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BIRATIONAL MAPS OF $X(1)$ INTO $\mathbb{P}^2$

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Abstract. In this paper we study birational maps of modular curve $X(1)$ attached to $SL_2(\mathbb{Z})$ into the projective plain $\mathbb{P}^2$. We prove that every curve of genus 0 and degree q in $\mathbb{P}^2$ can be uniformized by modular forms for $SL_2(\mathbb{Z})$ of weight $12q$ but not with modular forms of smaller weight, and that the corresponding uniformization can be chosen to be a birational equivalence. We study other regular maps $X(1) \longrightarrow \mathbb{P}^2$ and we compute the equation of obtained projective curve. We provide numerical examples in SAGE.

1. Introduction

The idea of using automorphic forms and uniformization theory (via Poincaré series) to construct holomorphic maps on compact Riemann surfaces is very old one ([12]). Regarding modular curves, the uniformization of the modular curves via theta functions has been studied extensively for example in [3–5, 7]. Furthermore, arithmetic aspects of the theory can be found in a well–known book of Shimura ([13]). The uniformization of modular curves is used to compute equations of modular curves $X_0(N)$ in [2, 6, 14, 16].

The usual plane models of curves $X_0(N)$ are derived using classical modular j–function [13] but the equations of obtained curves are rather difficult ([1]). Thus, it is a reasonable problem to search for other plane models ([11]). A related question is what kind of loci we can get when we uniformize with modular forms on $\Gamma_0(N)$ of a particular even weight $m \geq 2$. In general, this is rather messy ([11]) but in the case of the modular curve $X(1) = X_0(1)$ (which is a modular curve for $\Gamma_0(1) = SL_2(\mathbb{Z})$) this has a complete and satisfactory answer which is given in the present paper.

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Before we introduce our main result, we introduce some notation. Let M_m be the space of all modular forms of weight m for $SL_2(\mathbb{Z})$. We introduce the two Eisenstein series

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

of weight 4 and 6, where $q = \exp(2\pi i z)$. Then, for any $m \geq 4$, we have

$$M_m = \oplus_{\substack{\alpha,\beta \geq 0 \\ 4\alpha + 6\beta = m}} \mathbb{C} E_4^{\alpha} E_6^{\beta}. \tag{1.1}$$

Let $\mathbb{H}$ be the upper half–plane and let

$$X(1) = (\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}) / SL_2(\mathbb{Z})$$

be the corresponding modular curve. The curve $X(1)$ has genus zero. $X(1)$ is isomorphic to $\mathbb{P}^1$ via modular j–function.

Every irreducible complex projective curve is birationally equivalent to a plane curve. We say that an irreducible curve $\mathcal{C} \subset \mathbb{P}^2$ is uniformized by modular forms of weight m if there exists $f, g, h \in M_m$ such that $\mathcal{C}$ is the image of the holomorphic map $X(1) \longrightarrow \mathbb{P}^2$ given by

$$z \longmapsto (f(z) : g(z) : h(z)). \tag{1.2}$$

This forces that $\mathcal{C}$ has genus 0 (see Lemma 2.2). The main result of the present paper is the following theorem:

THEOREM 1.1. *Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible curve of degree q and genus 0. Then, $\mathcal{C}$ can be uniformized by the modular forms of weight $12q$ but not with modular forms of smaller weight. More precisely, the uniformization map by modular forms of weight $12q$ can be selected to be a birational equivalence.*

Theorem 1.1 is proved in Section 2. In Section 3 we give examples of uniformization for various classes of curves of genus 0.

In the spirit of [6], it is reasonable to study the following problem. Given three linearly independent modular forms $f, g, h \in M_m$, we construct the map (1.2). Then, it is reasonable to compute the (reduced) equation of the curve (of genus zero) obtained as the image of the map (1.2). We discuss these questions in Section 3 where we take f, g, h from the canonical bases of M_m (see Proposition 3.1), and in Section 4 where we explore the computational aspects of the problem using SAGE (see [15]).

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2. Proof of Theorem 1.1

In the proof we use standard results about complex algebraic curves ([9]). We begin the proof of Theorem 1.1 by recalling the notion of the divisor of a modular form (in the settings of $SL_2(\mathbb{Z})$) from [8, 2.3].

Let $m \geq 4$ be an even integer and $f \in M_m - \{0\}$. Then $\nu_{z-\xi}(f)$ is the order of the holomorphic function f at ξ. The number is constant on the $SL_2(\mathbb{Z})$–orbit of ξ.

