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OPTIMAL DAMPING OF THE INFINITE-DIMENSIONAL VIBRATIONAL SYSTEMS: COMMUTATIVE CASE

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Abstract. In this paper we treat the case of an abstract vibrational system of the form $M\ddot{x} + C\dot{x} + x = 0$, where the positive semi-definite selfadjoint operators M and C commute. We explicitly calculate the solution of the corresponding Lyapunov equation which enables us to obtain the set of optimal damping operators, thus extending already known results in the matrix case.

1. Introduction and preliminary results

The main object of the study in this paper is an abstract differential equation

$$M\ddot{x}(t) + C\dot{x}(t) + x(t) = 0, \; x(0) = x_0, \dot{x}(0) = \dot{x}_0, \tag{1.1}$$

where M and C are bounded non–negative selfadjoint operators on a Hilbert space $\mathcal{H}$, and $x_0, \dot{x}_0 \in \mathcal{H}$. We assume that operators M and C commute.

The equation (1.1) is a model of a vibrating system, with M corresponding to the mass, and C corresponding to the damping. The stiffness of the system is set to I, which is the case if, for example, the scalar product of $\mathcal{H}$ is defined by $\kappa(x, x)$ where κ is the stiffness form. Usually one uses the similar procedure to obtain $M = I$, but in the case of the systems for which the stiffness operator is selfadjoint and positive definite one can also study (1.1) as a mathematical model of the system.

Systems of the form (1.1) have been extensively studied in the context of the stability of mechanical structures, but they have applications in other fields also. For the basic introduction to these systems we refer the reader to [3, 21].

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The corresponding matrix differential equation

$$M\ddot{x} + C\dot{x} + Kx = 0,$$

with M,C,K positive semi–definite matrices is well-researched topic both in mathematical and engineering literature. The classic reference for a more mathematical point of view is the monograph [15]. The so–called modal damping (which in our setting is equivalent to the condition that M and C commute), is an especially popular topic in engineering literature, due to the fact that it is computationally approachable. The main result of this paper, the construction of the optimal damping, is a well–known result in the matrix case. This result has been generalized in [5] to the non–modal damping, obtaining the same set of optimal matrices. The results for the optimal damping without restrictions on the structure of damping matrices usually are not very useful in practical applications, but there is some recent progress in treating such problems (see [2, 18, 19]).

The assumption that the operators M and C are bounded is not as restrictive as it seems. Let us suppose that we have an abstract vibrational system given described by the equation

$$\mu(\ddot{x}, z) + \gamma(\dot{x}, z) + \kappa(x, z) = 0,\ \forall z \in \mathcal{X},$$

where μ, γ, κ are non–negative sesquilinear forms on a vector space $\mathcal{X}$. Here μ corresponds to the mass, γ to the damping, and κ to the stiffness of the system. We assume that the sesquilinear form κ is positive, i.e. $\kappa(x, x) > 0$ for all $x \neq 0$, and that $\mathcal{Y} = D(\mu) \cap D(\gamma) \cap D(\kappa)$ is a non–trivial subspace of $\mathcal{X}$. Let $\mathcal{H}$ denote the completion of the space $(\mathcal{Y}, \kappa(\cdot, \cdot))$. The norm generated by this scalar product will be denoted by $\|\cdot\|$. Obviously, $\mathcal{Y}$ is dense in $\mathcal{H}$.

We also assume that μ and γ are closable in $\mathcal{H}$, and we denote these closures also by μ and γ.

Then the second representation theorem [13, p. 331] implies the existence of selfadjoint non–negative operators M and C such that

$$D(M^{1/2}) = D(\mu),\ \ \mu(x, y) = (M^{1/2}x, M^{1/2}y),\ x, y \in D(\mu),$$
$$D(C^{1/2}) = D(\gamma),\ \ \gamma(x, y) = (C^{1/2}x, C^{1/2}y),\ x, y \in D(\gamma),$$

and (1.1) can be written as

$$(M^{1/2}\ddot{x}(t), M^{1/2}z) + (C^{1/2}\dot{x}(t), C^{1/2}z) + (x(t), z) = 0,$$

which can be written as

$$M\ddot{x}(t) + C\dot{x}(t) + x(t) = 0 \tag{1.2}$$

in the form–sum sense. In this setting, the boundedness of the operators M and C is equivalent to the existence of the constant Δ such that $\mu(x, x) \leq \Delta\kappa(x, x)$ and $\gamma(x, x) \leq \Delta\kappa(x, x)$, which is satisfied for a number of vibrational systems.

EXAMPLE 1.1. To illustrate the abstract setting of the problem, let us analyze a simple example of the Euler–Bernoulli beam with so–called structural damping and with hinged boundary conditions:

$$\frac{\partial^2}{\partial t^2}u(x,t) - \rho\frac{\partial^3}{\partial x^2 \partial t}u(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) = 0,\ x \in [0,1],\ t \geq 0,$$
$$u(0,t) = \frac{\partial^2}{\partial x^2}u(0,t) = 0, \quad u(1,t) = \frac{\partial^2}{\partial x^2}u(1,t) = 0,$$
$$u(x,0) = u_0(x),\ \ \frac{\partial}{\partial t}u(x,0) = u_1(x),$$

where ρ is the damping coefficient.

By the use of partial integration one readily sees that

$$\mu(u,v) = \int_0^1 u(x)\overline{v(x)}\mathrm{d}x,$$
$$\gamma(u,v) = \rho\int_0^1 u'(x)\overline{v'(x)}\mathrm{d}x,$$
$$\kappa(u,v) = \int_0^1 u''(x)\overline{v''(x)}\mathrm{d}x.$$

The corresponding space $\mathcal{X}$ is $\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])$, the standard space for the weak formulation of the problem. By standard arguments one can show that the norm generated by the form κ is equivalent to the standard norm of the space $\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])$.

