GLASNIK MATEMATIČKI
Vol. 48(68)(2013), 403 – 414

QUADRATIC OPERATORS ON AM-SPACES

Włodzimierz Fechner
University of Silesia, Poland

Abstract. Our purpose is to deal with quadratic operators acting between vector lattices of continuous mappings on a compact Hausdorff space. In our first main result we characterize quadratic-multiplicative operators, whereas in the second one we provide necessary and sufficient conditions for a quadratic operator to be proportional to the square of a continuous linear operator.

1. Introduction

The notion of quadratic mappings in a sense of Definition 1.1 below was introduced by J. Aczél and later extensively studied by several authors, among others by S. Kurepa (see [13–16]). However, the main attention was given to scalar-valued functions. In the present paper we discuss vector-valued quadratic operators and we provide counterparts to some results for scalar-valued quadratic functions.

Definition 1.1. *Let $(X,+),(Y,+)$ be Abelian groups. A map $Q\colon X \to Y$ is termed* quadratic *if it satisfies the following functional equation:*

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y), \quad x,y \in X. \tag{1.1}$$

The following characterization of quadratic maps as diagonalizations of bi-additive and symmetric mappings is due to J. Aczél (see [2], [3] and [4, Chapter 11, Proposition 1]).

Theorem 1.2 (Aczél). *Let $(X,+),(Y,+)$ be Abelian groups and assume that $(Y,+)$ is uniquely divisible by 2. A map $Q\colon X \to Y$ is quadratic if and*

2010 *Mathematics Subject Classification.* 39B52, 46A40, 46B42, 46E05, 47B60, 47H60.
Key words and phrases. Quadratic operator, vector lattice, AM-space.

only if there exists a bi-additive and symmetric mapping $B\colon X \times X \to Y$ *such that*

$$Q(x) = B(x, x), \quad x \in X.$$

Moreover, B *is uniquely determined via the following formula:*

$$B(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y)], \quad x, y \in X. \tag{1.2}$$

Assume that X is a linear space over a field of scalars K with characteristic different from 2 (we will write char$K \neq 2$ for short) and $T\colon X \to K$ is an additive function. It is easy to check that for an arbitrary constant $c \in K$ the map $Q\colon X \to K$ given by the formula:

$$Q(x) = cT(x)^2, \quad x \in X \tag{1.3}$$

is an example of a quadratic mapping. However, not every quadratic mapping defined on a real or complex linear space is of the form (1.3). This can be justified in quite a few ways. We will make use of a result of R. Ger ([9, Theorem 1]); see also the paper [8] by Z. Gajda and H. I. Miller. If X is a linear space over the rationals, then each set $H \subset X$ which possesses the property that every mapping $Q_0\colon H \to K$ has a unique extension to a quadratic mapping $Q\colon X \to K$ is called a *basic set* for equation (1.1). It is well-known that each Hamel basis (i.e the basis of X as a linear space over $\mathbb{Q}$) is a basic set for the Cauchy functional equation of additive mappings:

$$g(x + y) = g(x) + g(y), \quad x, y \in X$$

(see [12, Theorem 5.2.2]). More precisely [9, Theorem 1] states that if H is a Hamel basis of X, then the set $\frac{1}{2}(H + H)$ is a basic set for equation (1.1) of quadratic mappings. Clearly, $H \subsetneq \frac{1}{2}(H + H)$ since H is not a mid-convex set. From this it follows that there exist quadratic functions on X which do not verify (1.3) with any additive T and any constant c.

It is straightforward to observe that each mapping $Q\colon X \to K$ given by (1.3) with some additive $T\colon X \to K$ and some constant $c \in K$ satisfies the following auxiliary equation:

$$[Q(x + y) - Q(x - y)]^2 = 16Q(x)Q(y), \quad x, y \in X. \tag{1.4}$$

This equation was studied by S. Kurepa ([13]), see also [10]. Later, substantial generalizations have been obtained by B. R. Ebanks ([5, 6]). It turns out that quadratic functions with values in a field of characteristic different from 2 which are of the form (1.3) are precisely those which satisfy equality (1.4).

In what follows the Kurepa-Ebanks theorem ([5, Theorem 3]) will be needed.

