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QUASILINEAR ELLIPTIC EQUATIONS WITH POSITIVE EXPONENT ON THE GRADIENT

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Abstract. We study the existence and nonexistence of positive, spherically symmetric solutions of a quasilinear elliptic equation (1.1) involving p-Laplace operator, with an arbitrary positive growth rate e_0 on the gradient on the right-hand side. We show that $e_0 = p - 1$ is the critical exponent: for $e_0 < p-1$ there exists a strong solution for any choice of the coefficients, which is a known result, while for $e_0 > p - 1$ we have existence-nonexistence splitting of the coefficients $\tilde{f}_0$ and $\tilde{g}_0$. The elliptic problem is studied by relating it to the corresponding singular ODE of the first order. We give sufficient conditions for a strong radial solution to be the weak solution. We also examined when ω-solutions of (1.1), defined in Definition 2.3, are weak solutions. We found conditions under which strong solutions are weak solutions in the critical case of $e_0 = p - 1$.

1. Introduction

In the last several decades a lot of authors have studied the problems of solvability of quasilinear elliptic equations including p-Laplace operator. The existence of strong and weak solutions of quasilinear elliptic problems, where an arbitrary growth rate in the unknown function, as well as on the gradient, is allowed, have been studied by Žubrinić ([12]). This paper is a continuation of the paper [8] by Korkut, Pašić, and Žubrinić and it requires some additional conditions for solvability, which were found out as a sort of nonresonance conditions. In the paper [2] Abdellaoui, Dall'Aglio, and Peral analyzed existence, nonexistence, multiplicity and regularity of solutions of nonlinear elliptic equations, including the usual Laplacian on the open bounded set in $\mathbb{R}^N$, with a sufficiently smooth boundary. In [3] Abdellaoui,

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Giachetti, Peral, and Walias considered an elliptic problem which also includes dependence on the gradient, with homogeneous Dirichlet boundary condition, associated to the model of growth in a porous media. Dirichlet and Neumann problems for a class of nonlinear degenerate elliptic equations with general growth in the gradient have been studied by Tian and Li in [10]. In the paper [5] by Chen and Wang the existence, asymptotic behavior near the boundary and uniqueness of large solutions for a class of quasilinear elliptic equations with a nonlinear gradient have been studied. Analogous problems to those treated in this paper have been considered among others by Abdellaoui in [1] and Hansson, Maz'ya and Verbitsky in [7]. Very interesting results are given in [7], where the authors consider a nonlinear Dirichlet problem on a bounded domain. The authors studied existence of the positive solutions of $-\Delta u = a|\nabla u|^{q_1} + b|u|^{q_2} + \omega$, with $q_1, q_2 > 1$ in $\mathbb{R}^N$, with bounded nonnegative coefficients a and b and arbitrary nonnegative function or measure ω. The solvability of this problem is considered in the weak sense. From the known estimates of Green functions of uniformly elliptic differential operators, their main results remain true with this operator instead of the Laplacian. The mentioned paper does not consider the problem of nonexistence of solutions. The problem of existence and multiplicity of infinitely many radial solutions to the problem that includes the p-Laplacian and the gradient in the unknown function and a radial positive function on the right-hand side, has been considered by Abdellaoui in [1]. The author also gives conditions under which the mentioned problem has no positive solutions. The solutions are considered in the weak sense.

In our paper we consider problems of existence and nonexistence of spherically symmetric strong solutions of quasilinear elliptic equations in the case where the exponent on the gradient is any positive real number.