The point ξ is elliptic if the stabilizer $SL_2(\mathbb{Z})_\xi$ in $SL_2(\mathbb{Z})$ when divided by $\{\pm 1\}$ is not trivial. In any case, we let

$$e_\xi = \#\left(SL_2(\mathbb{Z})_\xi/\{\pm 1\}\right).$$

So, ξ is elliptic if and only if $e_\xi > 1$. We define

$$\nu_\xi(f) = \nu_{z-\xi}(f)/e_\xi.$$

The numbers e_ξ and $\nu_\xi(f)$ depend only on the $SL_2(\mathbb{Z})$–orbit of ξ. Thus, if $\mathfrak{a}_\xi$ is a projection of ξ to $X(1)$, we may let

$$\nu_{\mathfrak{a}_\xi}(f) = \nu_\xi(f).$$

There are just two orbits of elliptic points in $SL_2(\mathbb{Z})$: i and $e^{\pi i/3}$. We have $e_i = 2$ and $e_{e^{\pi i/3}} = 3$.

The cusps for $SL_2(\mathbb{Z})$ are $\mathbb{Q} \cup \{i\infty\}$. They form a single orbit. We define $\nu_{i\infty}(f)$ by using the Fourier expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

We let

$$\nu_{i\infty}(f) = N \geq 0,$$

where N is defined by $a_0 = a_1 = \cdots = a_{N-1} = 0$, $a_N \neq 0$. It is more technical to define $\nu_x(f)$ for $x \in \mathbb{Q}$ but it turns out that $\nu_x(f)$ does not depend on $x \in \mathbb{Q} \cup \{i\infty\}$.

Finally we define the divisor of f as follows:

$$\mathrm{div}(f) = \sum_{\mathfrak{a} \in X(1)} \nu_{\mathfrak{a}}(f)\mathfrak{a}.$$

This is a divisor with rational coefficient on the Riemann surface $X(1)$.

Using [8, 2.3], this sum is finite i.e., $\nu_{\mathfrak{a}}(f) \neq 0$ only for finitely many points. We let

$$\deg(\mathrm{div}(f)) = \sum_{\mathfrak{a} \in X(1)} \nu_{\mathfrak{a}}(f).$$

The particular case of [8, Theorem 2.3.3] is the following relation:

$$\deg(\mathrm{div}(f)) = \frac{m}{12}.$$

As in the proof [10, Lemma 4-1 (vi)] we prove that

LEMMA 2.1. *Assume that* $m \geq 12$ *is an even integer and* $f \in M_m$, $f \neq 0$. *Then there exists an integral effective divisor* $\mathfrak{c}_f \geq 0$ *of degree* $\dim M_m - 1$ *such that*

$$div(f) = \mathfrak{c}_f + \left(\frac{m}{4} - \left[\frac{m}{4}\right]\right)\mathfrak{a}_i + \left(\frac{m}{3} - \left[\frac{m}{3}\right]\right)\mathfrak{a}_{e^{\pi i/3}}.$$

Now, we begin the proof of Theorem 1.1. The first step in the proof of Theorem 1.1 is the following lemma:

LEMMA 2.2. *Assume that* $m \geq 12$ *is an even integer such that* $\dim M_m \geq 3$. *Let* $f, g, h \in M_m$ *be linearly independent. Then, the image of the map* $X(1) \to \mathbb{P}^2$ *given by* [1]

$$\mathfrak{a}_z \longmapsto (f(z) : g(z) : h(z))$$

is an irreducible projective curve of genus 0 *which we denote by* $\mathcal{C}(f, g, h)$. *Furthermore, the degree of* $\mathcal{C}(f, g, h)$ *is* $\leq \dim M_m - 1$ *but* > 1.

PROOF. $X(1)$ has a canonical structure of a smooth projective curve, and f/h and g/h are rational functions on $X(1)$. Thus, $\mathfrak{a}_z \longmapsto (f(z) : g(z) : h(z))$ is the rational map

$$\mathfrak{a}_z \longmapsto (f(z)/h(z) : g(z)/h(z) : 1).$$

But since $X(1)$ is smooth, this map is regular. Consequently, the image, which is our $\mathcal{C}(f, g, h)$, is an irreducible projective curve. By definition, the genus of $\mathcal{C}(f, g, h)$ is the genus of its desingularization (normalization), say $\mathcal{C}$. Thus, there exists a rational map $\varphi : \mathcal{C}(f, g, h) \longrightarrow \mathcal{C}$ which is a birational equivalence. This implies that the composition

$$X(1) \xrightarrow{\mathfrak{a}_z \longmapsto (f(z):g(z):h(z))} \mathcal{C}(f, g, h) \xrightarrow{\ \varphi\ } \mathcal{C}$$

is a non–constant rational map. Hence, the composition is regular and surjective. By Hurwitz's formula, the genus of $\mathcal{C}$ is less than or equal to the genus of $X(1)$. This implies that the genus of $\mathcal{C}$ is 0. Thus, $\mathcal{C}(f, g, h)$ has genus 0.