One can easily see that M is the inverse of the fourth–order differential operator $v \mapsto \frac{d^4}{dx^4}v$ with boundary coefficients $v(0) = v(1) = v''(0) = v''(1) = 0$. The operator C is the inverse of the second—order differential operator $v \mapsto -\rho^{-1}\frac{d^2}{dx^2}v$ with boundary coefficients $v(0) = v(1) = 0$. Hence C^{-1} is the Dirichlet Laplacian multiplied by $-1/\rho$ (on the space $\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])$). We also have $C^2 = \rho^2 M$, hence $C = \rho M^{1/2}$.

A straightforward calculation gives that the Green functions of the operators M and C are given by

$$G_M(x,s) = \left(\frac{s}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)x + \left(-\frac{1}{6} + \frac{s}{6}\right)x^3,\ x \leq s,$$
$$G_C(x,s) = \rho(1-x)s,\ x \leq s,$$

where for $x \geq s$ we have $G_*(x,s) = G_*(s,x)$, $* = M, C$.

The operator M is obviously compact.

The commutativity of M and C means that we are dealing with the so–called modal damping vibrational systems.

Let us go back to the equation (1.1).

Set $M_\lambda = \lambda^2 M + \lambda C + I$ for $\lambda \geq 0$. This operator can be obtained by plugging $x(t) = e^{\lambda t}x_0$ in (1.2). Obviously M_λ^{-1} exists as a bounded operator and $\|M_\lambda^{-1}\| \leq 1$. We also define

$$R_0(\lambda) = \begin{bmatrix} \frac{1}{\lambda}(M_\lambda^{-1} - I) & -M_\lambda^{-1}M^{1/2} \\ M_\lambda^{-1}M^{1/2} & -\lambda M M_\lambda^{-1} \end{bmatrix}. \tag{1.3}$$

By a straightforward computation one can easily check that $R_0(\lambda)$ is a pseudoresolvent, i.e.

$$R_0(\lambda) - R_0(\nu) = (\lambda - \nu)R_0(\lambda)R_0(\nu).$$

This implies that $\mathcal{N}(R_0(\lambda))$ is independent of the choice of λ.

PROPOSITION 1.2. *The null–space of the operator $R_0(\lambda)$ is given by*

$$\mathcal{N}(R_0(\lambda)) = \left\{ u \in \mathcal{H} \oplus \mathcal{H} : u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), x \in \mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}), y \in \mathcal{N}(M^{1/2}) \right\}.$$

PROOF. From the equation $R_0(\lambda)u = 0$, where $u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, it follows

$$\frac{1}{\lambda}M_\lambda^{-1}x - \frac{1}{\lambda}x - M_\lambda^{-1}M^{1/2}y = 0,$$
$$M^{1/2}M_\lambda^{-1}x - \lambda M_\lambda^{-1}My = 0.$$

Multiplying the first equation by λx, and second by λy and then conjugate, we get

$$(M_\lambda^{-1}x, x) - (x, x) - \lambda(M_\lambda^{-1}M^{1/2}y, x) = 0, \tag{1.4}$$

$$\lambda(M_\lambda^{-1}M^{1/2}y, x) - \lambda^2(M_\lambda^{-1}My, y) = 0. \tag{1.5}$$

Adding (1.4) and (1.5) we obtain

$$(M_\lambda^{-1}x, x) - (x, x) - \lambda^2(M_\lambda^{-1}My, y) = 0,$$

which implies

$$(M_\lambda^{-1}x, x) - (x, x) = \lambda^2(M_\lambda^{-1}My, y) \geq 0. \tag{1.6}$$

Since $\|M_\lambda^{-1}\| \leq 1$, it follows $(x, x) \geq (M_\lambda^{-1}x, x) \geq (x, x)$, hence

$$(M_\lambda^{-1}x, x) = (x, x), \tag{1.7}$$

and since $\mathcal{N}(R_0(\lambda))$ is independent of the choice of λ, this equation holds for all $\lambda > 0$. This implies $x \in \mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2})$. Also, from (1.6) follows $M^{1/2}y = 0$. ∎

Let us denote $\mathbf{Y} = (\mathcal{N}(R_0(\lambda)))^\perp$. This is the so–called *phase space.* Obviously, the subspace $\mathbf{Y}$ reduces the operator $R_0(\lambda)$. Let us denote by $P_\mathbf{Y} : \mathcal{H} \oplus \mathcal{H} \to \mathbf{Y}$ the corresponding orthogonal projector to the subspace $\mathbf{Y}$.

Let $R(\lambda) = P_\mathbf{Y} R_0(\lambda)|_\mathbf{Y}$ denote the corresponding restriction of the operator $R_0(\lambda)$ to the phase space. Then $R(\lambda)$ satisfies the resolvent equation and has trivial null space. Then from the theory of pseudoresolvents ([16]), it follows that there exists an unique closed operator $A : \mathbf{Y} \to \mathbf{Y}$ such that $R(\lambda) = (A - \lambda)^{-1}$ for all $\lambda \geq 0$.

The subspace $\mathbf{Y}$ can be decomposed by $\mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = (\mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}))^\perp$, $\mathcal{H}_2 = (\mathcal{N}(M^{1/2}))^\perp$. Since $\mathcal{N}(M^{1/2}) = \mathcal{N}(M)$ and $\mathcal{N}(C^{1/2}) = \mathcal{N}(C)$, we can also write $\mathcal{H}_1 = (\mathcal{N}(M) \cap \mathcal{N}(C))^\perp$, $\mathcal{H}_2 = (\mathcal{N}(M))^\perp$.

