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Acquiring the Concept / Concepts of Function through Programmed Instruction in a Computer Classroom

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Abstract

In teaching mathematics the concept/concepts of function are very important, but also difficult. Thus, they are gradually introduced in the teaching process depending on the students' age. The development of mathematics and education technologies allows different approaches for introducing students to these concepts as well as testing of their acquisition by students. The paper outlines the possibility of acquiring the concepts of function through programmed instruction using the Cartesian method, in which the educational space is represented by "a point" with six components. One of the specified components is media, which are used for acquiring new content, and also for testing what students have acquired, for example, in the field of function. This approach ensures the presence of the direct and feedback loop, and the teaching of mathematics is organized as a manageable process.
The above mentioned is effectively implemented in a computer classroom through programmed teaching using appropriate tools and Educational Computer Software (ECS).

Key words: *Cartesian method; direct and feedback loop; Educational Computer Software (ECS); media.*

Introduction

Mathematics as a fundamental subject in both elementary and secondary school in every grade is taught through the appropriate number of lessons, which is determined by the curriculum. The indicated status of the subject is determined by a syllabus which defines the content which is to be taught in mathematics lessons for each grade.

Due to the complexity of the subject content, syllabuses usually have a linear spiral structure. Procedures for the introduction and formation of mathematical concepts (rules and facts) by abstraction and generalization in teaching mathematics are definitely the problem of mathematics as a science, but also of its methodology.

A realistic example of the above is teaching the concept/concepts of a function and testing its acquisition. Students explicitly encounter the concept of function for the first time in the 7th grade in elementary school and then in all grades of grammar school specializing in mathematics, natural sciences, or social science. The encounter takes place much earlier in an intuitive approach.

As the concept of function is very important for teaching mathematics, but is also difficult, it needs to be introduced into teaching in an accessible manner and it should be easily understandable to students (Elia, Panaoura, Eracleus, & Gagatsis, 2007; Gagatsis & Shiakalli, 2004; Vinner & Dreyfus, 1989).

Within the teaching methods of mathematics, in general as well as in a segment of the teaching unit such as a function, an easy question can be taken as the starting point: TO WHOM?, but a simple answer to this question raises difficult questions: WHAT? and HOW? A function is one of the basic concepts in mathematics. As such, it needs to be appropriately placed in teaching mathematics, which is very important and significant in syllabuses of almost all levels.

Since the concept of a function is unclear and complicated to students, there is a need for finding the right way of its interpretation, introduction and formation in teaching mathematics.

Also, we have to consider the fact that the concept of function has undergone different transformations during its creation in mathematics (as a science). The most important are the following two stages:

- the first stage is characterized by reliance on the idea of dependency (the end of the 19th century and the first half of the 20th century) in the first phase;
- the second stage is characterized by reliance on the idea that a function is a special relation (mapping), taking the theory of sets and mathematical logic as a basis in the second phase.

In this paper, teaching mathematics and checking the results are based on the second interpretation of the concept of a function.

The teaching process of a teaching unit of functions includes: the concept of function (definition), the manner of assigning a task, domain and codomain, monotony, extreme values, odd and even functions, inverse functions, function composition (which are the items included in the syllabus of mathematics for the 4th grade of grammar school).

The paper is a segment of the more complex experimental research (ER), which was carried out by the authors within the topic ”From the introduction of the concept of a function to its formation”.

Overview of Previous Major Research

The rapid development of mathematics through centuries has not caused fast changes in teaching. Moreover, the methodology of teaching mathematics has not yet found its place among the scientific disciplines. Regardless of the fact that there is no congruence between mathematics as a science and its teaching, research in the teaching process is necessary and important in order to get the optimum results.

One of the most important projects on mathematical education is Nuffield's project (for students aged between five and fourteen). The first experimental verification within the project was conducted in the second half of the last century. The prime aim of Nuffield's project was to investigate how mathematics as a school subject is taught. The focus of the project is the view that students should be free in mathematics lessons and they should autonomously think and discover in order to achieve a higher level of understanding.

Briefly, the basic idea of Nuffield's project is: I do – I understand. Beside Nuffield's project, there have been several projects on mathematics education in England. The most important ones are: The School Mathematics Project (SMP), Midlands Mathematical Experiment (MME), Mathematics Curriculum Project, etc.

J.P. Galperin found out that teaching has to be organized as a process of guided thinking activities in teaching/learning. In that way, he entered the field of cybernetics of pedagogical processes and programmed instruction.

L.N. Landa argued in favour of making the process of teaching optimally manageable and of setting precise goals and tasks for the system of organization as well as for establishing the feedback loop.

E.G. Begle is an American mathematician heading the SMSG group (*School Mathematics Study Group*). In the mathematics textbooks published by the SMSG, a special attention is given to: structures, contemporary terminology, symbolism and a precise mathematical language as well as to autonomous learning through discovery. In that way, while transforming elementary syllabuses, the SMSG has been emphasizing the content and structures.

Research results of university professors of mathematics in Brussels (The centre for pedagogy of mathematics), F. Papy and G. Papy, made a creative contribution to forming the contemporary method of teaching mathematics based on relations (for children aged from 5 to 17). Ignacije Smolec, a Croatian mathematician and methodologist, continued their work.

CBM (*Computer based Mathematics Instruction*) is a project of the 2nd half of the previous century which has enabled the implementation of teaching mathematics via using computers in a computer classroom. The first experimental research was conducted at Stanford University, California (Suppes, 1963).

The contemporary methodological approach to mathematics teaching emphasizes greater student activity and autonomy. Special importance is given to the experiment in teaching (learning by discovery). Learning by discovery through experiments enables students to independently find problem solutions, new insights, to develop creativity, while they are working at their own pace (pace individualization).