We prove the last claim of the lemma. Let us write $(x_0 : x_1 : x_2)$ for homogeneous coordinates on $\mathbb{P}^2$. Since f, g and h are linearly independent, the degree of $\mathcal{C}(f, g, h)$ cannot be one. Let us show that it is $\leq \dim M_m - 1$. Let l be the line in $\mathbb{P}^2$ in general position with respect to $\mathcal{C}(f, g, h)$. Then, it intersects $\mathcal{C}(f, g, h)$ in different points a number of which is the degree of $\mathcal{C}(f, g, h)$. We can change the coordinate system so that the line l is $x_0 = 0$. In new coordinate system, the map $\mathfrak{a}_z \mapsto (f(z) : g(z) : h(z))$ is of the form

$$\mathfrak{a}_z \mapsto (F(z) : G(z) : H(z)),$$

[1] In this paper $\mathfrak{a}_z$ denotes the projection to $X(1)$ of the point $z \in \mathbb{H}$, and $\mathfrak{a}_x$ denotes the projection to $X(1)$ of the cusp $x \in \mathbb{Q} \cup \{i\infty\}$.

where $F, G, H \in M_m$ are again linearly independent. In particular, $F, G, H \neq 0$.

We write this map in the form of a regular map $X(1) \longrightarrow \mathcal{C}(F, G, H)$

$$\mathfrak{a}_z \mapsto (1 : G(z)/F(z) : H(z)/F(z)) . \tag{2.1}$$

Thanks to Lemma 2.1, the divisors of rational functions F/H and G/H are easy to compute. We obtain

$$\begin{aligned} \operatorname{div}(G/F) &= \operatorname{div}(G) - \operatorname{div}(F) = \mathfrak{c}_G - \mathfrak{c}_F \\ \operatorname{div}(H/F) &= \operatorname{div}(H) - \operatorname{div}(F) = \mathfrak{c}_H - \mathfrak{c}_F, \end{aligned} \tag{2.2}$$

where the divisors $\mathfrak{c}_F$, $\mathfrak{c}_G$, and $\mathfrak{c}_H$ are integral effective divisors of degree $\dim M_m - 1$.

Now, we intersect $\mathcal{C}(f, g, h)$ with the line $x_0 = 0$. Considering the map in the form (2.1), the intersection is determined by the poles of G/F and H/F. Since all divisors in (2.2) are effective, the poles of G/F and H/F are contained among the points in the support of $\mathfrak{c}_F$. The claim follows at once since the support of $\mathfrak{c}_F$ cannot have more than $\dim M_m - 1$ points because $\mathfrak{c}_F$ is effective and it has degree $\dim M_m - 1$. ∎

LEMMA 2.3. *Assume that $m \geq 12$ is an even integer such that $\dim M_m \geq 2$. Let $f, g, h \in M_m$ such that two of them are linearly independent but not all three. Then, the image of the map $X(1) \to \mathbb{P}^2$ given by*

$$\mathfrak{a}_z \longmapsto (f(z) : g(z) : h(z))$$

is a line.

PROOF. If for example f and g are linearly independent, then the map $X(1) \longrightarrow \mathbb{P}^1$ given by f/g is non–constant, and therefore surjective. In another words, the map $\mathfrak{a}_z \longmapsto (f(z) : g(z))$ is surjective. Since h is by the assumption a linear combination of f and g, $h = \lambda f + \mu g$, the claim follows from the fact that the map can be factored as follows:

$$X(1) \xrightarrow{\mathfrak{a}_z \mapsto (f(z):g(z))} \mathbb{P}^1 \xrightarrow{(s:t)\mapsto(s:t:\lambda s+\mu t)} \mathbb{P}^2.$$

∎

COROLLARY 2.4. *Under the assumptions of either Lemma 2.2 or Lemma 2.3, there exists unique up to a scalar homogeneous polynomial $P = P_{f,g,h}$ in three variables of degree $\leq \dim M_m - 1$ such that the locus $(P(x_0, x_1, x_2) = 0)$ is $\mathcal{C}(f, g, h)$.*

PROOF. This follows from Nullstellensatz. We remind the reader that the degree of P equals the degree of $\mathcal{C}(f, g, h)$. ∎

The critical step in the proof of Theorem 1.1 is the following lemma:

LEMMA 2.5. *Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible curve of degree $q \geq 1$ and genus 0. Then, $\mathcal{C}$ cannot be uniformized by modular forms of weight $< 12q$.*