To avoid technicalities and simplify the proofs, we assume $\mathcal{N}(M^{1/2}) \subset \mathcal{N}(C^{1/2})$, i.e. there is no damping on the positions where mass vanishes. All the results of this paper *remain valid* also in the case when $\mathcal{N}(M^{1/2}) \not\subseteq \mathcal{N}(C^{1/2})$.

This assumption implies $\mathcal{Y} := \mathcal{H}_1 = \mathcal{H}_2$, so now $\mathbf{Y} = \mathcal{Y} \oplus \mathcal{Y}$. Also, from now on let C denote the operator $C : \mathcal{Y} \to \mathcal{Y}$, and let M denote the operator $M : \mathcal{Y} \to \mathcal{Y}$. Letters M and C will denote operators in the spaces $\mathcal{H}$ or $\mathcal{Y}$ depending on the context.

Hence we can write

$$A^{-1} = \begin{bmatrix} -C & -M^{1/2} \\ M^{1/2} & 0 \end{bmatrix}.$$

From

$$A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -Cx - M^{1/2}y \\ M^{1/2}x \end{bmatrix}$$

follows that $D(A) = \mathcal{R}(A^{-1}) \subset (\mathcal{R}(C) + \mathcal{R}(M^{1/2})) \oplus \mathcal{R}(M^{1/2})$, hence the operator A is not bounded in general.

Example 1.3 (Continuation of example 1.1). If we denote the Dirichlet Laplacian on the space $\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])$ by Δ, then $C = -\rho\Delta^{-1}$, $M^{1/2} = -\Delta^{-1}$, and the operator A is given by

$$A(u,v)^T = (-\Delta v, \Delta(u + \rho v))^T$$

and $\mathbf{Y} = \big(\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])\big) \oplus \big(\mathcal{H}_2([0,1]) \cap \mathcal{H}_1^0([0,1])\big)$.

Let A be a linear operator with dense domain. The operator A is called dissipative operator if $\mathrm{Re}(Ax, x) \leq 0$ for all $x \in D(A)$.

Proposition 1.4. *The operator A generates a strongly continuous semigroup.*

Proof. It is easy to see that A^{-1} is dissipative, which implies that A is also dissipative. Since $R(\lambda)$ is everywhere defined for all $\lambda \geq 0$, it follows from Lumer–Phillips theorem [16, Theorem 4.3] that A generates a strongly continuous semigroup. ∎

Definition 1.5. *A (mild) solution of the Cauchy problem (1.1) is a function $x : [0, \infty) \to \mathcal{H}$ such that $x(t)$ is continuous, $Mx(t)$ is continuously differentiable, and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mx(t)) + Cx(t) + \int_0^t x(s)\mathrm{d}s - Cx_0 - M\dot{x}_0 = 0, \ \forall t \geq 0. \tag{1.8}$$

Here $x_0, \dot{x}_0 \in \mathcal{H}$.

PROPOSITION 1.6. *The Cauchy problem* (1.1) *has a solution if and only if* $x_0 \in \mathcal{Y}$. *If solution exists, it is unique. The solution is given by*

$$x(t) = PT(t) \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix} \right),$$

where $P : \mathbf{Y} \to \mathcal{Y}$ *is the orthogonal projector on the first component of* $\mathbf{Y}$ *and* $T(t)$ *denotes the semigroup generated by the operator* A.

PROOF. Since the operator A generates a strongly continuous semigroup, the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t), \\ u(0) &= u_0, \end{aligned} \tag{1.9}$$

has a unique mild solution for all $u_0 \in \mathbf{Y}$. We will connect Cauchy problems (1.1) and (1.9).

For the rest of the proof, let $\dot{x}_0 \in \mathcal{H}$ be arbitrary.

In case $x_0 \in \mathcal{Y}$, for $u_0 = \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix} \right)$ the Cauchy problem (1.9) in general has only a mild solution. Let us denote this solution by $u(t) = \left(\begin{smallmatrix} u_1(t) \\ u_2(t) \end{smallmatrix} \right)$. From $u(t) = A\int_0^t u(s)\mathrm{d}s + u_0$ follows

$$A^{-1} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \int_0^t \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + A^{-1} \begin{pmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{pmatrix}.$$

This implies

$$-Cu_1(t) - M^{1/2}u_2(t) = \int_0^t u_1(s)\mathrm{d}s - Cx_0 - M\dot{x}_0, \tag{1.10}$$

$$M^{1/2}u_1(t) = \int_0^t u_2(s)\mathrm{d}s + M^{1/2}x_0. \tag{1.11}$$

The relation (1.11) implies that $M^{1/2}u_1(t)$ (and hence $Mu_1(t)$) is continuously differentiable and that $u_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}u_1(t))$. Then (1.10) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}u_1(t)) + Cu_1(t) + \int_0^t u_1(s) - M\dot{x}_0 - Cx_0 = 0,$$

hence $u_1(t)$ is a mild solution of (1.1).

On the other hand, let $x(t)$ be a solution of (1.1) for $x_0 \in \mathcal{Y}$. Set $u(t) = \left(\begin{smallmatrix} x(t) \\ M^{1/2}x(t) \end{smallmatrix} \right)$ and $u_0 = \left(\begin{smallmatrix} x_0 \\ \dot{x}_0 \end{smallmatrix} \right)$. Obviously $u(t) \in \mathcal{Y}$ and $u(t)$ is continuous. One can easily prove that $A^{-1}u(t) = \int_0^t u(s)\mathrm{d}s + A^{-1}u_0$ holds, hence $u(t)$ is a mild solution of (1.9).