Dreyfus and Vinner (1982, 1989) tested 271 students and 36 teachers by asking them about a function. Questions consisted of definitions and interesting graphs and students were supposed to give correct answers (for example: Which of the graphs represents a function?) (Tall, 1991).

Frank Helmar has conducted vast research and generalization on the application of "the Cartesian method" and teaching management (…) creating "six-dimensional space" as the answer to the question: How do we learn?

In their research, the authors of this paper (Vukobratović, 2009) focus on the management in mathematics teaching with or without computers (programmed instruction of mathematics).

Positive results of the experiment, conducted either in programme or methodology sphere, provided an adequate basis for the transformation of mathematics teaching.

Changes in the syllabus solutions were mostly caused by the development of mathematics as a science. This development, as well as the development of technology, led to the modernization of teaching. It was the result of the requirement: *contemporary syllabus – contemporary interpretation – greater efficiency.*

Methodology of the Conducted Research

General Methodological Approach

Contemporary organization of teaching/learning mathematics, in terms of management and the application of contemporary technological research, is reflected in the following:

- the flow of information from a student to a teacher and from a teacher to a student should be considerably increased, and thus the efficiency of teaching mathematics will increase;
- the opportunity of a mathematics teacher to obtain information from each student through communication and/or through an intermediary (a computer);
- information (from a student) needs to be complete so that it reports on different parameters of students' activities, especially about mechanisms of psychological processes.

A mathematics teacher and/or a computer system in some way need to react on information provided by the students.

An important requirement for well-managed teaching of mathematics is an operative feedback loop.

The methodological background for studying management in teaching mathematics in the paper is an analytical method and programmed instruction (PI) in a computer classroom.

How the Research Was Conducted

Overview of Experimental Research Actions

Starting from the general methodological approach used in the experimental research, the following has been defined in order to enable it: subject, goals and tasks of the research, the description of the methodological background and the description of the experimental research.

Subject

The subject of the research in the paper is an experimental investigation of the influence of programmed instruction (PI) in a computer classroom and the application of analytical method on the optimization of the result (on the example of teaching a programme unit of a function in the 4^{th} grade of grammar school).

Goal and Tasks of Experimental Research

The development of mathematics, methodology and technology provided the basis for defining the following goals and tasks:

– to determine whether the use of programmed instruction/learning of mathematics in a computer classroom using the adequate applications of Educational Computer Software (ECS) achieve optimum teaching results which can serve as the basis for the transformation of mathematics teaching. Within the defined goal, the following specific tasks for the experimental research are defined:

– preparation of the technical and technological basis for the implementation of the experimental research in electronic communication culture (appropriate hardware and software);

– teachers and students of experimental classes prepare for the optimum use of teaching/learning environment;

– to evaluate ECS according to specific criteria in order to choose the one which is supposed to bring the optimum teaching results.

Description of the First Methodological Background of ER

Educational Space of Teaching Mathematics

The methodological background for studying management in teaching mathematics is an analytical method from the 1^{st} half of the 17^{th} century which was studied by Galileo Galilei and Rene Descartes. Mathematical problems are very difficult; therefore they need to be divided into less complex problems. Firstly, the simplest problems are solved. Modern sciences use the CARTESIAN METHOD for problem solving. The method was adapted to educational space by Paul Haimann in 1962 and he represented it as a point with six components (B, L, M, P, S, Z) (Figure 1). The components are connected by the feedback loop which is the contribution of the paper.

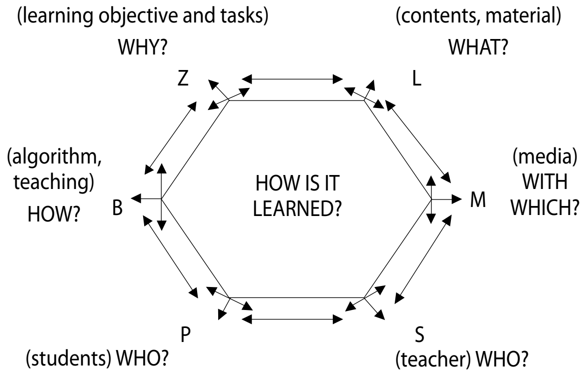


Figure 1. *The Cartesian method*

If a schema of educational space is analysed, the following can be observed:

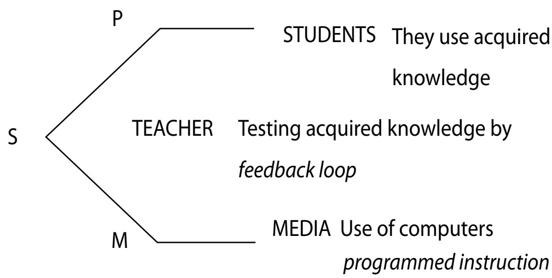
1) regular teaching can be divided into a learning system P (a student) and a learning environment S (a teacher);
2) the purpose of the regular teaching is in the learned content L and the hidden meaning of the learned things (objective and tasks) Z.

It is useful to consider the learning/teaching system Q as a pair of two components: "educational model" B and "applied media" M. Then, teaching becomes a quasi-point which represents an arranged set of six components (B, L, M, P, S, Z) in six-dimensional space.

By analysing each of the indicated components, they can be divided into their own components.

By inspecting the educational space, it can be noticed that the formally possible combinations are not always suitable for teaching mathematics: media (M) do not necessarily follow the method (B) in order to achieve a specific competency of a student (P) in the given environment (S) for specific learning content (L). A mathematics teacher should aspire to modern organization of teaching by using information technology (computers) (Cheng, 2000).