PROOF. We recall that

$$\dim M_m = \begin{cases} \left[\frac{m}{12}\right] & \text{if } m \equiv 2 \ (mod\ 12) \\ \left[\frac{m}{12}\right] + 1 & \text{if } m \not\equiv 2 \ (mod\ 12), \end{cases}$$

where $[x]$ denotes the largest integer $\leq x$. From this we see that if $m < 12q$, then

$$\dim M_m < \dim M_{12q} = q + 1.$$

Thus, if $\dim M_m \geq 2$ (to assure that normalization is possible at all), then, by Lemmas 2.2 and 2.3, it can uniformize the curves of degree $\leq \dim M_m - 1 < q$. $\square$

The following lemma completes the proof of Theorem 1.1:

LEMMA 2.6. *Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible curve of degree $q \geq 1$ and genus* 0*. Then, $\mathcal{C}$ can be uniformized by modular forms of weight* $12q$ *such that the corresponding map is a birational equivalence.*

PROOF. Since $\mathcal{C}$ has a genus 0, there exists a birational map

$$\mathbb{P}^1 \longrightarrow \mathcal{C}.$$

This map is necessary of the form

$$(s:t) \longmapsto (\alpha(s,t) : \beta(s,t) : \gamma(s,t)),$$

where α, β, and γ are homogeneous polynomials in two variables of the same degree of homogeneity.

Since $\mathcal{C}$ has degree q. We see that α, β, and γ have q as their degree of homogeneity. Indeed, to see this we just consider the number of points of the intersection of the curve with a line in a general position. For Zariski open subset of $(A : B : C) \in \mathbb{P}^2$, we have that the polynomial

$$A\alpha(s,t) + B\beta(s,t) + C\gamma(s,t)$$

must have degree q since it must have q–different solutions $(s : t) \in \mathbb{P}^1$. But since α, β, γ are homogeneous, they must have the same degree.

We observe that since above map is birational, at least two of α, β, γ are linearly independent. But they are homogeneous. Thus, at least two of $\alpha(T, 1), \beta(T, 1), \gamma(T, 1)$ are linearly independent, where T is indeterminate.

The field of rational functions $\mathbb{C}(X(1))$ is given by

$$\mathbb{C}(X(1)) = \mathbb{C}(j) \simeq \mathbb{C}(T),$$

where j is the classical modular j–function

$$j = E_4^3/\Delta.$$

We recall that E_4 is defined in the introduction, and Δ is the Ramanujan delta function

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = \exp(2\pi i z).$$

Thus, we see that at least two of the following modular functions $\alpha(j(z),1), \beta(j(z),1), \gamma(j(z),1)$ are linearly independent as elements of the field $\mathbb{C}(X(1))$. Hence, because of homogeneity, the same is true for the modular forms $\alpha(E_4^3(z),\Delta(z))$, $\beta(E_4^3(z),\Delta(z))$, and $\gamma(E_4^3(z),\Delta(z))$ of degree $12q$. Furthermore, Lemmas 2.2 and 2.3 are applicable to the map

$$X(1) \longrightarrow \mathbb{P}^1 \longrightarrow \mathcal{C}$$

given by

$$\begin{aligned}\mathfrak{a}_z \longmapsto &(\alpha(j(z),1) : \beta(j(z),1) : \gamma(j(z),1))\\ =&(\alpha(E_4^3(z),\Delta(z)) : \beta(E_4^3(z),\Delta(z)) : \gamma(E_4^3(z),\Delta(z))).\end{aligned}$$

Thus, the curve $\mathcal{C}\left(\alpha(E_4^3,\Delta),\beta(E_4^3,\Delta),\gamma(E_4^3,\Delta)\right)$ is contained inside $\mathcal{C}$. Hence, they are equal. This map is a birational equivalence. ∎

3. Examples of uniformization

In this section we consider polynomials in two variables x, y, which we homogenize in a usual manner $x = x_1/x_0$ and $y = x_2/x_0$. The examples constructed in this section are obtained by a direct application of Theorem 1.1. They give a birational equivalence. In the next section we will construct different type of uniformization and birational equivalence.

Let $x^q + a_{q-1}x^{q-1} + \cdots + a_1x + a_0 - y$ be a polynomial with complex coefficients a_i. It is easy to see its irreducibility in the ring $\mathbb{C}[x,y]$ just by considering it as a polynomial in y with coefficients in $\mathbb{C}[x]$. The affine curve $(y = x^q + a_{q-1}x^{q-1} + \cdots + a_1x + a_0)$ is irreducible and we have the obvious (affine) isomorphism $(x,y) \longmapsto x$ which has inverse $x \longmapsto (x, x^q + a_{q-1}x^{q-1} + \cdots + a_1x + a_0)$. The corresponding projective curve $\mathcal{C} = (x_2x_0^{q-1} = x_1^q + a_{q-1}x_1^{q-1}x_0 + \cdots + a_1x_1x_0^{q-1} + a_0x_0^q)$ is irreducible and has degree q. Moreover, its genus equals zero since the above affine isomorphisms induce birational map from $\mathcal{C}$ to $\mathbb{P}^1$.