The aim of this paper is to study existence and nonexistence of strong solutions of the following quasilinear elliptic problem with strong dependence on the gradient:

$$\begin{cases} -\Delta_p u = \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla u|^{e_0} \quad \text{in } B \setminus \{0\}, \\ \qquad u = 0 \text{ on } \partial B, \\ u(x) \text{ spherically symmetric and decreasing.} \end{cases} \tag{1.1}$$

Here B is an open ball of radius R centered at the origin in $\mathbb{R}^N$, $1 < p < \infty$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian. The Lebesgue measure (volume) of B in $\mathbb{R}^N$ is denoted by $|B|$, and the volume of the unit ball is denoted by C_N. The dual exponent of $p > 1$ is defined by $p' = \frac{p}{p-1}$. We assume that $\tilde{g}_0$, $\tilde{f}_0$ and e_0 are positive real numbers. By a strong solution we mean a function $u \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ which satisfies (1.1) pointwise. We shall also study the question under which conditions the ω-solutions of (1.1), (see Definition 2.3) are the weak solutions. In [8] Korkut, Pašić, Žubrinić studied the case when (1.1) has the natural growth in the gradient, that is, $e_0 = p$. The aim of this paper is to study the question of solvability and nonsolvability of (1.1) for all positive e_0.

We show that the critical exponent on the gradient in (1.1) is $e_0 = p-1$ in the following sence: if $0 < e_0 < p-1$ problem (1.1) is solvable for all positive $\tilde{g}_0$ and $\tilde{f}_0$,(which is known result), while for $e_0 > p-1$ we have nontrivial existence and nonexistence regions in the positive quadrant of $(\tilde{g}_0, \tilde{f}_0)$-plane.

THEOREM 1.1. *Assume that $m > \max\{-p, -N\}$, $N \geq 2$.*

(a) *If $0 < e_0 < p-1$ and $m \leq e_0/(p-e_0-1)$, then the problem (1.1) possesses a strong solution for all positive $\tilde{g}_0$ and $\tilde{f}_0$.*

(b) *Let $e_0 > p-1$. Then there exist two explicit positive constants $\tilde{C}_1$ and $\tilde{C}_2$, $\tilde{C}_1 < \tilde{C}_2$, such that*

(b1) *if $m \leq e_0/(p-e_0-1)$, $m \neq -1$ and if*

$$\tilde{f}_0\, \tilde{g}_0^{\frac{e_0}{p-1}-1} \leq \tilde{C}_1, \tag{1.2}$$

then there exists a strong solution of quasilinear elliptic problem (1.1),

(b2) *if $m < e_0/(p-1-e_0)$ and*

$$\tilde{f}_0\, \tilde{g}_0^{\frac{e_0}{p-1}-1} \geq \tilde{C}_2, \tag{1.3}$$

then problem (1.1) has no strong solutions.

The explicit values of $\tilde{C}_1$ and $\tilde{C}_2$ can be seen in (2.13) and (2.14) respectively.

2. Reduction to a singular ODE

We prove the existence result stated in Theorem 1.1 by studying the corresponding singular ODE (see [8])

$$\frac{d\omega}{dt} = g_0 \gamma t^{\gamma-1} + f_0 \frac{\omega(t)^\delta}{t^\varepsilon} \quad , \quad t \in (0, T], \tag{2.1}$$

where ε, g_0, f_0, δ, T are positive constants, that will be defined in Lemma 2.1. For $\varepsilon > 0$ this equation is singular at $t = 0$. In order to formulate the existence result of ODE (2.1) we shall seek solutions in the set:

$$D = \{\varphi \in C([0,T]) : \exists M > 0,\ 0 \leq \varphi(t) \leq M t^\gamma\}, \tag{2.2}$$

for $\gamma > 0$.

In order to formulate the nonexistence result of ODE (2.1) we introduce the set:

$$D^+ = \{\varphi \in C([0,T]) : \varphi(t) \geq 0 \text{ and nondecreasing}\}. \tag{2.3}$$

We shall use among others the following result which has been proved by Korkut, Pašić, Žubrinić in [8, Theorem 5 and Theorem 7]. This result shows that $\delta = 1$ is the critical case for (2.1).

LEMMA 2.1 (see [8]). *Assume that* $\gamma > 0$, $\delta > 0$, $\delta \geq \frac{\varepsilon-1}{\gamma} + 1$ *and* $\varepsilon \in \mathbb{R}$.

(a) *If* $\delta < 1$ *then (2.1) possesses a solution in* D *for all positive* g_0 *and* f_0.