In a mathematics lesson, as well as in lessons in general, a teacher and a student should work together to achieve the teaching objective and tasks. By analyzing the schema of six-dimensional educational space (Figure 1), we can separate the segments (Figure 2), as it has been done for media – M (1) and methods – B (2), which can be schematically presented in the following way:



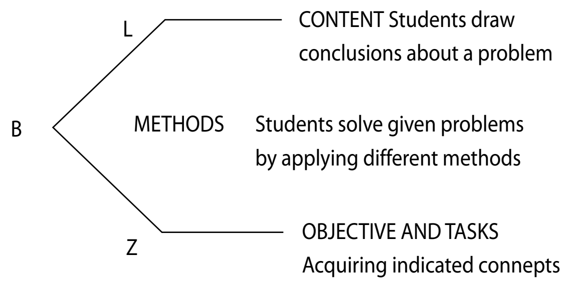


Figure 2. *Six-dimensional educational space, divided into segments*

Within the Cartesian method, one of the most important factors of successful teaching/learning is students' motivation (a motive – every reason the consequence of which is behaviour; motivation – a prerequisite to doing something).

Description of the Second Methodological Background of Experimental Research

Some Assumptions of Programmed Instruction

Programmed instruction (of mathematics) is defined as instruction in which students independently implement syllabuses which completely define the content and the way of learning, including procedures for systematic informing about the achieved results.

When designing a syllabus (for teaching mathematics), first of all, content is selected by eliminating everything that is unimportant. Secondly, it is logically structured. Then, the grouped material is divided into basic components – relatively small portions of information. Each portion is accompanied by an adequate task/tasks and finding the solution involves the application of knowledge and depends on its acquisition. The solutions of the tasks are given in a programme and students can compare their results with the programme in order to find out whether they can proceed to the acquisition of subsequent portions. Research has shown that programmed instruction is more efficient than the regular conventional instruction. The following advantages were found:

- in PI, students are always promptly informed about the results of their work;
- PI encourages students' independence in learning;
- PI discards passive learning and replaces it with learning based on the constant activity;
- the process of learning in PI is to a great extent individualized;
- PI combines group form of instruction with true individualization, which allows the adapting of mass instruction to individual students.

Programmed Instruction in a Computer Classroom

Nowadays, in the world, an exceptional attention is given to direct use of computers in teaching and learning. The best solutions (models, syllabuses, approaches) for education, learning and teaching are created by educators. There are different approaches to the use of informatics in education, teaching and learning. One of the approaches is based on the so-called intelligent mentor systems, which again are based on behavioural learning.

Programmed instruction/learning is an example of a learning concept with implemented solutions of a behavioural approach in the educational process. In this instruction, a mechanism works and ensures a continuous activity of a student with the presence of a direct and feedback loop in the form of:

STIMULUS – RESPONSE – SUPPORT.

In this day and age of the development of information technology, the media have taken an important place in regular lessons of mathematics (and in teaching in general). To systematize the concept of functions, we have chosen the programmed instruction of mathematics that is implemented using the computer in a computer classroom (Cheng, 2000).

The teaching process and teacher – student as well as computer – student communication are conducted in a computer classroom (Taylor, 1980). A teacher follows the work of students on the central computer (the server), and each student has their own computer (a client). Folders for each grade are open on the server (e.g. I1, I2, I3 ...), and each folder contains students' names (Figure 3). The communication is presented in the following schematic representation:

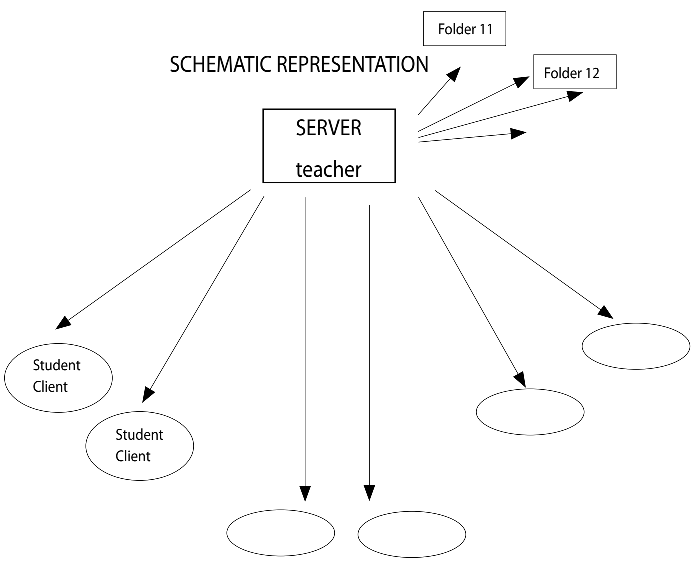


Figure 3. *A schematic representation of the functioning of a computer classroom*

The details of the communication are the following:

1) A teacher selects and/or prepares programmed material (information, tasks, tests).
2) A teacher sends prepared programmed material from the central computer (the server) to students' computers, clients.
3) Students become familiar with information and resolve the assigned tasks.
4) A teacher follows the work of students.
5) A student compares their solution with the solution provided by the computer. A teacher controls the work and if a student does not solve the problem correctly, a teacher notifies him/her and provides him/her with additional information.
6) If a student encounters a problem, a teacher decides to send the instruction via the computer.
7) The process is continuous for solving all tasks. A very important fact is that it is individual work.
8) Students can solve the tasks using the programme package GeoGebra.
9) If a comment is necessary to a large number of students, a teacher comments on typical errors via the system or verbally on the blackboard, while they take notes.

Within this kind of programmed instruction there is a continuous feedback loop between a teacher and a student, which is very important for the implementation of a modern methodological approach to teaching mathematics, in which an individualized approach and a subjective position of students are central.

A mathematics teacher, who runs the teaching process, is not concerned with an individual (a student), but the collective (entire class). Through computers, students are trained for effective communication, cooperation and teamwork. Graphics capabilities of computers help math to be "seen", an algebraic part of the software ensures that mathematics is being done, a programming language ensures that mathematics is created. The computer is the stimulus of motivation, encourages students' creativity and it is an efficient tool for testing students' knowledge.