In terms of projective coordinates, the birational equivalence $\mathbb{P}^1 \longrightarrow \mathcal{C}$ is given by

$$(s:t) \longmapsto (s^q : s^{q-1}t : t^q + a_{q-1}st^{q-1} + \cdots + a_1s^{q-1}t + a_0s^q). \tag{3.1}$$

Thus, the birational equivalence between $X(1)$ and $\mathcal{C}$ is obtained from the factorization

$$X(1) \xrightarrow{\mathfrak{a}_z \longmapsto (E_4^3(z):\Delta(z))} \mathbb{P}^1 \xrightarrow{\text{map (3.1)}} \mathcal{C}.$$

Thus, $\mathcal{C}$ is uniformized and birationally equivalent to $X(1)$ with aid of $E_4^{3q}, \Delta E_4^{3q-3}, \Delta^q + a_{q-1}\Delta^{q-1}E_4^3 + \cdots + a_1\Delta E_4^{3q-3} + a_0 E_4^{3q} \in M_{12q}$.

Let $m, n \geq 1$ be relatively prime integers. Then the curve $(x^m - y^n = 0)$ is irreducible and birationally equivalent to the affine line $\mathbb{A}^1$. Indeed, we consider the map

$$\mathbb{A}^1 \xrightarrow{x \mapsto (x^n, x^m)} (x^m - y^n = 0). \tag{3.2}$$

This map is a birational equivalence since applying the Euclid algorithm it can be decomposed into a sequence of birational equivalences:

$$\begin{array}{ll}
m = k_1 n + r_1 & (x^n - y^{r_1} = 0) \xrightarrow{(x,y)\mapsto(y, xy^{k_1})} (x^m - y^n = 0), \\
n = k_2 r_1 + r_2 & (x^{r_1} - y^{r_2} = 0) \xrightarrow{(x,y)\mapsto(y, xy^{k_2})} (x^n - y^{r_1} = 0), \\
\dots & \dots \\
r_{i-1} = k_{i+1} r_i + r_{i+1} & (x^{r_i} - y^{r_{i+1}} = 0) \xrightarrow{(x,y)\mapsto(y, xy^{k_{i+1}})} (x^{r_{i-1}} - y^{r_i} = 0), \\
r_i = k_{i+2} r_{i+1} + 1 & (x^{r_{i+1}} - y = 0) \xrightarrow{(x,y)\mapsto(y, xy^{k_{i+2}})} (x^{r_i} - y^{r_{i+1}} = 0), \\
 & \mathbb{A}^1 \xrightarrow{x \mapsto (x, x^{r_{i+1}})} (x^{r_{i+1}} - y = 0).
\end{array}$$

Let us assume $m > n$. We remark that the polynomial $x^m - y^n$, or equivalently $x_1^m - x_2^n x_0^{m-n}$ is irreducible. This is so because the curve $(x^m - y^n = 0)$ is irreducible. This implies that $(x_1^m - x_2^n x_0^{m-n} = 0)$ is irreducible. Thus, by Nullstellensatz, $x_1^m - x_2^n x_0^{m-n}$ is a power of an irreducible polynomial, a degree of which determines the degree of $(x_1^m - x_2^n x_0^{m-n} = 0)$. But if we intersect with the line $(x_0 - x_2 = 0)$, we get m different points of intersection. This proves the claim.

Thus, still assuming $m > n$ for definiteness, the birational isomorphism between $X(1)$ and $\mathcal{C} = (x_1^m - x_2^n x_0^{m-n} = 0)$ is obtained from the factorization

$$X(1) \xrightarrow{\mathfrak{a}_z \longmapsto (E_4^3(z):\Delta(z))} \mathbb{P}^1 \xrightarrow{(s:t)\mapsto(s^m : t^n s^{m-n} : t^m)} \mathcal{C},$$

where the last map is the birational isomorphism (3.2) in its projective form. Thus, $\mathcal{C}$ is unformized and birationally equivalent to $X(1)$ with aid of $E_4^{3m}, \Delta^n E_4^{3m-3n}, \Delta^m \in M_{12m}$.