(b) *Let* $\delta > 1$. *Then there exist two explicit positive constants* C_1 *and* C_2, $C_1 < C_2$, *such that*

(b1) *If*

$$f_0 \, g_0^{\delta-1} \leq C_1, \tag{2.4}$$

then there exists a solutin of (2.1) in D.

(b2) *If* $\delta > \frac{\varepsilon-1}{\gamma} + 1$ *and if*

$$f_0 \, g_0^{\delta-1} \geq C_2, \tag{2.5}$$

then the ODE (2.1) has no solutions in D^+.

In [8] the following explicit values of constants C_1 and C_2 have been obtained:

$$C_1 = \frac{\gamma\delta - \varepsilon + 1}{\delta T^{\gamma(\delta-1)-\varepsilon+1}(\delta')^{\delta-1}}, \tag{2.6}$$

and

$$C_2 = \begin{cases} \dfrac{[\gamma(\delta-1)-\varepsilon+1]\delta^{\delta'}}{(\delta-1)T^{\gamma(\delta-1)-\varepsilon+1}} & \text{for } \varepsilon < 1, \\ \dfrac{\gamma\,\delta^{\delta'}}{T^{\gamma(\delta-1)-\varepsilon+1}} & \text{for } \varepsilon \geq 1. \end{cases} \tag{2.7}$$

In the critical case, that is, when $\delta = 1$, we have the ordinary differential equation of the first order.

To prove Theorem 1.1 we state two lemmas. The first one enables to generate solutions of quasilinear elliptic equation (1.1) using solutions of singular ODE (2.1) with a suitable choice of coefficients which we shall define later. It represents a modification of [8, Lemma 1]. The second lemma will be used in the proof of nonexistence part of Theorem 1.1 and represents a modification of [8, Lemma 2]. Their proofs can be obtained using simple changes of the corresponding proofs in [8] and therefore we omit them.

LEMMA 2.2 (see [8]). *Let* $\tilde{g}_0$ *and* $\tilde{f}_0$ *be positive real numbers. Assume that* $1 < p < \infty$, *and* $m > \max\{-p, -N\}$. *Let us define the constants:*

$$\gamma = 1 + \frac{m}{N}, \quad \delta = \frac{e_0}{p-1}, \quad \varepsilon = \delta\left(1 - \frac{1}{N}\right), \tag{2.8}$$

and

$$g_0 = \frac{\tilde{g}_0}{C_N^{\frac{m+p}{N}} N^{p-1}(m+N)}, \quad f_0 = \frac{\tilde{f}_0}{N^{p-e_0} C_N^{\frac{p-e_0}{N}}}. \tag{2.9}$$

Then for any solution $\omega \in D$ of ODE (2.1), where D is defined by (2.2) and $T = |B|$, with some $M > 0$, the corresponding function $u : B \to \mathbb{R}$, defined by

$$u(x) = \int_{C_N|x|^N}^{|B|} \frac{\omega(t)^{p'-1}}{t^{p'(1-\frac{1}{N})}}\, dt, \tag{2.10}$$

is a strong solution of quasilinear elliptic problem (1.1). Furthermore, the following relation holds for all $r \in (0, R]$:

$$\tilde{u}'(r) = -|\nabla u| = -NC_N^{1/N} \left(\frac{\omega(s)}{s^{1-\frac{1}{N}}}\right)^{p'-1}, \quad s = C_N|x|^N. \tag{2.11}$$

where $\tilde{u}(r)$ is defined with $\tilde{u}(r) = u(x)$, $r = |x|$,

Following the terminology introduced in [8], we provide the next definition.

DEFINITION 2.3. A function $u : B_R(0) \to \mathbb{R}$ is an *ω-solution* of quasilinear elliptic problem (1.1), if it is a strong solution which can be obtained as described in Lemma 2.2, generated by a solution ω of equation (2.1), with additional requirement that $0 \le \omega(t) \le Mt^{\gamma}$, for some $M > 0$.

The open question is if there are any other solutions of quasilinear elliptic problem (1.1) which are not ω-solutions. Here we identified the function $u(x)$ with the function $\tilde{u}(r)$, for $r = |x|$. Let as mention that in [1] the condition of existence of infinitely many positive radial solutions for the problem involving the p-Laplacian, with positive radial function and the gradient in the unknown function on the right-hand side have been given, but the solutions are considered in the weak sense.