Description of the Third Methodological Background of Experimental Research

Specific Contents in Experimental Teaching

The paper entails teaching and testing the contents of degree function which encompass a definition, domain, codomain, monotony and extreme values, inverted functions and a graphical representation.

The research entails degree functions of the following forms:

$$f(x)=\sqrt{x}\,,\; f(x)=\sqrt{x}-1\,,\; f(x)=\sqrt{x-1}\,,\; f(x)=\sqrt{x-k}\,,\quad k\in R$$

This segment of the research was conducted before the end of the school year 2010/2011, immediately before the university entrance exams.

Sample Used for Experimental Research

By defining the subject of ER, goals, tasks and methodological background (variables-experimental factors) were specified, which affected the choice of a research model. The model of the main experiment is an experiment with parallel groups (Mužić, 1973). The following two parallel groups are formed for ER in the 4th grades of grammar school specializing in natural sciences and mathematics:

- an experimental group (three classes in which the contents were taught using PI in a computer classroom);
- the control group (three classes in which the contents were taught using the classical methods).

The experimental research model focused on the group uniformity or equivalence which had to be fulfilled by following and preserving school requirements. Therefore, all the students from the classes comprising the sample were included in the experiment, but they did not have to be included into the experimental or control group. Because of the group equivalence, a certain number of students from some classes were not included in the group. During the experiment, all students were treated in the same way, even during the testing. However, when the results about the influence of an experimental factor were summarized, we took into account only the results of students who were the members of the experimental and the control group.

The selected sample was a non-probability sample and it consisted of the classes from different schools in Novi Sad.

Two criteria influenced the choice of classes which were to be included into the sample:

- uniformity of teachers who teach mathematics in the sampled classes (the uniformity was measured according to professional qualifications, years of experience and working results)
- uniformity of sampled students according to gender, success in mathematics, the results of placement testing and general academic achievement.

  The uniformity of the sample is considered to be good according to the defined characteristics.

One of the prerequisites for the objectivity of the results obtained in the experimental research is to ensure the control and to standardize experimental situations. The experimental research was organized under such circumstances that the complete standardization was difficult to achieve. In order to ensure a high level of standardization of experimental conditions, a preparation phase in the form of seminars in every phase of the experiment (preparation for the implementation of PI in a computer classroom) was organized.

Results

Knowledge (outcomes of experimental teaching) was tested by tasks of objective type, T3. The testing lasted 35 minutes and the programme package GeoGebra was used (Herceg & Herceg, 2007), as follows:

Typical Examples

A test for checking the acquired knowledge related to the concept of a function.

Questionnaire

Let us present the questionnaire given to these three groups of high school students.

Prior to this questionnaire, the students had solved a few more exercises than those given later in the questionnaire.

The questionnaire had the following form:

1. The function $f(x)=\sqrt{x}$, is given.
   a) Determine the domain and the range of the function.
   b) Determine the monotony and extremes of the function.
   c) Determine the inverse function of the given function.
   d) Draw the graph of the function.
2. The function $f(x)=\sqrt{x}-1$, is given.
   a) Determine the domain and the range of the function.
   b) Determine the monotony and extremes of the function.
   c) Determine the inverse function of the given function.
   d) Draw the graph of the function.
3. The function $f(x)=\sqrt{x-1}$, is given.
   a) Determine the domain and the range of the function.
   b) Determine the monotony and extremes of the function.
   c) Determine the inverse function of the given function.
   d) Draw the graph of the function.
4. The function $f(x)=\sqrt{x-k},\quad k\in R$, is given.
   a) Determine the domain and the range of the function.
   b) Determine the monotony and extremes of the function.
   c) Determine the inverse function of the given function.
   d) Draw the graph of the function.

The answers of the given examples solved by the GeoGebra programme package.