THEOREM 1.3 (Kurepa-Ebanks). *Let* $(X, +)$ *be a group,* K *a field with characteristic different from* 2 *and* $Q\colon X \to K$ *a quadratic map. Then* Q *satisfies equation* (1.4) *if and only if there exist an additive map* $T\colon X \to K$ *and a (nonzero) constant* $c \in K$ *for which the formula* (1.3) *holds true.*

REMARK 1.4. Let us join (1.2) with (1.4) and use the Kurepa-Ebanks theorem to see that if for a given quadratic mapping $Q\colon X \to K$, formula (1.3) is valid with some additive $T\colon X \to K$ and some $c \in K$, then we have

$$B(x,y) = cT(x)T(y), \quad x,y \in X.$$

Moreover, an inspection of the Ebanks' proof allows us to derive precise formulas for the additive map T and for constant c (see [5, Proof of Theorem 1, page 178]). If $Q = 0$, then $T = 0$ and c can be arbitrary. If this is not the case, then one can pick any point $x_0 \in X$ such that $Q(x_0) \neq 0$ and define $c = Q(x_0)^{-1}$ and $T(x) = B(x_0, x)$, where B is given by (1.2).

Now assume that X is an algebra over the field $K = \{\mathbb{R}, \mathbb{C}\}$ and $T\colon X \to K$ is an additive and multiplicative map. If $Q\colon X \to K$ is defined as the square of T, then Q is a quadratic-multiplicative function. One can ask under which conditions the converse implication is also true. The first positive result is due to C. Hammer and P. Volkmann ([11]). They described real-to-real quadratic-multiplicative functions.

THEOREM 1.5 (Hammer-Volkmann). *Assume that $Q\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary mapping. Then Q is a quadratic-multiplicative function if and only if there exists an additive-multiplicative function $T\colon \mathbb{C} \to \mathbb{C}$ such that Q is of the form*

$$Q(x) = |T(x)|^2, \quad x \in \mathbb{R}.$$

A generalization of this theorem is due to Z. Gajda ([7]). If K is a field then we denote by $\overline{K}$ the algebraic closure of K and if $\zeta \in \overline{K}$ then $K(\zeta)$ stands for the smallest field such that $K \subseteq K \cup \{\zeta\} \subseteq \overline{K}$.

THEOREM 1.6 (Gajda). *Assume that X is a commutative unitary ring, K is a field with characteristic different from 2 and $Q\colon X \to K$ is an arbitrary mapping. Then Q is a quadratic-multiplicative function if and only if there exist an element $\zeta \in \overline{K}$ such that $\zeta^2 \in K$ and additive-multiplicative mappings $u\colon X \to K(\zeta)$ and $v\colon X \to K(\zeta)$ such that*

$$u(x) + v(x) \in K, \qquad u(x) - v(x) \in \zeta K,\ x \in X$$

and Q is of the form

$$Q(x) = u(x)v(x), \quad x \in X. \tag{1.5}$$

REMARK 1.7. One can ask whether about the relation between quadratic mappings which are of the form (1.3) and those which are of the form (1.5). In view of Theorem 1.3 and Theorem 1.6, to answer this question we need to establish a connection between additive-multiplicative mappings u, v postulated by Theorem 1.6 and the additive-multiplicative function T mentioned in Theorem 1.3. An inspection of the Gajda's proof of Theorem 1.6 indicates that such a connection exists but is not direct. Assume that X is a commutative unitary ring and K is a field with characteristic different from

2. Further, assume that $Q\colon X \to K$ is a non-zero quadratic-multiplicative function, there exists some $\zeta \in \overline{K}$ such that $\zeta^2 \in K$ and u and v are described in Theorem 1.6.