LEMMA 2.4 (see [8]). *Let u be a strong solution of quasilinear elliptic problem (1.1), where the constants $\tilde{f}_0$, $\tilde{g}_0$ are positive real numbers. Assume that $m > \max\{-p, -N\}$. Let us define the function $V : (0, T] \to \mathbb{R}$ with*

$$V(s) = u(x), \quad s = C_N|x|^N,$$

and also the function $\omega : (0, T] \to \mathbb{R}$ with

$$\omega(s) = s^{p(1-\frac{1}{N})} \left|\frac{dV(s)}{ds}\right|^{p-1}, \quad s \in (0, T], \quad T = |B|.$$

Then the function ω satisfies the following ODE:

$$\begin{cases} \frac{d\omega(s)}{ds} = g_0\gamma s^{\gamma-1} + f_0\frac{\omega(s)^{\delta}}{s^{\varepsilon}}, & s \in (0, T), \\ \omega \in D^{+}, \end{cases} \tag{2.12}$$

where the constants g_0, f_0 are given by (2.9), D^{+} is defined by (2.3) and the constants γ, δ, ε are given by (2.8). Also, the function u has the following representation:

$$u(x) = V(s) = \int_s^T \frac{\omega(t)^{p'-1}}{t^{p'(1-\frac{1}{N})}}\, dt.$$

Note that Lemma 2.4 does not claim that any strong solution of (1.1) is ω-solution, since the condition $\omega(t) \le Mt^\gamma$ does not have to be fulfilled.

Proof of Theorem 1.1. To prove the existence of solutions stated in (a) and (b), we first define constants γ, δ, ε and T as in Lemma 2.2.

(a) Since $0 < e_0 < p-1$, we have that $0 < \delta < 1$. It is easy to see that conditions $m \le \frac{e_0}{p-e_0-1}$ is equivalent to $\delta \ge \frac{\varepsilon-1}{\gamma} + 1$, where we have used that $m > \max\{-p, -N\}$. From Lemma 2.1 we obtain that there exists a solution ω of singular equation (2.1) for each positive constants f_0 and g_0. By Lemma 2.2 the corresponding function u defined by (2.10) is a strong solution of (1.1).

(b1) Here $e_0 > p-1$ implies that $\delta > 1$. The condition on m is equivalent to $\delta > \frac{\varepsilon-1}{\gamma} + 1$. Let us define

$$\tilde{C}_1 = N^{p-e_0} C_N^{\frac{p-e_0}{N}} \left[C_N^{\frac{m+p}{N}} N^{p-1}(m+N) \right]^{\delta-1} C_1, \tag{2.13}$$

where C_1 is defined by (2.6). Because of the (2.9) and (2.8), inequality (1.2) is then equivalent to (2.4), so by Lemma 2.1 and Lemma 2.2 the existence part (b1) of Theorem 1.1 has been proved.

(b2) We prove the nonexistence part by contradiction. Let us define:

$$\tilde{C}_2 = N^{p-e_0} C_N^{\frac{p-e_0}{N}} \left[C_N^{\frac{m+p}{N}} N^{p-1} (m+N) \right]^{\delta-1} C_2\,, \tag{2.14}$$

where C_2 is given by (2.7). Condition (1.3) is equivalent to (2.5) and by Lemma 2.4 the nonexistence part (b2) of Theorem 1.1 has been proved. □

Remark 2.5. *Using a rescaling argument, problem (1.1) can be related to the following simpler one:*

$$\begin{cases} -\Delta_p u = \tilde{g}_0 \tilde{f}_0^{\frac{p-1}{e_0-p+1}} |x|^m + |\nabla u|^{e_0} \quad \textit{in } B \setminus \{0\}, \\ \quad u = 0 \ \textit{on } \partial B, \\ u(x) \ \textit{spherically symmetric and decreasing.} \end{cases} \tag{2.15}$$

For the critical case when $e_0 = p-1$, exploiting Ascoli's theorem as well as Schauder's fixed point theorem it is easy to see that under some conditions, problem (1.1) has at least one strong solution. We consider the following problem:

$$\begin{cases} -\Delta_p u = \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla u|^{p-1} \quad \text{in } B \setminus \{0\}, \\ \quad u = 0 \text{ on } \partial B, \\ u(x) \text{ spherically symmetric and decreasing.} \end{cases} \tag{2.16}$$

The main result is given in the following theorem.