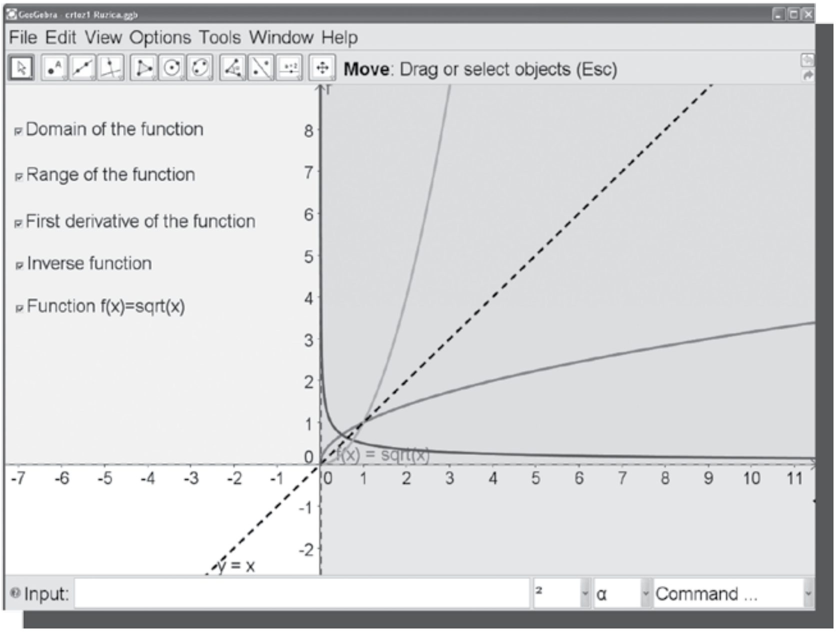


Figure 4. *Answers to the Test 1*

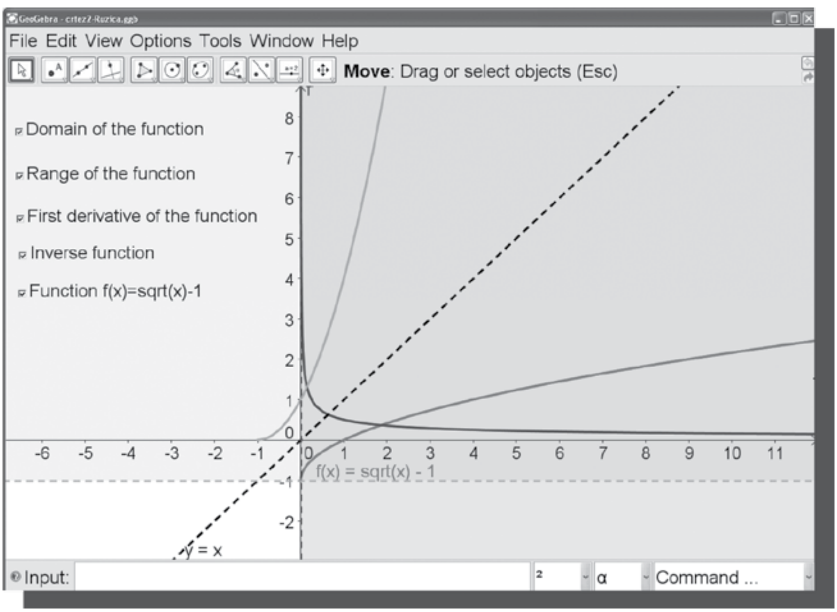


Figure 5. *Answers to the Test 2*

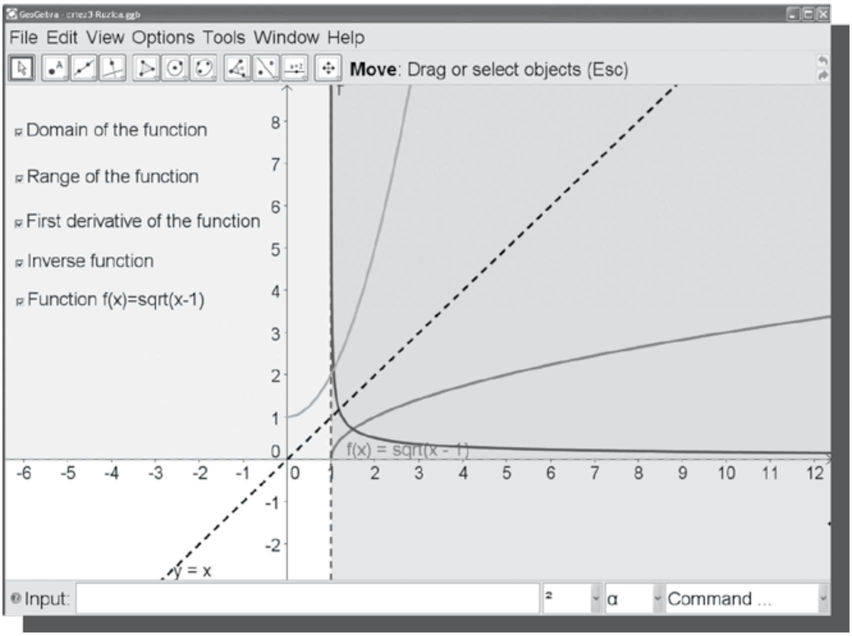


Figure 6. *Answers to the Test 3*

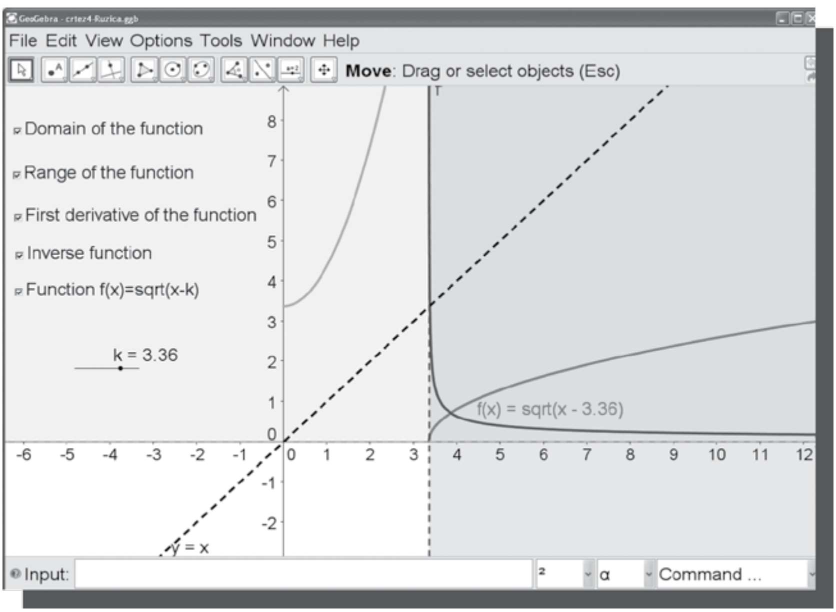


Figure 7. *Answers to the Test 4*

Discussion

The testing (T3) included 126 students of the 4th grade of grammar school specializing in natural sciences and mathematics, i.e. 63 students in the experimental group and 63 students in the control group.

Having analysed the obtained results, it was observed that the test results of the experimental group were better than the test results of the control group (as it can be seen from percentage: 41%, 11%, 49%, 29%) which is represented in Table 1 and Figure 8.