In the following proposition we describe all curves that can be obtained by the uniformization using using three forms in the canonical basis (see (1.1))

$$E_4^{3q},\ E_4^{3q-3}E_6^2, \dots, E_6^{2q} \tag{3.3}$$

of M_{12q} for $q \geq 2$.

PROPOSITION 3.1. *Let $q \geq 2$. We consider the basis of M_{12q} described in (3.3). Then all curves up to the order of basis elements and uniformization by smaller $M_{12q'}$, $q' \geq 2$, that can be uniformized by three basis forms of M_{12q}*

are given by $(x_0^{q'-j'}x_2^{j'} - x_1^{q'} = 0)$, *where* $0 < j < q$, $(j,q) = d$, *and* $q' = q/d$, *and* $j' = j/d$. *The uniformization is a birational equivalence if and only if* $d = 1$.

Proof. First, let us consider the case $q = 2$. In this case we are dealing with M_{12q} and the canonical basis is $E_4^6, E_4^3E_6^2, E_6^4$. It is obvious that we have $E_4^6E_6^4 - \left(E_4^3E_6^2\right)^2 = 0$. Thus, we obtain the curve $x_0x_2 - x_1^2$. This proves the claim for $q = 2$.

In general, let us consider the curve obtained from $E_4^{3q-3i}E_6^{2i}$, $E_4^{3q-3j}E_6^{2j}$, $E_4^{3q-3k}E_6^{2k}$, where $0 \le i < j < k \le q$, in that order. If $i > 0$ or $k < q$, then every form is divisible by E_6^2 or E_4^3, respectively. But this means that the resulting equation comes from the corresponding forms in $M_{12(q-1)}$. So, it is already on the list. Thus, we conclude that a contribution of M_{12q} is by means of the triples $E_4^{3q},\ E_4^{3q-3j}E_6^{2j},\ E_6^{2q}$, where $0 < j < q$. In this case, we let $(j,q) = d$, $q' = q/d$, and $j' = j/d$. Then we obtain the following:

$$\begin{aligned}&\left(E_4^{3q}\right)^{q'-j'}\left(E_6^{2q}\right)^{j'} - \left(\,E_4^{3q-3j}E_6^{2j}\right)^{q'}\\ &\qquad= \left(E_4^{3q'd}\right)^{q'-j'}\left(E_6^{2q'd}\right)^{j'} - \left(\,E_4^{3q'd-3j'd}E_6^{2j'd}\right)^{q'} = 0.\end{aligned}$$

Using the second example in this section, we conclude that

$$\mathcal{C}(E_4^{3q},\ E_4^{3q-3j}E_6^{2j},\ E_6^{2q}) = (x_0^{q'-j'}x_2^{j'} - x_1^{q'} = 0).$$

We discuss the birational equivalence. Put $f = E_4^{3q}$, $g = E_4^{3q-3j}E_6^{2j}$, and $h = E_6^{2q}$. The map $\mathfrak{a}_z \longmapsto (f(z) : g(z) : h(z))$ can be considered as a regular map from the smooth projective curve $X(1)$ (explained in the proof of Lemma 2.2) which is surjective. On the level of fields of rational functions, this implies the following:

$$\mathbb{C}\left(x_0^{q'-j'}x_2^{j'} - x_1^{q'} = 0\right) = \mathbb{C}\left(\mathcal{C}(f,g,h)\right) \simeq \mathbb{C}\left(f/h, g/h\right) \subset \mathbb{C}(X(1)) = \mathbb{C}(j).$$

By the standard characterization of birational equivalence, the map $\mathfrak{a}_z \longmapsto (f(z) : g(z) : h(z))$ is a birational equivalence if $\mathbb{C}\left(f/h, g/h\right) = \mathbb{C}(X(1)) = \mathbb{C}(j)$. Equivalently, reverting back to original notation, we must have

$$\mathbb{C}(j) = \mathbb{C}\left(\left(\frac{E_4^3}{E_6^2}\right)^j, \left(\frac{E_4^3}{E_6^2}\right)^q\right).$$

But, there exists $m, n \in \mathbb{Z}$ such that $jn + qm = d$. This means that we have the following: $\mathbb{C}(j) = \mathbb{C}\left(\left(E_4^3/E_6^2\right)^{-d}\right)$. Hence

$$\mathbb{C}(j^{-1}) = \mathbb{C}(j) = \mathbb{C}\left(\left(\frac{E_4^3}{E_6^2}\right)^d\right) = \mathbb{C}\left(\left(\frac{E_6^2}{E_4^3}\right)^d\right) = \mathbb{C}\left(\left(1 - j^{-1}\right)^d\right).$$

This forces $d = 1$ since $\mathbb{C}(j^{-1}) \simeq \mathbb{C}(T)$, where T is indeterminate. $\square$