Theorem 2.6. *Assume that $m > \max\{-p, -N\}$, $N \ge 2$. Let $\tilde{g}_0$ be any positive real number and let $0 < \tilde{f}_0 < NC_N^{\frac{1}{N}}(\gamma + \frac{1}{N})T^{-\frac{1}{N}}$. Then the problem (2.16) has at least one strong solution.*

3. Existence of weak solutions

In Theorem 1.1 we have proved more than just existence of strong solutions: there exists ω-solutions of quasilinear elliptic problem (1.1). This follows from the fact that in existence part of Theorem 1.1 we have $\omega \in D$. In this section we want to examine when ω-solutions of (1.1) are weak solutions. We shall also find conditions under which the ω-solutions are the weak solutions in the critical case of $e_0 = p - 1$.

Let us recall that the function u is the weak solution of quasilinear elliptic problem (1.1) if $u \in W_0^{1,p}(B) \cap L^\infty(B)$ and equation (1.1) is satisfied in the weak sense:

$$\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \tilde{g}_0 \int_B |x|^m \varphi(x) \, dx + \tilde{f}_0 \int_B |\nabla u|^{e_0} \varphi(x) \, dx,$$

for all test functions $\varphi \in W_0^{1,p}(B) \cap L^\infty(B)$.

The main result in this section is given in the Theorem 3.3 in which we found general sufficient conditions for any strong ω-solution to be a weak solution. We shall consider a more general quasilinear elliptic problem, given in the following form:

$$\begin{cases} -\Delta_p u = F(|x|, u, |\nabla u|) \quad \text{in } B \setminus \{0\}, \\ \qquad u = 0 \text{ on } \partial B, \\ u(x) \text{ spherically symmetric and decreasing.} \end{cases} \tag{3.1}$$

Next we shall use [4, Theorem IX.1.7], which provides the conditions under which a function $u \in W^{1,p}(\Omega)$ belongs to the space $W_0^{1,p}(\Omega)$.

Theorem 3.1. *Assume that Ω is a domain of class C^1 in $\mathbb{R}^N$ and let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, where $1 \le p < \infty$. Then the following two assertions are equivalent:*

(i) $u = 0$ *on* $\partial\Omega$

(ii) $u \in W_0^{1,p}(\Omega)$.

The next Lemma represents a generalization of [8, Proposition 11]. We give a sufficient conditions for a strong radial solution to be the weak solution.

Lemma 3.2. *Let $N \ge 2$. Assume that $F : (0, R] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, where $\mathbb{R}_+ = [0, \infty)$. Let $u \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ be a strong solution of quasilinear elliptic problem (3.1). Assume that the following four conditions are fulfilled:*

(i) $F(|x|, u, |\nabla u|) \in L^1(B)$;

(ii) $I_\varepsilon := \varepsilon^{N-1} \tilde{u}(\varepsilon) \to 0$, *when* $\varepsilon \to 0$, *where* $\tilde{u}(\varepsilon) = u(x)$, $\varepsilon = |x|$;

(iii) $J_\varepsilon := \varepsilon^{N-1} |\tilde{u}'(\varepsilon)|^{p-1} \to 0$, *when* $\varepsilon \to 0$;

(iv) $\int_B |\nabla u|^p dx < \infty$.

Then u is the weak solution of quasilinear elliptic problem (3.1).