The results obtained vary from task to task due to the different level of task difficulty for students as well as differences in the organization of conducting experimental teaching in the groups. Therefore:

1) In the experimental group, teaching/learning of contents was conducted after the preparatory phase in which students were completely equipped for teaching/learning in a computer classroom which provided them with additional motivation and therefore better results.
2) In the control group, programmed experimental contents were implemented in a traditional way without a preparatory phase which resulted in a lack of additional motivation and much weaker results.

Table 1. *Results of testing experimental and control group*

test	experimental	control
1.	78%	37%
2.	51%	40%
3.	87%	38%
4.	53%	24%

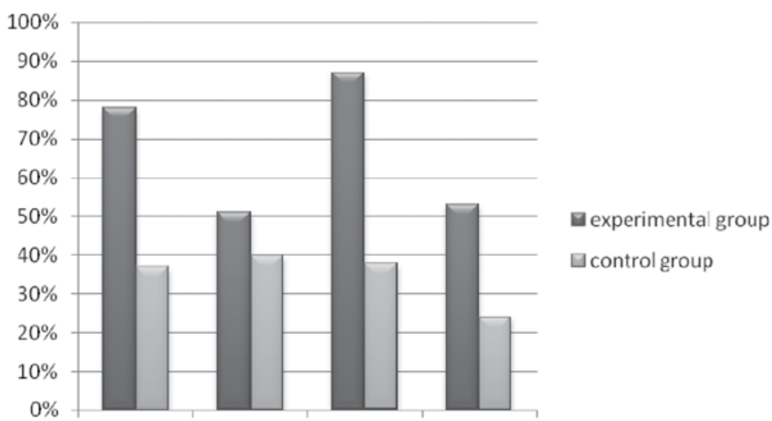


Figure 8. *Graphic presentation of the obtained results*

Conclusion

In the process of experimental research on the segment of a degree function, we confirmed the assumption that the outcomes of programmed instruction in a computer classroom are significantly better than the outcomes achieved through traditional mathematics teaching within the non-probability sample of students of the 4th grade of grammar school.

Considering the presented results obtained in the experimental research, we can determine contributions within the answers to these two questions:

1) What are the actual contributions of the conducted experimental research?
2) What other experimental research is suggested within teaching and learning mathematics?

Answers:

1) The emphasis in the answer to the first question is on the following:
   a) The most important contribution of the paper is the final construction of educational space which is represented by the arranged set of six elements (B, L, M, P, S, Z) according to "the Cartesian method" by establishing the feedback loop between the six elements.
   b) The second contribution is the applied methodology in the conducted research and the evaluation of the results within that.
2) The emphasis in the answer to the second question is on the following:

   Further research is suggested under the conditions of:
   – automated frontal teaching (managed by computer)
   – programmed instruction with the use of GeoGebra programme package
   – additional research could be related to the further modification of educational space of mathematics teaching by dividing the components into their parts.

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Usvajanje pojma/pojmova funkcije u programiranoj nastavi koja se izvodi u računalnom kabinetu

Sažetak

U nastavi matematike pojam/pojmovi funkcije vrlo su bitni, no također i vrlo zahtjevni. Stoga se postupno uvode u nastavni proces ovisno o dobi učenika. Razvoj matematike i obrazovne tehnologije omogućava raznolike pristupe pri upoznavanju učenika s tim pojmovima, kao i raznolike pristupe testiranju stupnja njihove usvojenosti od učenika. Ovaj rad navodi mogućnost usvajanja pojmova funkcije putem programirane nastave, primjenom kartezijanske metode, u kojoj se obrazovni prostor predstavlja „točkom" koja se sastoji od šest komponenti. Jedna od tih navedenih komponenti su mediji, koji se koriste za usvajanje novoga nastavnog sadržaja, no i za testiranje onoga što su učenici uistinu usvojili, npr. u području funkcije. Ovaj pristup osigurava postojanje izravne petlje povratne veze, pa se nastava matematike organizira kao praktičan proces.
Navedeno se učinkovito provodi u računalnom kabinetu putem programirane nastave i primjenom odgovarajućih alata i obrazovnoga računalnog softvera.

Ključne riječi: *izravna petlja povratne veze; kartezijanska metoda; mediji; obrazovni računalni softver.*

Uvod

Matematika je temeljni predmet i u osnovnoj i u srednjoj školi, predaje se određen broj nastavnih sati koji je utvrđen kurikulom. Navedeni status matematike kao nastavnog predmeta određen je nastavnim planom i programom koji određuje nastavne sadržaje koji će se poučavati na satima matematike u svakom razredu.

Zbog složenosti nastavnog sadržaja tog nastavnog predmeta, nastavni plan i program obično ima linearnu spiralnu strukturu. Postupci za uvođenje i oblikovanje matematičkih pojmova (pravila i činjenice) putem apstrakcije i generalizacije u nastavi su matematike svakako problem matematike kao znanosti, ali također i problem metodike nastave matematike.

Jedan stvaran primjer navedenoga jest poučavanje pojma/pojmova funkcije i testiranje njegova usvajanja. Učenici se prvi put eksplicitno susreću s pojmom funkcije u sedmom razredu osnovne škole, a zatim i u svim razredima prirodoslovno-matematičke gimnazije ili opće gimnazije, a intuitivno puno prije.

Budući da je pojam funkcije bitan za nastavu matematike, no da je također i vrlo zahtjevan, potrebno ga je u nastavu uvesti na pristupačan način kako bi učenicima bio lako razumljiv (Elia, Panaoura, Eracleus i Gagatsis, 2007; Gagatsis i Shiakalli, 2004; Vinner i Dreyfus, 1989).

U sklopu nastavnih metoda koje se koriste u nastavi matematike, i općenito, i u dijelu nastavne cjeline kao što je funkcija, kao polaznu točku možemo uzeti jednostavno pitanje: KOME?, no jednostavan odgovor na to pitanje otvara teška pitanja kao što su: ŠTO? i KAKO? Funkcija je jedan od temeljnih pojmova u matematici i mora se na odgovarajući način uklopiti u nastavu matematike, što je jako važno u nastavnim planovima i programima na svim razinama.