Finally, let us assume that there exists a solution of (1.1) for $x_0 \in \mathcal{H}$. We decompose x_0 as $x_0 = y_0 + w_0$, where $y_0 \in \mathcal{Y}$ and $w_0 \in \mathcal{N}(M)$. For the initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{x}_0$ there exists a unique solution $y(t)$ of (1.1).

Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}x(t)) + Cx(t) + \int_0^t x(s) - M\dot{x}_0 - Cx_0 = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}y(t)) + Cy(t) + \int_0^t y(s) - M\dot{x}_0 - Cy_0 = 0.$$

By subtracting these two equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}z(t)) + Cz(t) + \int_0^t z(s) = 0,$$

where $z(t) = x(t) - y(t)$. This implies that $z(t)$ is a solution of (1.1) for initial conditions $z(0) = 0$, $\dot{z}(0) = 0$. From the uniqueness of the solutions, it follows $z(t) \equiv 0$, hence $w_0 = 0$, i.e. $x_0 \in \mathcal{Y}$. ∎

Next we will explain what does the term "optimal damping" from the title means.

By optimal damping we understand the choice of the damping operator C such that the total energy of the system, defined by

$$\int_0^\infty E(t; x_0, \dot{x}_0)\mathrm{d}t = \int_0^\infty \frac{1}{2}\|M^{1/2}\dot{x}(t)\|^2 + \frac{1}{2}\|x(t)\|^2\mathrm{d}t, \tag{1.12}$$

is minimal. One can easily see that $E(t)$ corresponds to the kinetic energy of the system. Here, as before, $x_0, \dot{x}_0 \in \mathcal{H}$.

From Proposition 1.6 follows that the total energy of the system is given by

$$\int_0^\infty (T(t)^* T(t) u_0, u_0)\mathrm{d}t,$$

where $u_0 = \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix}\right)$.

Let us assume that the operators M and C are such that the corresponding operator A generates a uniformly exponentially stable semigroup $T(t)$. Later (Remark 2.2) we will obtain a necessary and sufficient condition for the uniform exponential stability of modally damped vibrational systems. Under this assumption, the following result can be derived from [17].

THEOREM 1.7. *The following operator equation*

$$A^* X x + X A x = -x, \quad \textit{for all}\, x \in D(A), \tag{1.13}$$

has a bounded solution, and the solution X can be expressed by

$$Xx = \int_0^\infty T(t)^* T(t) x \mathrm{d}t. \tag{1.14}$$

Theorem 1.7 immediately implies that the total energy of the system (1.1) is given by (Xu_0, u_0), where $u_0 = \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix}\right)$.

To make our minimization process independent of the initial conditions, we would like to minimize the average total energy over the set of admissible damping operators C, i.e.

$$\int_{\|u_0\|=1} (Xu_0, u_0)\mu(\mathrm{d}u_0) \to \min, \tag{1.15}$$

where X is regarded as a function of C, and μ is some measure on the unit sphere in $\mathbf{Y}$. The natural choice for μ are surface measures generated by a Gaussian measure with zero mean and a covariance operator $\left(\begin{smallmatrix} K & 0 \\ 0 & K \end{smallmatrix}\right)$, where K is a trace class operator which commutes with M (and hence with C).

Our aim is to explicitly calculate X and then to find the optimal C.

2. Main result

In this section we will prove the following result.

Theorem 2.1. *For an abstract vibrational system described by* (1.1) *with the fixed operator M and a variable damping operator C which commutes with M and for which the system is exponentially stable, the optimal damping operator in the sense of* (1.15) *is given by* $C_{\mathrm{opt}} = 2M^{1/2}$.

We start with the well-known formula [9, Corollary 3.5.15]

$$T(s)x = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\varepsilon-in}^{\varepsilon+in} e^{\lambda s} R(\lambda, A)x\mathrm{d}\lambda,\ x \in D(A),$$

where $\varepsilon > 0$ is arbitrary chosen, $n \in \mathbb{N}$ and $s \geq 0$. Since it is always $T(0) = I$, in the sequel we consider only $s > 0$.

Since M and C commute, there exists a bounded selfadjoint operator G such that the operators M and C are functions of G ([1, Theorem 76.2]), hence there exists a spectral function $E(t)$ and $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ measurable functions for all Stieltjes measures ([1, Section 75], [14, Section 36.1]) generated by $(E(t)x, x)$, $x \in \mathcal{Y}$, such that

$$M = \int_0^{\Xi} \alpha(t)\mathrm{d}E(t), \quad C = \int_0^{\Xi} \beta(t)\mathrm{d}E(t), \tag{2.1}$$

where $\Xi = \|G\|$. Since M and C are bounded, so are also the functions α and β. We have also $\alpha(t) > 0$ a.e. It follows that the resolvent $R(\lambda, A)$ can be written as

$$R(\lambda, A) = \int_0^{\Xi} \begin{bmatrix} \frac{-\lambda\alpha(t)-\beta(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1} & \frac{-\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1} \\ \frac{\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1} & \frac{-\lambda\alpha(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1} \end{bmatrix} \mathrm{d}E(t),$$

hence

$$T(s)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\varepsilon-in}^{\varepsilon+in} e^{\lambda s}\cdot$$
$$\cdot \begin{bmatrix} \int_0^\Xi \frac{-\lambda\alpha(t)-\beta(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)x + \int_0^\Xi \frac{-\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)y \\ \int_0^\Xi \frac{\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)x + \int_0^\Xi \frac{-\lambda\alpha(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)y \end{bmatrix} \mathrm{d}\lambda. \tag{2.2}$$

We first treat the case when $x, y \in \mathcal{R}(M^{1/2})$.