PROOF. (a1) First, let us show that for $i = 1, \dots, N$, the pointwise derivative $\frac{\partial u}{\partial x_i}$ of solution u of the quasilinear elliptic problem (3.1) is also the weak derivative. Since u is continuous on $\overline{B}$, it is integrable on B. For any $i = 1, \dots, N$ we can write

$$\int_B u \frac{\partial \varphi}{\partial x_i}\, dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u \frac{\partial \varphi}{\partial x_i}\, dx,$$

where $\varphi \in C_0^\infty(B)$ is any test function. Here, $\Omega_\varepsilon = B \setminus B_\varepsilon(0)$, where $B_\varepsilon(0)$ is the open ball of radius $\varepsilon > 0$, centered at the origin in $\mathbb{R}^N$. Let S_ε be the inner bounding sphere of Ω_ε. So, the set S_ε is the boundary of the ball $B_\varepsilon(0)$. Since Ω_ε is open and of class C^1, using Green's formula (formula of integration by parts) we have

$$\int_{\Omega_\varepsilon} u \frac{\partial \varphi}{\partial x_i}\, dx \; = -\int_{\Omega_\varepsilon} \frac{\partial u}{\partial x_i} \varphi\, dx + \int_{S_\varepsilon} u\, \varphi\, \nu_i\, dS.$$

Here, ν_i is the i-component of the unit normal outward vector $\nu = (\nu_1, \nu_2, \dots, \nu_N)$ at the point x, $|x| = \varepsilon$, with respect to Ω_ε. We have

$$\left| \int_{S_\varepsilon} u\, \varphi\, \nu_i\, dS \right| \le C |\tilde{u}(\varepsilon)| \varepsilon^{N-1},$$

where C is some positive constant. Using the assumption (ii) we see that

$$\lim_{\varepsilon \to 0} \left| \int_{S_\varepsilon} u\, \varphi \nu_i\, dS \right| = 0,$$

so, the last integral tends to zero as $\varepsilon \to 0$. We have

$$\int_B u \frac{\partial \varphi}{\partial x_i}\, dx = -\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{\partial u}{\partial x_i} \varphi\, dx = -\int_B \frac{\partial u}{\partial x_i} \varphi\, dx.$$

(a2) Let us prove now that any strong solution u of (3.1) is also a weak solution. Since u is the strong solution, it satisfies (3.1) pointwise in $\Omega_\varepsilon \subseteq B$. Let φ be any function that belongs to $W_0^{1,p}(B) \cap L^\infty(B)$. This together with Green's formula yields:

$$\int_{\Omega_\varepsilon} u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{\Omega_\varepsilon} \frac{\partial u}{\partial x_i} \varphi\, dx + \int_{S_\varepsilon} u \varphi \nu_i\, dS.$$

In our case we have:

$$\int_{\Omega_\varepsilon} F(|x|, u, |\nabla u|)\, \varphi\, dx = -\int_{\Omega_\varepsilon} \Delta_p u\, \varphi\, dx = -\int_{\Omega_\varepsilon} \operatorname{div}(|\nabla u|^{p-2} \nabla u) \varphi\, dx.$$

Since

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) &= \operatorname{div}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_1}, \dots, |\nabla u|^{p-2} \frac{\partial u}{\partial x_N} \right) \\ &= \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right), \end{aligned}$$

we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon}\Delta_p u\,\varphi\,dx &= \sum_{i=1}^N\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)\varphi\,dx\\ &= \sum_{i=1}^N\left[-\int_{\Omega_\varepsilon}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_i}\,dx+\int_{S_\varepsilon}|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\varphi\nu_i\,dS\right]\\ &= -\int_{\Omega_\varepsilon}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx+\int_{S_\varepsilon}\sum_{i=1}^N|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\varphi\nu_i\,dS,\end{aligned}$$

from which we conclude that

$$\int_{\Omega_\varepsilon}F(|x|,u,|\nabla u|)\,\varphi\,dx=\int_{\Omega_\varepsilon}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx-\int_{S_\varepsilon}\sum_{i=1}^N|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\,\varphi\,\nu_i\,dS.$$