Let $A\colon X \times X \to K$ be the symmetric bi-additive mapping which corresponds to Q. Next, define $T_0\colon X \to K$ as $T_0(x) = A(x,1)$ for all $x \in X$ and then $B \times B\colon X \to K$ as

$$B(x,y) = A(x,1)A(y,1) - A(x,y), \quad x,y \in X.$$

It is checked that B is bi-additive and symmetric and moreover

$$B(x,y)^2 = B(x,x) \cdot B(y,y), \quad x,y \in X.$$

Consequently, by Theorem 1.3 applied for the diagonalization of B there exist an additive map $T_1\colon X \to K$ and a constant $c_1 \in K$ such that

$$c_1 T_1(x)^2 = T_0(x)^2 - Q(x), \quad x \in X.$$

Finally, the following equalities hold true:

$$u(x) = T_0(x) + \zeta T_1(x), \quad x \in X$$

and

$$v(x) = T_0(x) - \zeta T_1(x), \quad x \in X.$$

From now on, we will be using the following terminology. A quadratic map Q defined on a linear space over $K \in \{\mathbb{R}, \mathbb{C}\}$ which additionally possess the following homogeneity property:

$$Q(tx) = t^2 Q(x), \quad x \in X,\ t \in K \tag{1.6}$$

will be termed *quadratic functional* or *quadratic operator*, respectively, depending whether the target space is a field or a vector space.

It is worth to mention that condition (1.6) imposed upon a quadratic map Q does not imply that the bi-additive mapping B associated with Q is bilinear (or sesquilinear). This problem was discussed by S. Kurepa; see [13] and [16]. However, one can see that from Theorem 1.3 it follows that if additionally (1.4) is satisfied, then this implication is true.

Now, assume that X and Y are vector lattices (Riesz spaces). It is to recall that a positive operator $T\colon X \to Y$ is termed *lattice (Riesz) homomorphism* if

$$T(x \vee y) = Tx \vee Ty, \quad x,y \in X.$$

Definition 1.8. *Assume that X is a Banach lattice. Then X is called AM*-space *if*

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}, \quad x,y \in X^+ \text{ such that } x \wedge y = 0.$$

The Kakutani-Bohnenblust-Krein theorem says that every AM-space with unit is lattice isometric to the space $C(\Omega)$ of all real continuous mappings defined on some compact Hausdorff space Ω. Moreover every AM-space is lattice isometric to some closed vector sublattice of $C(\Omega)$ (see e.g. the monograph of Abramovich and Aliprantis, [1, Theorem 3.6]). In view of this result, later on we will restrict our studies to spaces $C(\Omega)$.

2. Quadratic operators

Our purpose is to obtain representations of quadratic operators acting between $C(\Omega)$-spaces, analogous to the scalar ones of Kurepa, Ebanks and Gajda, which were recalled in the Introduction. We start with a result which is an analogue of Theorem 1.5 and of Theorem 1.6.

Theorem 2.1. *Assume that Ω_1, Ω_2 are compact Hausdorff spaces. Then $Q\colon C(\Omega_1) \to C(\Omega_2)$ is a quadratic and multiplicative operator if and only if there exist a clopen subset $B \subseteq \Omega_2$ and mappings $\tau, \sigma\colon \Omega_2 \to \Omega_1$ which are continuous on B such that:*

$$Q(x)(t) = \chi_B(t)x(\tau(t))x(\sigma(t)), \quad x \in C(\Omega_1),\ t \in \Omega_2. \tag{2.1}$$

Proof. If Q is of the form (2.1), then it is straightforward to calculate that it is a quadratic-multiplicative operator having values in $C(\Omega_2)$. To prove the converse we may assume that $Q \neq 0$. Fix arbitrarily $t \in \Omega_2$ and define $q_t\colon C(\Omega_1) \to \mathbb{R}$ as

$$q_t(x) = Q(x)(t), \quad x \in C(\Omega_1). \tag{2.2}$$

It is clear that q_t is a quadratic-multiplicative functional. We can assume that $q_t \neq 0$. Therefore, by Theorem 1.6 there exist two additive-multiplicative mappings $u_t, v_t\colon C(\Omega_1) \to \mathbb{C}$ such that

$$q_t(x) = u_t(x)v_t(x), \quad x \in C(\Omega_1).$$

We will show that u_t and v_t are $\mathbb{R}$-linear. Due to the multiplicativity it is sufficient to show that mappings $f, g\colon \mathbb{R} \to \mathbb{C}$ given by

$$f(\lambda) = u_t(\lambda e), \quad g(\lambda) = v_t(\lambda e), \quad \lambda \in \mathbb{R}$$

are equal to the identity mapping, where e is the unit of $C(\Omega_1)$. We have

$$f(\lambda)g(\lambda) = q_t(\lambda e) = \lambda^2 q_t(e) = \lambda^2, \quad \lambda \in \mathbb{R}.$$