4. Computation using SAGE

In this section we compute some uniformizations using the open source mathematics software SAGE. We compute irreducible polynomials given by Corollary 2.4. For simplicity, we denote forms in canonical basis for M_{12q} given by (3.3) with:

$$e_0,\ e_1, \ldots, e_i, \ldots e_q \tag{4.1}$$

We compute this base in SAGE as follows:

$$\mathrm{sage:\ \ E4 = eisenstein\_series\_qexp(4, prec)}$$
$$\mathrm{sage:\ \ E6 = eisenstein\_series\_qexp(6, prec)}$$

This returns the q-expansions of the normalized weight 4 and 6 Eisenstein series to precision *prec*. Then we get basis forms:

$$\mathrm{sage:\ \ e_i = E4\hat{}(3 * (q - i)) * E6\hat{}(2 * i)}$$

for $0 \leq i \leq q$. We calculate equation for curve obtained from linearly independent forms $f, g, h \in M_{12q}$ as follows.

First, using simple routines we calculate all monoms of degree q obtained from f, g, h. Then, using SAGE command linear_dependence we calculate dependences of monoms:

$$\mathrm{sage:\ \ L = V.linear\_dependence(vectors, zeros =' left').}$$

This gives us the equation of the curve. We use SAGE command factor to check irreducibility of the corresponding polynomial.

$$\mathrm{sage:\ \ F = factor(pol).}$$

Since all curves obtained by three forms in the canonical basis are described in Proposition 3.1, we make some elementary operations on elements of canonical basis to obtain some other three linearly independent forms. The following are some irreducible homogeneous polynomials we computed in SAGE:
1. M_{120}, $q = 10$. For $f = e_0,\ g = e_0 + e_1,\ h = e_0 + e_1 + e_{10}$ we get

$$\begin{aligned}&2x_0^{10} - 9x_0^9x_1 + 45x_0^8x_1^2 - 120x_0^7x_1^3 + 210x_0^6x_1^4 - 252x_0^5x_1^5\\ &+ 210x_0^4x_1^6 - 120x_0^3x_1^7 + 45x_0^2x_1^8 - 10x_0x_1^9 + x_1^{10} - x_0^9x_2.\end{aligned}$$

For $f = e_0,\ g = e_0 + e_3,\ h = e_0 + e_3 + e_{10}$ we get

$$\begin{aligned}&2x_0^{10} - 7x_0^9x_1 + 48x_0^8x_1^2 - 119x_0^7x_1^3 + 210x_0^6x_1^4 - 252x_0^5x_1^5 + 210x_0^4x_1^6\\ &- 120x_0^3x_1^7 + 45x_0^2x_1^8 - 10x_0x_1^9 + x_1^{10} - 3x_0^9x_2 - 6x_0^8x_1x_2 - 3x_0^7x_1^2x_2\\ &+ 3x_0^8x_2^2 + 3x_0^7x_1x_2^2 - x_0^7x_2^3.\end{aligned}$$

For $f = e_0 + e_8,\ g = e_7 + e_8,\ h = e_{10} + e_8$ we get

$$x_0^2x_1^8 - 2x_0x_1^9 + 2x_1^{10} - 8x_0^2x_1^7x_2 + 9x_0x_1^8x_2 - 8x_1^9x_2 + 35x_0^2x_1^6x_2^2$$
$$- 34x_0x_1^7x_2^2 + 20x_1^8x_2^2 - 90x_0^2x_1^5x_2^3 + 103x_0x_1^6x_2^3 - 48x_1^7x_2^3 + 142x_0^2x_1^4x_2^4$$
$$- 182x_0x_1^5x_2^4 + 82x_1^6x_2^4 - 126x_0^2x_1^3x_2^5 + 178x_0x_1^4x_2^5 - 80x_1^5x_2^5 + 53x_0^2x_1^2x_2^6$$
$$- 86x_0x_1^3x_2^6 + 40x_1^4x_2^6 - x_0^3x_2^7 - 8x_0^2x_1x_2^7 + 16x_0x_1^2x_2^7 - 8x_1^3x_2^7 + x_0^2x_2^8$$
$$- 2x_0x_1x_2^8 + x_1^2x_2^8.$$