We want to change the order of integration in (2.2) by the use of Fubini theorem. By the change of variables, we obtain

$$\begin{aligned}(T(s)\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)) = {} & - e^{\varepsilon s} \lim_{n\to\infty} \frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda s}\Bigg[\int_0^\Xi p_1(\lambda,t)\mathrm{d}(E(t)x_1, x_2) \\ & - \int_0^\Xi p_2(\lambda,t)\mathrm{d}(E(t)y_1,x_2) + \int_0^\Xi p_2(\lambda,t)\mathrm{d}(E(t)x_1,y_2) \\ & + \int_0^\Xi p_3(\lambda,t)\mathrm{d}\left(E(t)y_1,y_2\right)\Bigg]\mathrm{d}\lambda,\end{aligned}$$

where

$$\begin{aligned} p_1(\lambda,t) &= \frac{-(i\lambda+\varepsilon)\alpha(t)-\beta(t)}{(i\lambda+\varepsilon)^2\alpha(t)+(i\lambda+\varepsilon)\beta(t)+1}, \\ p_2(\lambda,t) &= \frac{\sqrt{\alpha(t)}}{(i\lambda+\varepsilon)^2\alpha(t)+(i\lambda+\varepsilon)\beta(t)+1}, \\ p_3(\lambda,t) &= \frac{-(i\lambda+\varepsilon)\alpha(t)}{(i\lambda+\varepsilon)^2\alpha(t)+(i\lambda+\varepsilon)\beta(t)+1}. \end{aligned}$$

The integrals

$$\int_{-n}^{n}\int_0^\Xi e^{i\lambda s}p_j(\lambda,t)\mathrm{d}(E(t)x,x)\mathrm{d}\lambda,\ j=1,2,3, \tag{2.3}$$

can be viewed as a double Lebesgue integrals in $\mathbb{R}\times\mathbb{R}_+$, with the product measure generated by the standard Lebesgue measure in $\mathbb{R}$ and by (real–valued) Stieltjes measure $(E(t)x,x)$ in $\mathbb{R}_+$. Let us now fix $n\in\mathbb{N}$. In order to use Fubini theorem on (2.3) we have to prove ([14, pp. 361,362]):

(i) the functions p_j, $j=1,2,3$ are measurable in the product measure, and

(ii) the integrals

$$\int_0^\Xi \left(\int_{-n}^{n} \left|e^{i\lambda s}p_j(\lambda,t)\right| \mathrm{d}\lambda\right) \mathrm{d}(E(t)x,x),\ j=1,2,3$$

exist.

To prove (i) it suffices to show that the function $g(\lambda,t) = (i\lambda+\varepsilon)^2\alpha(t) + (i\lambda+\varepsilon)\beta(t)+1$ is measurable and vanishes only on the set of the measure zero. One can easily see that the function g does not vanish. Set $A_n = \alpha^{-1}([n,n+1])$, which is a measurable set in $\mathbb{R}_+$. Then $\mathbb{R}_+ = \cup_n A_n$. To prove that g is measurable, observe that for an arbitrary $\delta > 0$ the following holds

$$\{(\lambda,t) : |g(\lambda,t)| < \delta\} = \bigcup_n \left(A_n \times \bigcup_{t\in A_n} \{\lambda : |g(\lambda,t)| < \delta\} \right).$$

For fixed $t \in A_n$, one can easily see that $\{\lambda : |g(\lambda,t)| < \delta\}$ is either an empty set or an open interval or an union of two open intervals, hence always an open set. It follows that $\bigcup_{t\in A_n}\{\lambda : |g(\lambda,t)| < \delta\}$ is an open set as a union of open sets. From this immediately follows that $\{(\lambda,t) : |g(\lambda,t)| < \delta\}$ is measurable for all $\delta > 0$, hence g is a measurable function.

Now we prove (ii). We have

$$\int_{-n}^{n} |p_j(\lambda,t)|\mathrm{d}\lambda = \int_{\Upsilon_n} |p_j(\lambda,t)|\mathrm{d}\lambda + \int_{\Gamma_n} |p_j(\lambda,t)|\mathrm{d}\lambda, \tag{2.4}$$

where Γ_n is lower semi–circle connecting $-n$ and n, and Υ_n is the contour consisting of the segment $[-n,n]$ and the curve Γ_n. The first integral can be calculated by the use of residue theorem. The poles of the functions $\lambda \mapsto p_j(\lambda,t)$ are the zeros of the polynomial g. We calculate the zeros of g. We have

$$\lambda_{1,2} = \pm\frac{\sqrt{4\alpha(t)-\beta(t)^2}}{2\alpha(t)} + i\frac{2\varepsilon\alpha(t)+\beta(t)}{2\alpha(t)}, \tag{2.5}$$

in the case $4\alpha(t) \geq \beta(t)^2$, and

$$\lambda_{1,2} = i\frac{2\varepsilon\alpha(t)+\beta(t) \pm \sqrt{\beta(t)^2-4\alpha(t)}}{2\alpha(t)}, \tag{2.6}$$

in the case $4\alpha(t) < \beta(t)^2$. Hence the first integral on the right hand side in (2.4) is

$$\int_{\Upsilon_n} |p_j(\lambda,t)|\mathrm{d}\lambda = 0,\ j = 1,2,3. \tag{2.7}$$

To estimate the second integral on the right hand side in (2.4) we proceed as follows.