Therefore, expanding the equality

$$f(\lambda+\mu)g(\lambda+\mu) = (\lambda+\mu)^2,$$

which is valid for arbitrary $\lambda, \mu \in \mathbb{R}$, we achieve

$$f(\lambda)g(\mu) + f(\mu)g(\lambda) = 2\lambda\mu, \quad \lambda, \mu \in \mathbb{R}.$$

Put $\mu = 1$ to get

$$f(\lambda) + g(\lambda) = 2\lambda, \quad \lambda \in \mathbb{R}.$$

Now, one can easily obtain

$$[f(\lambda) - g(\lambda)]^2 = [f(\lambda) + g(\lambda)]^2 - 4f(\lambda)g(\lambda) = 0, \quad \lambda \in \mathbb{R},$$

which is what we need.

Next, define operators U, V on the space $C(\Omega_1)$ as

$$U(x)(t) = u_t(x), \quad V(x)(t) = v_t(x), \quad x \in C(\Omega_1),\, t \in \Omega_2.$$

We will check whether the target space of both U and V is the space $C_{\mathbb{C}}(\Omega_2)$ of complex continuous functions on Ω_2. We need both maps

$$\Omega_2 \ni t \mapsto U(x)(t), \qquad \Omega_2 \ni t \mapsto V(x)(t)$$

to be continuous for every $x \in C(\Omega_1)$. This follows from Remarks 1.4 and 1.7. Indeed, in Theorem 1.6 we take $\zeta = i$. Further, let A_t, B_t and T_0^t be the mappings corresponding to q_t defined in Remark 1.7. We can assume that $B_t \neq 0$. Next, pick some $x_0 \in C(\Omega_1)$ such that $B_t(x_0, x_0) \neq 0$ and define T_1^t as $T_1^t(x) = B_t(x_0, x)$. Now, from this and from the formulas mentioned in Remark 1.7 we see that for every fixed $x \in C(\Omega_1)$ the continuity of the mapping $\Omega_2 \ni t \mapsto Q(x)(t)$ implies the continuity of $\Omega_2 \ni t \mapsto T_0^t(x)$ and $\Omega_2 \ni t \mapsto T_1^t(x)$ and consequently, of $U(x)$ and $V(x)$.

Our next step is to extend operators U and V on the space $C_{\mathbb{C}}(\Omega_1)$ of complex continuous functions on Ω_1 in an usual way:

$$U_{\mathbb{C}}(x + iy) = U(x) + iU(y), \quad V_{\mathbb{C}}(x + iy) = V(x) + iV(y),$$

for $x, y \in C(\Omega_1)$. It is easy to check that the extended operators are $\mathbb{R}$-linear and multiplicative. Moreover, it is clear that $U_{\mathbb{C}}(ix) = iU_{\mathbb{C}}(x)$ and $V_{\mathbb{C}}(ix) = iV_{\mathbb{C}}(x)$ for all $x \in C(\Omega_1)$. Therefore, by a standard argumentation one can show that both operators are $\mathbb{C}$-linear (e.g. one may consider for fixed $x \in C(\Omega_1)$ and $t \in \Omega_2$ mappings $\mathbb{C} \ni z \mapsto f(z) := U_{\mathbb{C}}(zx)(t)$ and $\mathbb{C} \ni z \mapsto g(z) := V_{\mathbb{C}}(zx)(t)$ and apply the description of additive mappings on $\mathbb{C}$ from [12, Chapter 5.6]). Further, by the representation theorem of linear-multiplicative operators (see e.g. [1, Theorem 4.27]), there exist two clopen subsets $B_u, B_v \subseteq \Omega_1$ and two mappings $\tau, \sigma \colon \Omega_2 \to \Omega_1$ which are continuous on B_u and B_v, respectively, such that

$$U_{\mathbb{C}}(x)(t) = \chi_{B_u}(t)x(\tau(t)), \quad V_{\mathbb{C}}(x)(t) = \chi_{B_v}(t)x(\sigma(t)), \quad x \in C_{\mathbb{C}}(\Omega_1),\, t \in \Omega_2.$$

Note that in formula (2.1) we do not need the complex extensions of U and V, since both $U_{\mathbb{C}}$ and $V_{\mathbb{C}}$ maps $C(\Omega_1)$ into $C(\Omega_2)$. To finish the proof it is sufficient to put $B = B_u \cap B_v$. $\square$

Now, we will provide an analogue of Theorem 1.3.