Since the function φ is bounded on S_ε, we have

$$\left|\int_{S_\varepsilon}\sum_{i=1}^N|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\,\varphi\nu_i\,dS\right|\le C\int_{S_\varepsilon}\sum_{i=1}^N|\nabla u|^{p-1}\,dS\le C\sum_{i=1}^N|\tilde u'(\varepsilon)|^{p-1}\int_{S_\varepsilon}dS,$$

where C is some positive constant which changes from line to line. Here we use that $|\tilde u'(\varepsilon)|=|\nabla u(x)|$, where $\tilde u(\varepsilon)=u(x)$ for $|x|=\varepsilon$ and that $|\frac{\partial u}{\partial x_i}|\le|\nabla u|$. Since $\int_{S_\varepsilon}dS=C\varepsilon^{N-1}$, where C is some positive constant, from assumption (iii) in Lemma 3.2 it follows that $\sum_{i=1}^N|\tilde u'(\varepsilon)|^{p-1}\varepsilon^{N-1}\to 0$ when $\varepsilon\to 0$. Passing to the limit as $\varepsilon\to 0$, we see that

$$\int_B F(|x|,u,|\nabla u|)\varphi\,dx=\int_B|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx,$$

for every test function φ in $W_0^{1,p}(B)\cap L^\infty(B)$. The left-hand side is well defined by the assumption (i) in Lemma 3.2, so it follows that $-\Delta_p u=F(|x|,u,|\nabla u|)$ is satisfied in the weak sense. Finally, from (iv) in Lemma 3.2 and from (a1), we conclude that the solution u belongs to $W^{1,p}(B)$. Since the open ball B is of class C^1, the function u as a strong solution belongs to $C^2(B\setminus\{0\})\cap C(\overline{B})$ and since $u=0$ on ∂B, it follows from the Theorem 3.1 that u belongs to $W_0^{1,p}(B)$. ∎

The claim of the above lemma clearly holds also for radial solutions of (1.1) without any assumption that u be positive and decreasing.

In the next theorem we shall give the conditions for any ω-solution of quasilinear elliptic problem (1.1) to be a weak solution. It extends [11, Lemma 6.1]

Theorem 3.3. *Assume that $N\ge 2$, $m>\max\{-p,-N\}$, and $m>-1-\frac{N(p-1)}{e_0}$. Then any ω-solution of quasilinear elliptic problem (1.1) is also the weak solution.*

PROOF. In the proof of Theorem 1.1, the strong solutions were also the ω-solutions given by integral representation $u(x) = \int_{C_N|x|^N}^{|B|} \frac{\omega(t)^{p'-1}}{t^{p'(1-\frac{1}{N})}}\,dt$, with $0 \le \omega(t) \le Mt^\gamma$, where M was big enough positive constant and $\gamma = 1+\frac{m}{N}$. If we compare (1.1) with the general problem (3.1), we see that $F(|x|, u, |\nabla u|) = \tilde{g}_0|x|^m + \tilde{f}_0|\nabla u|^{e_0}$. It suffices to show that for $F(|x|, u, |\nabla u|)$, the conditions (i)-(iv) of Lemma 3.2 are satisfied. Let $u \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ be any ω-solution of (1.1). First, let us show that the condition (i) from Lemma 3.2 is satisfied. Since $m > -N$, for any $x \in B\backslash\{0\}$ the condition $\int_{B\backslash\{0\}} |x|^m\,dx < \infty$ is fulfilled. It follows that

$$\int_B |x|^m\,dx = \int_0^R r^m r^{N-1}\,dr \int_{S_1(0)} dS(y) = \frac{R^{m+N}}{m+N} \cdot \omega_N < \infty,$$

so, $|x|^m \in L^1(B)$. From (2.11) we have

$$\begin{aligned}\int_B |\nabla u|^{e_0}\,dx &= N^{e_0} C_N^{\frac{e_0}{N}} \int_0^T \left(\frac{\omega(t)}{t^{1-\frac{1}{N}}}\right)^{e_0(p'-1)} dt \\ &\le N^{e_0} C_N^{\frac{e_0}{N}} M^{\frac{e_0}{p-1}} \int_0^T t^{(\gamma-1+\frac{1}{N})e_0(p'-1)}\,dt.\end{aligned}$$

