{aligned} \tag{5.5}$$

$$t^2 D(x) G_{-j}(t,x) = \sum_{n=2}^{\infty} n(n-1)\left(B_{n-1}^{(-j)}(x) - xB_{n-2}^{(-j)}(x)\right)\frac{t^n}{n!}. \tag{5.6}$$

Combining (5.1), (5.2), (5.5), (5.6) with right-hand side of $G_{-k}(t,x)/(e^t-1)^2$, the identity (2.8) can be established for $n \geq 2$.

6. A generalization of the Arakawa–Kaneko zeta function

Let k be an integer and m be a positive integer. We introduce zeta functions by means of the Laplace–Mellin integral.

Definition 6.1. *For $k \in \mathbb{Z}$ and $m \geq 1$, define*

$$\xi_{k,m}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{t}{e^t-1}\right)^{m-1} \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1} dt,$$

$$\xi_{k,m}(s) = \xi_{k,m}(s,1) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{t}{e^t-1}\right)^{m-1} \frac{Li_k(1-e^{-t})}{e^t-1} t^{s-1} dt.$$

The zeta function $\xi_{k,m}(s,x)$ is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k|+1$ if $k \leq 0$. Hence $\xi_{k,m}(s)$ is defined for $\mathrm{Re}(s) > 0$ and $k \geq 1$. It should be noted that $\xi_{k,1}(s,x)$ is just the zeta function $\xi_k(s,x)$, and $\xi_{k,1}(s)$ is the zeta function $\xi_k(s)$ defined in Section 1.

Theorem 6.2. *When $k \geq 1$ (resp. $k \leq 0$), suppose $x > 0$ (resp. $x > |k|+1$). Then the function $s \mapsto \xi_{k,m}(s,x)$ can be analytically continued to the whole complex s-plane as an entire function and its values at negative integers are given by*

$$\xi_{k,m}(-n,x) = (-1)^n S_m^{(k)}(n;-x) \qquad (n = 1,2,3,\ldots).$$

Proof. We express $\xi_{k,m}(s,x)$ as the sum of two integrals:

$$\xi_{k,m}(s,x) = \frac{1}{\Gamma(s)} \int_0^1 \left(\frac{t}{e^t-1}\right)^{m-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^\infty \left(\frac{t}{e^t-1}\right)^{m-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1} dt.$$

For any $s \in \mathbb{C}$, the second integral converges absolutely and the second term on the right-hand side becomes zero thanks to $\Gamma(s)^{-1}$. If $\mathrm{Re}(s) > 0$, then the first term on the right-hand side is written as

$$\frac{1}{\Gamma(s)} \sum_{i=0}^\infty \frac{S_m^{(k)}(i;-x)}{i!} \cdot \frac{1}{i+s}.$$

From this, for a non-negative integer n, we get

$$\xi_{k,m}(-n,x) = \left(\lim_{s\to -n} \frac{1}{\Gamma(s)(n+s)}\right) \frac{S_m^{(k)}(n;-x)}{n!} = (-1)^n S_m^{(k)}(n;-x).$$

□

Letting $x = 1$ in Theorem 6.2, we get an extension of a result by Arakawa–Kaneko (see [1, Theorem 6 (i)]).

Theorem 6.3. *Assume $k \geq 1$ and $x > 0$. Then the function $s \mapsto \xi_{k,m}(s,x)$ can be analytically continued to the whole complex s-plane as an entire function and its values at negative integers are given by*

$$\xi_{k,m}(-n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} S_m^{(k)}(r) \qquad (n = 1, 2, 3, \ldots).$$

Proof. The part of analytic continuation of the zeta function follows from the last theorem. Noting the generating function of $S_m^{(k)}(n; -1)$ is

$$\left(\sum_{n=0}^{\infty} S_m^{(k)}(n) \frac{t^n}{n!} \right) e^{-t} = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} S_m^{(k)}(r) \right) \frac{t^n}{n!},$$

we have the result about special values from the last theorem. ∎

We conclude this section by giving a few identities for $\xi_{k,m}(s,x)$.

Theorem 6.4 (Difference identity). *Let $m \geq 2$. With the hypothesis of Theorem 6.2, we have*

$$\xi_{k,m}(s, x+1) - \xi_{k,m}(s,x) = -s\xi_{k,m-1}(s+1, x). \tag{6.1}$$

Proof. The left-hand side becomes

$$-\frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{t}{e^t - 1} \right)^{m-1} \frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{-xt} t^{s+1} dt,$$

which is the right-hand side. ∎

Putting $s = -n$ in (6.1), we obtain the following.

Corollary 6.5. *Let $m \geq 2$. With the hypothesis of Theorem 6.2, we have*

$$S_m^{(k)}(n; -x-1) - S_m^{(k)}(n; -x) = -nS_{m-1}^{(k)}(n-1; -x) \quad (n = 1, 2, 3, \ldots).$$

Theorem 6.6 (Raabe's identity). *Let $m \geq 2$. With the hypothesis of Theorem 6.2, we have*

$$\int_0^1 \xi_{k,m}(s, x+w) dw = \xi_{k,m-1}(s, x+1), \tag{6.2}$$

$$\int_0^1 S_m^{(k)}(n; -x-w) dw = S_{m-1}^{(k)}(n; -x-1) \qquad (n = 1, 2, 3, \ldots). \tag{6.3}$$

Proof. As for (6.2), the left-hand side is

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{t}{e^t - 1} \right)^{m-1} \frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} \left(\int_0^1 e^{-(x+w)t} dw \right) t^{s-1} dt,$$

which is equal to $\xi_{k,m}(s, x+1)$.

As for (6.3), it is sufficient to combine Theorem 6.2 and (6.2) for $s = -n$. ∎

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Y. Hamahata
Institute for Teaching and Learning
Ritsumeikan University
1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577
Japan
E-mail: hamahata@fc.ritsumei.ac.jp

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