THEOREM 2.2. *Assume that Ω_1, Ω_2 are compact Hausdorff spaces and Ω_2 is metrizable. and $Q \colon C(\Omega_1) \to C(\Omega_2)$ is an arbitrary map. Then Q is a nonnegative and continuous quadratic operator which satisfies equality* (1.4) *jointly with the following auxiliary condition:*

$$(2.3) \qquad Q(x + y) = Q(x - y) \text{ for all } x, y \in \mathbb{C}(\Omega_1) \text{ such that } x \wedge y = 0$$

if and only if there exist a mapping $\tau\colon \Omega_2 \to \Omega_1$ *and a nonnegative function* $w \in C(\Omega_2)$ *such that* τ *is continuous on the set* $\{t \in \Omega_2 : w(t) > 0\}$ *and:*

$$Q(x)(t) = w(t)[x(\tau(t))]^2, \quad x \in C(\Omega_1),\, t \in \Omega_2. \tag{2.4}$$

Proof. If Q is defined as above, then it is straightforward to check that Q given by (2.4) is a nonnegative quadratic operator and Q satisfies both (1.4) and (2.3). To prove the converse, fix arbitrary $t \in \Omega_2$ and define $q_t\colon C(\Omega_1) \to \mathbb{R}$ by (2.2). Then q_t is a nonnegative quadratic functional which satisfies the assumptions of Theorem 1.3. Thus, we derive the existence of a linear functional $T_t\colon C(\Omega_1) \to \mathbb{R}$ and some real constant $c_t \geq 0$ such that

$$q_t(x) = c_t T_t(x)^2, \quad x \in C(\Omega_1).$$

It is clear that there is more than one possibility for the choice of the constant c_t and of the map T_t. What we need is to ensure that both mappings

$$\Omega_2 \ni t \mapsto c_t, \qquad \Omega_2 \ni t \mapsto T_t(x)$$

can be taken as continuous for every fixed $x \in C(\Omega_1)$. We will use the description of the constant c_t and of the map T_t provided in Remark 1.4. Note that for every fixed t, if the constant c_t is positive, then thanks to the homogeneity property (1.6) of Q, c_t can be taken as arbitrary positive number. Therefore, in this case let us take $c_t = 1$.

The set

$$A = \{t \in \Omega_2 : Q(x)(t) = 0\}$$

is closed. If $A = \Omega_2$, then both mappings in question can be taken as zero mappings and we are done. Assume that there exists some $t_0 \in \Omega_2 \setminus A$. Since Ω_2 is a compact metrizable space, then A and t_0 can be separated by a continuous function. Therefore, there exists a separating function $c\colon \Omega_2 \to [0,1]$ which is equal 0 precisely on A and equal 1 on t_0. Thus, we have $c(t) = c_t$ for $t \in \Omega_2$ and further, if we take $T_t = 0$ for all $t \in A$, then we are sure that for every $x \in C(\Omega_1)$ the map $\Omega_2 \ni t \mapsto T_t(x)$ is continuous.

Now, define a linear operator T_0 on the space $C(\Omega_1)$ by the formula

$$T_0(x)(t) = T_t(x) \quad x \in C(\Omega_1),\, t \in \Omega_2.$$

From the previous argumentation it follows that the target space of T_0 is $C(\Omega_2)$.

Next, fix arbitrarily $x, y \in C(\Omega_1)$ such that $x \wedge y = 0$. Then, from (1.4) and (2.3) it follows that

$$Q(x)(t) \cdot Q(y)(t) = 0, \quad t \in \Omega_2.$$

From this we see that $Q(x) \wedge Q(y) = 0$. Consequently, we obtain

$$(T_0 x)^2 \wedge (T_0 y)^2 = 0.$$

Next, we will make use of the Riesz-Kantorovich formulas (see e.g. [1, Theorem 1.16]). We define operators T_0^+ and T_0^- as follows:

$$T_0^+x = \sup\{T_0y : 0 \le y \le x\},$$
$$T_0^-x = \sup\{-T_0y : 0 \le y \le x\}.$$