Note that the integrability condition $(\gamma - 1 + \frac{1}{N})e_0(p'-1) + 1 > 0$ in $t = 0$ is equivalent to $m > -1 - \frac{N(p-1)}{e_0}$. It follows that $F(|x|, u, |\nabla u|) = \tilde{g}_0|x|^m + \tilde{f}_0|\nabla u|^{e_0} \in L^1(B)$. Let us prove that the condition (ii) in Lemma 3.2 is satisfied. Using (2.10) we have

$$\begin{aligned}I_\varepsilon = \varepsilon^{N-1}\,\tilde{u}(\varepsilon) &= \varepsilon^{N-1} \int_{C_N\varepsilon^N}^{|B|} \frac{\omega(t)^{p'-1}}{t^{p'(1-\frac{1}{N})}}\,dt \\ &\le M^{p'-1}\,\varepsilon^{N-1} \int_{C_N\varepsilon^N}^{|B|} t^{\gamma(p'-1)-p'(1-\frac{1}{N})}\,dt.\end{aligned}$$

Since $m > -p$ we have that $\gamma(p'-1) - p'(1-\frac{1}{N}) + 1 > 0$ and the integrability condition for $\varepsilon = 0$ is satisfied. We obtain

$$\begin{aligned}I_\varepsilon &\le M^{p'-1}\varepsilon^{N-1} \int_{C_N\varepsilon^N}^{|B|} t^{\frac{m+N-Np+p}{N(p-1)}}\,dt \\ &\le M^{p'-1}\,\frac{N(p-1)}{m+p}\left[T^{\frac{m+p}{N(p-1)}}\varepsilon^{N-1} - C_N^{\frac{m+p}{N(p-1)}}\varepsilon^{N-1+\frac{m+p}{p-1}}\right].\end{aligned}$$

From the assumptions $m > -p$ and $N \ge 2$ we conclude that $I_\varepsilon \to 0$ when $\varepsilon \to 0$. We show that the condition (iii) from Lemma 3.2 is satisfied.

Using (2.11) with $t = C_N\varepsilon^N$ we have

$$J_\varepsilon = \varepsilon^{N-1}\,|\tilde{u}'(\varepsilon)|^{p-1} = \varepsilon^{N-1}\,N^{p-1}\,C_N^{\frac{p-1}{N}}\left(\frac{\omega(t)}{t^{1-\frac{1}{N}}}\right)^{(p'-1)(p-1)}$$
$$= N^{p-1}\,C_N^{\frac{p+m}{N}}\,M\varepsilon^{m+N}.$$

From $m > -N$, it follows that $J_\varepsilon \to 0$ when $\varepsilon \to 0$. The condition (iv) in Lemma 3.2 follows from (2.11). We obtain

$$\int_B |\nabla u|^p\,dx = \int_0^T N^p\,C_N^{\frac{p}{N}}\left(\frac{\omega(t)}{t^{1-\frac{1}{N}}}\right)^{p(p'-1)}$$
$$\le N^p\,C_N^{\frac{p}{N}}\,M^{p(p'-1)}\int_0^T t^{\frac{mp+p}{N(p-1)}}\,dt.$$

The last integral is finite due to $\frac{mp+p}{N(p-1)}+1 > 0$. Since all conditions (i)-(iv) in Lemma 3.2 are satisfied for $F(|x|, u, |\nabla u|) = \tilde{g}_0|x|^m + \tilde{f}_0|\nabla u|^{e_0}$, we conclude that u is also the weak solution. $\square$

For the critical exponent on the gradient $e_0 = p-1$ we consider the special case (3.1), where $F(|x|, u, |\nabla u|) = \tilde{g}_0|x|^m + \tilde{f}_0|\nabla u|^{p-1}$. In this case we have the following result.

Theorem 3.4. *Assume that $m > \max\{-p, -N\}$, $N \ge 2$. Let $\tilde{g}_0$ be any positive constant and $0 < \tilde{f}_0 < NC_N^{\frac{1}{N}}\left(\gamma + \frac{1}{N}\right)T^{-\frac{1}{N}}$. Then every ω-solution u of quasilinear elliptic problem (2.16) is also the weak solution.*

The proof is similar to that of Theorem 3.3, so we omit it.

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