Budući da je pojam funkcije učenicima nejasan i kompliciran, postoji potreba da se pronađe ispravan način njegova tumačenja, uvođenja i oblikovanja u nastavi matematike.

Također, moramo razmotriti i činjenicu da je pojam funkcije prošao različite transformacije tijekom svojeg nastajanja u matematici (kao znanosti). Najvažnije su transformacije ova dva stupnja:

– prvi stupanj karakterizira oslanjanje na ideju ovisnosti (kraj 19. stoljeća i prva polovina 20. stoljeća) u prvoj fazi
– drugi stupanja karakterizira oslanjanje na ideju da je funkcija poseban odnos, uzimajući teoriju skupova i matematičke logike kao osnovu u drugoj fazi.

U ovom radu nastava matematike i provjeravanje rezultata temelje se na drugoj interpretaciji pojma funkcije.

Nastavni proces za nastavnu cjelinu funkcije uključuje: pojam funkcije (definicija), način zadavanja zadatka, domenu i kodomenu, neprekidnost, ekstremne vrijednosti, parne i neparne funkcije, recipročne funkcije, kompozicije funkcije (što su elementi nastavnog plana i programa matematike za četvrti razred gimnazije).

Rad je dio složenijeg eksperimentalnog istraživanja koje su proveli autori u sklopu teme „Od uvođenja pojma funkcije do njegova formiranja".

Pregled važnijih prethodnih istraživanja

Brz razvoj matematike tijekom stoljeća nije uzrokovao brze promjene u nastavi. Štoviše, metodika nastave matematike još uvijek nije pronašla svoje mjesto među znanstvenim disciplinama. Bez obzira na činjenicu da ne postoji kongruencija između matematike kao znanosti i njezine nastave, istraživanje provedeno o nastavnom procesu neophodno je i važno da bi se došlo do optimalnih rezultata.

Jedan od najvažnijih projekata u nastavi matematike je Nuffieldov projekt (za učenike u dobi od pet do četrnaest godina). Prva eksperimentalna verifikacija projekta provedena je u drugoj polovini prošloga stoljeća. Glavni je cilj Nuffieldova projekta bio ispitati kako se poučava matematika kao nastavni predmet. U središtu projekta bilo je stajalište da bi učenici na nastavi matematike trebali biti slobodni i da bi samostalno trebali razmišljati i otkrivati s ciljem postizanja višeg stupnja razumijevanja.

Ukratko, osnovna ideja Nuffieldova projekta jest: činim – razumijem. Osim Nuffieldova projekta u Engleskoj je provedeno nekoliko projekata o nastavi matematike. Najznačajniji su: Školski matematički projekt, Midlands matematički eksperiment, Matematički kurikulski projekt, itd.

J. P. Galperin je zaključio da se nastava matematike mora organizirati kao proces vođenih misaonih aktivnosti u poučavanju/učenju. Na taj je način ušao u područje kibernetike pedagoških procesa i programirane nastave.

L. N. Landa je zagovarao potrebu da se nastavni proces učini što je više moguće praktičnim i da se postave precizni ciljevi i zadatci, i u sustavu organizacije, i u stvaranju petlje povratne veze.

E. G. Begle je američki matematičar koji je bio voditelj školske matematičke grupe. U matematičkim udžbenicima koje je ta grupa objavila, posebna pažnja poklanja se: strukturama, suvremenoj terminologiji, simbolizmu i preciznom matematičkom jeziku, kao i autonomnom učenju otkrivanjem. Na taj način, transformacijom nastavnih planova i programa na osnovnom stupnju, školska matematička grupa naglašava sadržaj i strukture.

Rezultati istraživanja sveučilišnih profesora matematike u Briselu (Centar za matematičku pedagogiju) F. Papyja i G. Papyja doprinijeli su stvaranju suvremene metode nastave matematike koja se temelji na odnosima (za djecu u dobi od pet do sedamnaest godina). Ignacije Smolec, hrvatski matematičar i metodičar, nastavio je njihov rad.

Računalno potpomognuta nastava matematike (*Computer based Mathematics Instruction*) projekt je iz druge polovine prošloga stoljeća, koji je omogućio izvođenje nastave matematike korištenjem računala u računalnom kabinetu. Prvo eksperimentalno istraživanja provedeno je na Sveučilištu Stanford u Kaliforniji (Suppes, 1963).

Suvremeni metodički pristup nastavi matematike naglašava veću aktivnost i autonomnost učenika. Posebna važnost daje se eksperimentima u nastavi (učenje otkrivanjem). Učenje otkrivanjem preko eksperimenata omogućuje učenicima da samostalno pronađu rješenja matematičkih problema, steknu nove uvide, razviju kreativnost, a sve to dok rade svojim vlastitim tempom (individualizacija tempa učenja).

Dreyfus i Vinner (1982, 1989) testirali su 271 učenika i 36 nastavnika ispitujući ih o funkcijama. Pitanja su se sastojala od definicija i zanimljivih grafova, a učenici su trebali dati točne odgovore (npr. Koji od grafova predstavlja funkciju?) (Tall, 1991).

Frank Helmar proveo je opsežno istraživanje i generalizaciju o primjeni kartezijanske metode u vođenju nastavnog procesa (...) stvaranjem šesterodimenzionalnog prostora kao odgovora na pitanje: Kako učimo?

Autori ovoga rada (Vukobratović, 2009) u svojem su se istraživanju usredotočili na izvođenje nastave matematike s upotrebom računala ili bez nje (programirana nastava matematike).