The Riesz-Kantorovich formulas define a natural lattice structure on the space of all bounded linear operators between $C(\Omega_1)$ and $C(\Omega_2)$. In particular, T_0^+ and T_0^- are positive operators, $T_0^+ \wedge T_0^- = 0$, $T_0 = T_0^+ - T_0^-$ and $|T_0| = T_0^+ + T_0^-$. Put $T = |T_0|$; obviously T is a positive operator and moreover

$$\begin{aligned}(T_0)^2 = (T_0^+ - T_0^-)^2 &= (T_0^+)^2 - (T_0^+)(T_0^-) - (T_0^-)(T_0^+) + (T_0^-)^2\\ &= (T_0^+)^2 + (T_0^-)^2 = |T_0|^2 = T^2,\end{aligned}$$

since $(T_0^+)(T_0^-) = (T_0^-)(T_0^+) = 0$. Now, we can prove that T is a lattice homomorphism. To do this it is enough to show that $Tx \wedge Ty = 0$ whenever $x \wedge y = 0$ (see e.g. [1, Theorem 1.34]). We already have that

$$(Tx)^2 \wedge (Ty)^2 = 0,$$

if $x \wedge y = 0$. Since T is a positive operator, then from this we derive

$$Tx \wedge Ty = 0,$$

as claimed. In the last step of the proof we apply a representation theorem for lattice homomorphisms (see [1, Theorem 4.25]). Therefore, T is of the form

$$T(x)(t) = w_0(t)x(\tau(t)), \quad x \in C(\Omega_1),\ t \in \Omega_2,$$

for some mapping $\tau\colon \Omega_1 \to \Omega_2$ and for a nonnegative weight function $w_0 \in C(\Omega_2)$. Moreover, the map τ is continuous on the set $\{t \in \Omega_2 : w(t) > 0\}$. To finish the proof one needs to put $w = c \cdot w_0^2$. □

In what follows we will focus on the auxiliary condition (2.3) imposed upon Q in the statement of Theorem 2.2. From the proof of this theorem we can notice that condition (2.3) plays the role of the respective condition defining lattice homomorphisms. In Theorems 2.4 and 2.7 below we will provide two sets of equivalent conditions related to (2.3). We begin with the following easy observation.

Proposition 2.3. *Assume that X is a vector lattice, Y is an Abelian group and $Q\colon X \to Y$ is a quadratic mapping. Then*

$$Q(|x|) + Q(x) = 2Q(x^+) + 2Q(x^-), \quad x \in X.$$

Proof. It suffices to apply equation (1.1) with $x = x^+$ and $y = x^-$. □

Theorem 2.4. *Assume that X is a vector lattice, Y is an Abelian group, $Q\colon X \to Y$ is a quadratic mapping and $B\colon X \times X \to Y$ is the corresponding bi-additive and symmetric mapping. Then, the following conditions are equivalent:*

(*i*) $B(x, y) = 0$ *for all* $x, y \in X$ *such that* $x \wedge y = 0$*;*
(*ii*) $B(x^+, x^-) = 0$ *for all* $x \in X$*;*
(*iii*) $Q(x^+) + Q(x^-) = Q(|x|)$ *for all* $x \in X$*;*
(*iv*) $Q(x) = Q(x^+) + Q(x^-)$ *for all* $x \in X$*;*
(*v*) $Q(x) = Q(|x|)$ *for all* $x \in X$*;*
(*vi*) $Q(x + y) = Q(x - y)$ *for all* $x, y \in X$ *such that* $x \wedge y = 0$.

PROOF. $(i) \Rightarrow (ii)$. The implication follows immediately from the fact that $x^+ \wedge x^- = 0$ for every $x \in X$.

$(ii) \Rightarrow (iii)$. Fix arbitrary $x \in X$. We have

$$\begin{aligned} Q(|x|) &= B(|x|, |x|) = B(x^+ + x^-, x^+ + x^-) \\ &= B(x^+, x^+) + B(x^+, x^-) + B(x^-, x^+) + B(x^-, x^-) \\ &= Q(x^+) + Q(x^-), \end{aligned}$$

which proves (iii).