2. M_{180}, $q = 15$. For $f = e_0 + e_{14},\ g = e_{13} + e_{14}, h = e_{15} + e_{14}$ we get

$$x_1^{15} - x_0x_1x_2^{13} - x_0x_2^{14} + x_1x_2^{14}$$

For $f = e_0 + e_3,\ g = e_2 + e_3, h = e_{15} + e_3$ we get

$$x_0^{12}x_1^3 - 12x_0^{11}x_1^4 + 66x_0^{10}x_1^5 - 220x_0^9x_1^6 + 495x_0^8x_1^7 - 792x_0^7x_1^8 + 926x_0^6x_1^9$$
$$- 768x_0^5x_1^{10} + 450x_0^4x_1^{11} - 208x_0^3x_1^{12} + 84x_0^2x_1^{13} - 24x_0x_1^{14} + 4x_1^{15} - 3x_0^{12}x_1^2x_2$$
$$+ 36x_0^{11}x_1^3x_2 - 198x_0^{10}x_1^4x_2 + 660x_0^9x_1^5x_2 - 1485x_0^8x_1^6x_2 + 2376x_0^7x_1^7x_2$$
$$- 2787x_0^6x_1^8x_2 + 2339x_0^5x_1^9x_2 - 1362x_0^4x_1^{10}x_2 + 570x_0^3x_1^{11}x_2 - 190x_0^2x_1^{12}x_2$$
$$+ 48x_0x_1^{13}x_2 - 6x_1^{14}x_2 - x_0^{13}x_2^2 + 15x_0^{12}x_1x_2^2 - 102x_0^{11}x_1^2x_2^2 + 418x_0^{10}x_1^3x_2^2$$
$$- 1155x_0^9x_1^4x_2^2 + 2277x_0^8x_1^5x_2^2 - 3300x_0^7x_1^6x_2^2 + 3564x_0^6x_1^7x_2^2 - 2871x_0^5x_1^8x_2^2$$
$$+ 1705x_0^4x_1^9x_2^2 - 726x_0^3x_1^{10}x_2^2 + 210x_0^2x_1^{11}x_2^2 - 37x_0x_1^{12}x_2^2 + 3x_1^{13}x_2^2.$$

3. M_{228}, $q = 19$. For $f = e_0 + e_{18},\ g = e_{17} + e_{18}, h = e_{19} + e_{18}$ we get

$$x_1^{19} - x_0x_1x_2^{17} - x_0x_2^{18} + x_1x_2^{18}.$$

For $f = e_0 + e_{17},\ g = e_{16} + e_{17}, h = e_{19} + e_{17}$ we get

$$x_0^2x_1^{17} - 2x_0x_1^{18} + x_1^{19} - 17x_0^2x_1^{16}x_2 + 34x_0x_1^{17}x_2 - 17x_1^{18}x_2 + 152x_0^2x_1^{15}x_2^2$$
$$- 224x_0x_1^{16}x_2^2 + 136x_1^{17}x_2^2 - 903x_0^2x_1^{14}x_2^3 + 1050x_0x_1^{15}x_2^3 - 595x_1^{16}x_2^3$$
$$+ 3910x_0^2x_1^{13}x_2^4 - 4324x_0x_1^{14}x_2^4 + 2142x_1^{15}x_2^4 - 12876x_0^2x_1^{12}x_2^5 + 14620x_0x_1^{13}x_2^5$$
$$- 6800x_1^{14}x_2^5 + 32918x_0^2x_1^{11}x_2^6 - 38692x_0x_1^{12}x_2^6 + 17646x_1^{13}x_2^6 - 65779x_0^2x_1^{10}x_2^7$$
$$+ 79988x_0x_1^{11}x_2^7 - 36193x_1^{12}x_2^7 + 102416x_0^2x_1^9x_2^8 - 129136x_0x_1^{10}x_2^8$$
$$+ 58208x_1^{11}x_2^8 - 122760x_0^2x_1^8x_2^9 + 161208x_0x_1^9x_2^9 - 72624x_1^{10}x_2^9$$
$$+ 110960x_0^2x_1^7x_2^{10} - 152584x_0x_1^8x_2^{10} + 68952x_1^9x_2^{10} - 73432x_0^2x_1^6x_2^{11}$$
$$+ 106360x_0x_1^7x_2^{11} - 48416x_1^8x_2^{11} + 34268x_0^2x_1^5x_2^{12} - 52556x_0x_1^6x_2^{12}$$
$$+ 24208x_1^7x_2^{12} - 10798x_0^2x_1^4x_2^{13} + 17586x_0x_1^5x_2^{13} - 8228x_1^6x_2^{13} + 2190x_0^2x_1^3x_2^{14}$$
$$- 3784x_0x_1^4x_2^{14} + 1802x_1^5x_2^{14} - 269x_0^2x_1^2x_2^{15} + 491x_0x_1^3x_2^{15} - 238x_1^4x_2^{15}$$
$$- x_0^3x_2^{16} + 19x_0^2x_1x_2^{16} - 35x_0x_1^2x_2^{16} + 17x_1^3x_2^{16}.$$

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