$$\int_{\Gamma_n^-} |p_j(\lambda,t)|\mathrm{d}\lambda = n\int_{\pi}^{2\pi} |p_j(ne^{i\varphi},t)|\mathrm{d}\varphi \leq cn^2\int_{\pi}^{2\pi} \frac{\mathrm{d}\varphi}{|g(ne^{i\varphi},t)|},$$

where $c = \varepsilon + \sup \alpha(t) + \sup \beta(t) + 1,\ j = 1, 2, 3$. We have

$$\begin{aligned} g(ne^{i\varphi}, t) = &- n^2 e^{2i\varphi} \alpha(t) + ine^{i\varphi} \widetilde{\beta}(t) + \widetilde{\gamma}(t) \\ = &- n^2 e^{2i\varphi} \alpha(t) \cos 2\varphi - ne^{i\varphi} \widetilde{\beta}(t) \sin \varphi + \widetilde{\gamma}(t) \\ &+ i(-n^2 \alpha(t) \sin 2\varphi + n\widetilde{\beta}(t) \cos \varphi), \end{aligned}$$

where $\widetilde{\beta}(t) = 2\varepsilon + \beta(t)$, $\widetilde{\gamma}(t) = \varepsilon^2 \alpha(t) + \varepsilon \beta(t) + 1$. Hence

$$\begin{aligned} |g(ne^{i\varphi}, t)|^2 = n^4 \alpha(t)^2 + n^2 \widetilde{\beta}(t)^2 + \widetilde{\gamma}(t)^2 - 2n^3 \alpha(t) \widetilde{\beta}(t) \sin \varphi \\ - 2n^2 \alpha(t) \widetilde{\gamma}(t) \cos 2\varphi - 2n \widetilde{\beta}(t) \widetilde{\gamma}(t) \sin \varphi. \end{aligned}$$

Since $\sin \varphi \le 0$ for $\pi \le \varphi \le 2\pi$ and $\widetilde{\beta}(t) \ge 0$, $\widetilde{\gamma}(t) \ge 1$, we obtain an estimate

$$|g(ne^{i\varphi}, t)|^2 \ge n^2 \alpha(t). \tag{2.8}$$

So, we have obtained

$$\int_{-n}^{n} |p_j(\lambda, t)| \mathrm{d}\lambda \le c\alpha(t)^{-1/2},$$

where c does not depend on t. Hence the integrals in (ii) exist for all $x \in \mathcal{R}(M^{1/2})$. We are now in position to use Fubini theorem on (2.3), which leads to

$$\int_{-n}^{n} \int_{0}^{\Xi} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}(E(t)x, x) \mathrm{d}\lambda = \int_{0}^{\Xi} \left(\int_{-n}^{n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)x, x).$$

Set $f_n^j(t) = \int_{-n}^{n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda$, $j = 1, 2, 3$. Then

$$f_n^j(t) = \int_{\Upsilon_n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda + \int_{\Gamma_n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda, \tag{2.9}$$

where Υ_n and Γ_n are as in (2.4). From (2.5) and (2.6) we obtain

$$\int_{\Upsilon_n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda = 0,\ j = 1, 2, 3, n \in \mathbb{N}.$$

To estimate the second integral in (2.9) we use the well–known Jordan lemma ([10, Lemma 9.2]) which implies

$$\left| \int_{\Gamma_n} e^{i\lambda s} p_j(\lambda, t) \mathrm{d}\lambda \right| \le c \max\{|p_j(\lambda, t)| : \lambda \in \Gamma_n\}.$$

Now (2.8) implies

$$\max\{|p_j(\lambda, t)| : \lambda \in \Gamma_n\} \le (1 + \varepsilon)\alpha(t)^{1/2} + \frac{\beta(t)}{\alpha(t)^{1/2}},\ n \in \mathbb{N},$$

hence $|f_n^j(t)| \le f(t)$, where

$$f(t) = (1 + \varepsilon)\alpha(t)^{1/2} + \frac{\beta(t)}{\alpha(t)^{1/2}}.$$

Since f is integrable for all Stieltjes measures generated by $x \in \mathcal{R}(M^{1/2})$, we can use Lebesgue dominate convergence theorem to obtain

$$\begin{aligned}\lim_{n\to\infty} \int_0^{\Xi} \left(\int_{-n}^{n} e^{i\lambda s} p_j(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)x,x) \\ = \int_0^{\Xi} \left(\int_{-\infty}^{\infty} e^{i\lambda s} p_j(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)x,x),\end{aligned}$$

for all $x \in \mathcal{R}(M^{1/2})$, in the sense of the principal value integral.

Since $(E(t)x,y)$ can be expressed by the polarization formula

$$\begin{aligned}(E(t)x,y) = \frac{1}{4}(E(t)(x+y,x+y) - (E(t)(x-y),x-y) \\ + i(E(t)(x+iy),x+iy) - i(E(t)(x-iy),x-iy)),\end{aligned}$$

we obtain

$$\begin{aligned}(T(s)\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)) = & -e^{\varepsilon s}\frac{1}{2\pi}\left[\int_0^{\Xi} \left(\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)x_1,x_2) \right. \\ & - \int_0^{\Xi} \left(\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)y_1,x_2) \\ & + \int_0^{\Xi} \left(\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}(E(t)x_1,y_2) \\ & \left. + \int_0^{\Xi} \left(\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t) \mathrm{d}\lambda \right) \mathrm{d}\,(E(t)y_1,y_2) \right],\end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in \mathcal{R}(M^{1/2})$.

Hence we can write

(2.10)
$$\begin{aligned}& T(s)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \\ & = -\frac{1}{2\pi}e^{\varepsilon s}\begin{bmatrix} \int_0^{\Xi}\int_{-\infty}^{\infty} e^{i\lambda s}p_1(\lambda,t)\mathrm{d}\lambda \mathrm{d}E(t)x - \int_0^{\Xi}\int_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda \mathrm{d}E(t)y \\ \int_0^{\Xi}\int_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda \mathrm{d}E(t)x + \int_0^{\Xi}\int_{-\infty}^{\infty} e^{i\lambda s}p_3(\lambda,t)\mathrm{d}\lambda \mathrm{d}E(t)y \end{bmatrix}\end{aligned}$$

in the sense of Pettis integral (for the definition and the basic properties see [11, Chapter 3]). Moreover, the formula (2.10) holds for all $x, y \in \mathcal{Y}$, which easily follows from the fact that $T(s)$ is a bounded operator.