$(iii) \Leftrightarrow (iv)$. Fix arbitrary $x \in X$. With the aid of Proposition 2.3 we easily transform equivalently (iii) as follows:

$$Q(x^+) + Q(x^-) = Q(|x|) = 2Q(x^+) + 2Q(x^-) - Q(x),$$

and this is identical with (iv).

$(iii)\,\&\,(iv) \Rightarrow (v)$: obvious.

$(v) \Rightarrow (vi)$. Fix arbitrary $x, y \in X$ such that $x \wedge y = 0$ and denote $\xi = x - y$. Then $\xi^+ = x$ and $\xi^- = y$, so $|\xi| = x + y$. Now, apply (v) with $x = \xi$ to obtain

$$Q(x - y) = Q(\xi) = Q(|\xi|) = Q(x + y).$$

$(vi) \Rightarrow (i)$. Follows immediately from formula (1.2). □

Now, we will provide two counterexamples. Firstly, we will show that there exists a continuous quadratic mapping which does not satisfy the equivalent conditions spoken of in Theorem 2.4, and further we will show that there exist discontinuous quadratic mappings which satisfy them.

EXAMPLE 2.5. Let $X = \mathbb{R}^2$ equipped with the standard operations and pointwise order and let $Y = \mathbb{R}$. Then the mapping $Q\colon X \to Y$ given by

$$Q(x_1, x_2) = (x_1 - x_2)^2, \quad (x_1, x_2) \in X,$$

is a quadratic mapping. Clearly, Q is nonnegative and continuous. Moreover, it is easy to see that Q fails to satisfy the equivalent conditions of Theorem 2.4.

EXAMPLE 2.6. Assume that $X = \mathbb{R}^n$ for some positive integer n. Then X is a vector lattice and simultaneously a real Hilbert space (with the standard inner product). Moreover, it is easy to see that every pair x, y of elements of X which satisfies $x \wedge y = 0$ is in particular orthogonal. Further, let $a\colon \mathbb{R} \to \mathbb{R}$

be a discontinuous additive mapping. It is clear that a is unbounded (from both sides) on any interval of positive length. Next, let us define

$$B(x,y) = a((x|y)), \quad x, y \in X$$

and

$$Q(x) = B(x,x) = a(\|x\|^2), \quad x \in X.$$

It is straightforward to check that Q is a quadratic mapping and (1.2) is satisfied by Q and B. Moreover, Q is discontinuous and clearly fails to be nonnegative. However, one can see that B satisfies condition (i) of Theorem 2.4, so Q satisfies conditions (iii)–(vi) of this statement.

The next result is similar to Theorem 2.4 and provides a set of stronger equivalent conditions, which for a nonnegative quadratic mapping Q reduce to the respective conditions of Theorem 2.4.

Theorem 2.7. *Assume that X, Y are vector lattices, $Q\colon X \to Y$ is a quadratic mapping and $B\colon X \times X \to Y$ is the corresponding bi-additive and symmetric mapping. Then, the following conditions are equivalent:*

(i) $2B(x,y) = Q(x-y)^-$ *for all $x, y \in X$ such that $x \wedge y = 0$;*
(ii) $2B(x^+, x^-) = Q(x)^-$ *for all $x \in X$;*
(iii) $Q(x^+) + Q(x^-) + Q(x)^- = Q(|x|)$ *for all $x \in X$;*
(iv) $Q(x)^+ = Q(x^+) + Q(x^-)$ *for all $x \in X$;*
(v) $|Q(x)| = Q(|x|)$ *for all $x \in X$;*
(vi) $|Q(x+y)| = Q(x-y)$ *for all $x, y \in X$ such that $x \wedge y = 0$.*

Proof. $(i) \Rightarrow (ii)$. Follows from the identities $x^+ \wedge x^- = 0$ and $x^+ - x^- = x$.

$(ii) \Rightarrow (iii)$. Fix arbitrary $x \in X$. We obtain

$$\begin{aligned} Q(|x|) = B(|x|,|x|) &= Q(x^+) + Q(x^-) + 2B(x^+, x^-) \\ &= Q(x^+) + Q(x^-) + Q(x)^-, \end{aligned}$$

which proves (iii).

$(iii) \Leftrightarrow (iv)$. Fix arbitrary $x \in X$. With the aid of Proposition 2.3 we easily transform equivalently (iii) as follows:

$$Q(x^+) + Q(x^-) + Q(x)^- = Q(|x|) = 2Q(x^+) + 2Q(x^-) - Q(x)^+ + Q(x)^-,$$

and this is identical with (iv).

$(iii) \,\&\, (iv) \Rightarrow (v)$. Obvious.

$(v) \Rightarrow (vi)$. Fix arbitrary $x, y \in X$ such that $x \wedge y = 0$ and denote $\xi = x - y$. Then $\xi^+ = x$ and $\xi^- = y$, so $|\xi| = x + y$. Then, apply (v) with $x = \xi$ to obtain

$$|Q(x-y)| = |Q(\xi)| = Q(|\xi|) = Q(x+y).$$

$(vi) \Rightarrow (i)$. Fix arbitrary $x, y \in X$ such that $x \wedge y = 0$. We have

$$\begin{aligned} 4B(x,y) &= Q(x+y) - Q(x-y) \\ &= Q(x+y) - Q(x-y)^+ - Q(x-y)^- + 2Q(x-y)^- \\ &= Q(x+y) - |Q(x-y)| + 2Q(x-y)^- = 2Q(x-y)^-. \end{aligned}$$

□

Let us point out that the map Q provided in Example 2.6 does not satisfy the conditions of Theorem 2.7. Indeed, for Q defined as in Example 2.6 we have by Theorem 2.4 (v) that $Q(|x|) = Q(x)$ for all $x \in X$ and, since Q fails to be nonnegative, then the condition (v) of Theorem 2.7 is violated.

Acknowledgements.

Research of the author was supported by the Polish Ministry of Science and Higher Education in the years 2012-2013.

References

[1] Y. A. Abramovich and C. D. Aliprantis, An invitation to operator theory, American Mathematical Society, Providence, 2002.
[2] J. Aczél, *The general solution of two functional equations by reductions to functions additive in two variables and with the aid of Hamel bases*, Glasnik Mat.-Fiz. Astronom. Ser. II **20** (1965), 65–73.
[3] J. Aczél, J.K. Chung and C. T. Ng, *Symmetric second differences in product form on groups*, World Scientific Publ. Co., Teaneck, 1989, 1–22.
[4] J. Aczél and J. Dhombres, Functional equations in several variables, Cambridge Univ. Press, Cambridge, 1989.
[5] B. R. Ebanks, *Factoring multiadditive mappings*, Linear Multilinear Algebra **33** (1993), 175–184.
[6] B. R. Ebanks, *Factoring biquadratic maps on modules*, Forum Math. **7** (1995), 29–43.
[7] Z. Gajda, *On multiplicative solutions of the parallelogram functional equation*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 59–66.
[8] Z. Gajda and H. I. Miller, *On quadratic functionals and some properties of Hamel bases*, J. Math. Anal. Appl. **145** (1990), 45–51.
[9] R. Ger, *Some remarks on quadratic functionals*, Glas. Mat. (III) **23(43)** (1988), 315–330.
[10] A. Grząślewicz, *On the solution of the system of functional equations related to quadratic functionals*, Glas. Mat. Ser. III **14(34)** (1979), 77–82.
[11] C. Hammer and P. Volkmann, *Die multiplikativen Lösungen der Parallelogrammgleichung*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 197–201.
[12] M. Kuczma, An introduction to the theory of functional equations and inequalities, Birkhäuser, Basel, 2009.
[13] S. Kurepa, *Quadratic and sesquilinear functionals*, Glasnik Mat.-Fiz. Astronom. Ser. II **20** (1965), 79–92.
[14] S. Kurepa, *Note on inequalities associated with hermitian functionals*, Glasnik Mat. Ser. III **3(23)** (1968), 197–206.
[15] S. Kurepa, *Quadratic functionals conditioned on an algebraic basic set*, Glasnik Mat. Ser. III **6(26)** (1971), 265–275.
[16] S. Kurepa, *On quadratic forms*, Aequationes Math. **34** (1987), 125–138.

W. Fechner
Institute of Mathematics
University of Silesia
Bankowa 14, 40-007 Katowice
Poland
E-mail: fechner@math.us.edu.pl & wlodzimierz.fechner@us.edu.pl

Received: 18.8.2012.
Revised: 16.12.2012.