Our next aim is to compute the integrals $\int_{-\infty}^{\infty} e^{i\lambda s} p_j(\lambda,t)\mathrm{d}\lambda$, $j = 1,2,3$ and hence to obtain an integral representation of $T(s)$ in terms of the spectral function $E(t)$. Since $p_j(\cdot,t)$ are rational functions such that the degree of the denominator is greater of the degree of the nominator, the standard result from the calculus (see [10]) implies that

$$\int_{-\infty}^{\infty} e^{i\lambda s} p_j(\lambda,t)\mathrm{d}\lambda = 2\pi i \left(\sum_{\lambda\in S_+} \mathrm{Res}(e^{is\cdot}p_j(\cdot,t);\lambda) + \frac{1}{2}\sum_{\lambda\in S_0} \mathrm{Res}(e^{is\cdot}p_j(\cdot,t);\lambda)\right),$$

where S_+ is the set of all poles of the function $e^{is\cdot}p_j(\cdot,t)$ in the upper half plane, and S_0 is the set of all real poles of the function $e^{is\cdot}p_j(\cdot,t)$. The poles of the functions $e^{is\cdot}p_j(\cdot,t)$ are exactly the zeros of the function $g(\cdot,t)$ which are calculated in (2.5) and (2.6).

From the straightforward calculation we obtain:
(i) in the case $4\alpha(t) > \beta(t)^2$

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t)\mathrm{d}\lambda \\
&\qquad = -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cos(\varrho(t)s) + \frac{\beta(t)}{\sqrt{4\alpha(t)-\beta(t)^2}}\sin(\varrho(t)s)\right),\\
&\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t)\mathrm{d}\lambda = 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \frac{\sqrt{\alpha(t)}}{\sqrt{4\alpha(t)-\beta(t)^2}}\sin(\varrho(t)s),\\
&\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t)\mathrm{d}\lambda \\
&\qquad = -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cos(\varrho(t)s) - \frac{\beta(t)}{\sqrt{4\alpha(t)-\beta(t)^2}}\sin(\varrho(t)s)\right),
\end{aligned}$$

where $\varrho(t) = \frac{\sqrt{4\alpha(t)-\beta(t)^2}}{2\alpha(t)}$,
(ii) in the case $4\alpha(t) < \beta(t)^2$

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t)\mathrm{d}\lambda \\
&\qquad = -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cosh(\widetilde{\varrho}(t)s) + \frac{\beta(t)}{\sqrt{\beta(t)^2-4\alpha(t)}}\sinh(\widetilde{\varrho}(t)s)\right),\\
&\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t)\mathrm{d}\lambda = 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \frac{\sqrt{\alpha(t)}}{\sqrt{\beta(t)^2-4\alpha(t)}}\sinh(\widetilde{\varrho}(t)s),\\
&\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t)\mathrm{d}\lambda \\
&\qquad = -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cosh(\widetilde{\varrho}(t)s) - \frac{\beta(t)}{\sqrt{\beta(t)^2-4\alpha(t)}}\sinh(\widetilde{\varrho}(t)s)\right),
\end{aligned}$$

where $\widetilde{\varrho}(t) = \frac{\sqrt{\beta(t)^2-4\alpha(t)}}{2\alpha(t)}$,
(iii) and in the (limit) case $4\alpha(t) = \beta(t)^2$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t)\mathrm{d}\lambda &= -\,2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(1 + s\frac{\beta(t)}{2\alpha(t)}\right),\\
\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t)\mathrm{d}\lambda &= 2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} s\alpha(t)^{-1/2},\\
\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t)\mathrm{d}\lambda &= -\,2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(1 - s\frac{\beta(t)}{2\alpha(t)}\right).
\end{aligned}$$

Let us define functions

$$\widetilde{\sin}(t,s) = \begin{cases} \frac{\sin(\varrho(t)s)}{\sqrt{4\alpha(t)-\beta(t)^2}}, & \varrho(t) \in \mathbb{R}\setminus\{0\},\\ \frac{s}{2\alpha(t)}, & \varrho(t) = 0,\\ \frac{\sinh(\widetilde{\varrho}(t)s)}{\sqrt{\beta(t)^2-4\alpha(t)}}, & \widetilde{\varrho}(t) \in \mathbb{R}\setminus\{0\}, \end{cases}$$

and

$$\widetilde{\cos}(t,s) = \begin{cases} \cos(\varrho(t)s), & \varrho(t) \in \mathbb{R},\\ \cosh(\widetilde{\varrho}(t)s), & \widetilde{\varrho}(t) \in \mathbb{R}, \end{cases}.$$

Then we can write

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t)\mathrm{d}\lambda &= -\,2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s)\right),\\
\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t)\mathrm{d}\lambda &= 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \sqrt{\alpha(t)}\,\widetilde{\sin}(t,s),\\
\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t)\mathrm{d}\lambda &= -\,2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s)\right),
\end{aligned}$$

in all three cases.

Hence we have obtained

$$T(s) = \int_0^{\Xi} e^{-s\frac{\beta(t)}{2\alpha(t)}} \begin{bmatrix} \widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s) & 2\sqrt{\alpha(t)}\,\widetilde{\sin}(t,s)\\ -2\sqrt{\alpha(t)}\,\widetilde{\sin}(t,s) & \widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s) \end{bmatrix} \mathrm{d}E(t).$$

Now we are in position to use the formula (1.14) in order to calculate the operator X. Set

$$\begin{aligned}
q_1(t,s) &= e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s)\right),\\
q_2(t,s) &= 2e^{-s\frac{\beta(t)}{2\alpha(t)}} \sqrt{\alpha(t)}\,\widetilde{\sin}(t,s),\\
q_3(t,s) &= e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s)\right).
\end{aligned}$$

Then

$$\begin{aligned}(X\left(\begin{smallmatrix}x_1\\ y_1\end{smallmatrix}\right),\left(\begin{smallmatrix}x_2\\ y_2\end{smallmatrix}\right)) = \int_0^\infty \Bigg(&\int_0^\Xi (q_1(t,s)^2 + q_2(t,s)^2)\mathrm{d}(E(t)x_1, x_2) \\ &+ \int_0^\Xi (q_1(t,s) - q_3(t,s))q_2(t,s)\mathrm{d}(E(t)x_1, y_2) \\ &+ \int_0^\Xi (q_1(t,s) - q_3(t,s))q_2(t,s)\mathrm{d}(E(t)y_1, x_2) \\ &+ \int_0^\Xi (q_3(t,s)^2 + q_2(t,s)^2)\mathrm{d}(E(t)y_1, y_2)\Bigg)\mathrm{d}s.\end{aligned} \tag{2.11}$$

As before, we would like to change the order of integration in the previous formula. To do that, it is sufficient to prove that conditions (i) and (ii) from page 381 are satisfied. The condition (i) is obviously satisfied.

Note that $(q_1(t,s)-q_3(t,s))q_2(t,s) \geq 0$ for all $t, s > 0$, hence all functions in (2.11) are positive. By the use of the standard integration formulas one obtains

$$\int_0^\infty (q_1(t,s)^2 + q_2(t,s)^2)\mathrm{d}s = \frac{1}{2}\beta(t) + \frac{\alpha(t)}{\beta(t)},$$

$$\int_0^\infty (q_1(t,s) - q_3(t,s))q_2(t,s)\mathrm{d}s = \frac{1}{2}\sqrt{\alpha(t)},$$

$$\int_0^\infty (q_3(t,s)^2 + q_2(t,s)^2\mathrm{d}s = \frac{\alpha(t)}{\beta(t)}.$$

Hence to be able to change the order of integration the function α/β has to be bounded measurable.

Remark 2.2. The previous considerations and Datko–Pazy theorem ([8]) imply that the system decays exponentially if and only if there exist representations (2.1) such that α/β is bounded measurable. In terms of the operators M and C this condition is equivalent to the condition $C \geq \Delta M$, for some $\Delta > 0$.

From now on we assume that the function α/β is bounded measurable. Then we can write

$$X = \int_0^\Xi \begin{bmatrix} \frac{1}{2}\beta(t) + \frac{\alpha(t)}{\beta(t)} & \frac{1}{2}\sqrt{\alpha(t)} \\ \frac{1}{2}\sqrt{\alpha(t)} & \frac{\alpha(t)}{\beta(t)} \end{bmatrix} \mathrm{d}E(t).$$

Note that this formula is a direct generalization of the formula in the matrix case given in [7] (see also [6], [22]).

Now it is obvious that to obtain the optimal C it is enough to solve 2×2 matrix case with

$$Y = \begin{bmatrix} \frac{1}{2}\beta(t) + \frac{\alpha(t)}{\beta(t)} & \frac{1}{2}\sqrt{\alpha(t)} \\ \frac{1}{2}\sqrt{\alpha(t)} & \frac{\alpha(t)}{\beta(t)} \end{bmatrix}.$$

Hence we have to solve the following minimization problem:

$$\int_{\|y\|=1} y^* Y y \mu(\mathrm{d}y) \to \min.$$

Since the map $Y \to \int_{\|y\|=1} y^* Y y \mu(\mathrm{d}y)$ is a linear functional on the space of symmetric matrices with trace inner product, by Riesz representation theorem there exists a symmetric matrix Z such that $\int_{\|y\|=1} y^* Y y \mu(\mathrm{d}y) = \mathrm{Trace}(YZ)$. Our choice of μ implies that Z is a diagonal matrix. By putting formally $Z = I$, we see that the optimal energy decay on the set of damping operators C which commute with M and for which the system is exponentially stable is attained for the operator C_{opt} which has a spectral function β_{opt} such that $\frac{1}{2}\beta(t) + 2\frac{\alpha(t)}{\beta(t)} \to \min$ for all $t > 0$. One can easily see that $\beta_{\mathrm{opt}}(t) = 2\sqrt{\alpha(t)}$, i.e.

$$C_{\mathrm{opt}} = 2M^{1/2},$$

which corresponds to the well-known result in the matrix case.

Remark 2.3. This result is mainly of theoretical nature, because, in most cases, the damping $C = 2M^{1/2}$ is not physically realizable. But it gives an upper bound for the best possible dissipation of vibration energy, and can also have value for the numerical treatment of the problem.

Example 2.4. An important class of vibrational systems are systems with so–called analytic damping (see [12], [4], [20]). Analytic damping corresponds to the case where $C = \beta M^{\alpha}$, hence it is a special case of the modal damping. The main result of this paper shows that the optimal analytic damping is given for the exponent $1/2$, which is the smallest number for which the corresponding semigroup is holomorphic.

An example of a system with analytic damping is the system from example 1.1. Hence the optimal damping coefficient in this case is $\rho = 2$.

The main conclusion of the paper is that the well–known fact that the optimal modal damping matrix is given by $C_{\mathrm{opt}} = 2M^{1/2}$ also holds in the infinite–dimensonal case.

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