Pozitivni rezultati eksperimenta, provedenog ili u programiranoj ili u metodičkoj sferi, pružili su dovoljno dobru osnovu za transformiranje nastave matematike.

Promjene u nastavnim planovima i programima većinom su bile uzrokovane razvojem matematike kao znanosti. Ovaj razvoj, kao i razvoj tehnologije, doveo je do osuvremenjivanja nastave, te je bio rezultat zahtjeva: *suvremeni nastavni plan i program – suvremeno tumačenje – veća uspješnost*.

Metodologija provedenog istraživanja

Opći metodološki pristup

Suvremena organizacija poučavanja/učenja matematike, u smislu provođenja i primjene suvremenoga tehnološkog istraživanja, ogleda se u sljedećem:

– protok informacija od učenika do nastavnika i od nastavnika do učenika trebao bi se znatno povećati, a tako će se povećati i uspješnost nastave matematike
– mogućnost nastavnika matematike da prikupi informacije od svakog učenika izravnom komunikacijom i/ili preko posrednika (računalo)
– informacija (koja dolazi od učenika) treba biti potpuna tako da pruža uvid u razne parametre aktivnosti učenika, posebno u mehanizme psiholoških procesa.

Nastavnik matematike i/ili računalni sustav na neki način treba reagirati na informacije koje je dobio od učenika.

Važan zahtjev za dobro provedenu nastavu matematike je operativna petlja povratne veze.

Metodička podloga za proučavanje izvođenja nastave matematike u ovom je radu analitička metoda i programirana nastava u računalnom kabinetu.

Kako je provedeno istraživanje

Pregled etapa u eksperimentalnom istraživanju

Počevši od općeg metodološkog pristupa korištenog u eksperimentalnom istraživanju, da bi se omogućilo provođenje istraživanja, morali su biti utveđeni: predmet, ciljevi i zadatci istraživanja, opis metodološke pozadine i opis eksperimentalnog istraživanja.

Predmet

Predmet istraživanja u ovome je radu eksperimentalno ispitivanje utjecaja programirane nastave u računalnom kabinetu i primjena analitičke metode na optimizaciju rezultata (na primjeru poučavanja programske cjeline funkcije u četvrtom razredu gimnazije).

Ciljevi i zadatci eksperimentalnog istraživanja

Razvoj matematike, metodike i tehnologije pruža osnovu za definiranje zadataka kojima je cilj odrediti postiže li provedba programirane nastave/učenja matematike u računalnom kabinetu korištenjem adekvatnih aplikacija obrazovnog računalnog softvera optimalne nastavne rezultate koji se mogu koristiti kao osnova za promjenu nastave matematike. Unutar definiranog cilja mogu se definirati sljedeći specifični zadatci eksperimentalnog istraživanja:

- priprema tehničke i tehnološke osnove za provođenje eksperimentalnog istraživanja o kulturi elektroničke komunikacije (prigodan hardver i softver)
- nastavnici i učenici u eksperimentalnim razredima pripremaju se za optimalno korištenje okoline poučavanja/učenja
- procijeniti obrazovni računalni softver prema specifičnim kriterijima da bi se odabrao onaj koji će dovesti do najboljih obrazovnih rezultata.

Opis prve metodološke pozadine eksperimentalnog istraživanja

Obrazovni prostor nastave matematike

Metodološka pozadina za proučavanje izvođenja nastave matematike jest analitička metoda iz prve polovine 17. stoljeća koju su proučavali Galileo Galilei i Rene Descartes. Matematički problemi su vrlo teški, pa ih stoga treba razdijeliti na manje kompleksne probleme. Najprije se rješavaju najjednostavniji problemi. Moderne znanosti koriste kartezijansku metodu za rješavanje problema. Tu metodu je za obrazovni prostor prilagodio Paul Haimann 1962. i predstavio je kao točku sa šest komponenti (B, L, M, P, S, Z) (Slika 1). Komponente su povezane petljom povratne veze, što je doprinos ovoga rada.

Slika 1.

Ako se analizira prikaz obrazovnog prostora, može se primijetiti sljedeće:

1) uobičajena nastava može se podijeliti na sustav učenja P (učenik) i sustav okoline u kojoj se učenje odvija S (nastavnik);
2) svrha uobičajene nastave je u naučenom sadržaju L i skrivenom značenju naučenoga gradiva (ciljevi i zadatci) Z.

Korisno je promatrati sustav učenja/poučavanja Q kao par dviju komponenti: „obrazovnog modela“ B i „primijenjenih medija“ M. Tada poučavanje postaje kvazi-točka koja predstavlja uređeni skup šest komponenti (B, L, M, P, S, Z) u šesterodimenzionalnom prostoru.

Svaku od navedenih komponenti možemo nakon analize podijeliti na njihove komponente.

Ispitivanjem obrazovnog prostora može se primijetiti da formalno moguće kombinacije nisu uvijek pogodne za nastavu matematike: mediji (M) ne slijede nužno metodu (B) da bi se postigla specifična kompetencija učenika (P) u danoj okolini (S) za određeni obrazovni sadržaj (L). Nastavnik matematike trebao bi težiti modernoj organizaciji nastave koja podrazumijeva korištenje informacijske tehnologije (računala) (Cheng, 2000).

Na nastavnom satu matematike, kao i na ostalim nastavnim satima, nastavnik i učenik trebali bi zajedno raditi da bi postigli obrazovne ciljeve i zadatke. Analiziranjem prikaza šesterodimenzionalnog obrazovnog prostora (Slika 1) možemo odvojiti segmente (Slika 2), kao što je to bilo u slučaju medija – M (1) i metoda – B (2), što se može zorno prikazati na sljedeći način:

Slika 2.

U kartezijanskoj je metodi jedan od najvažnijih čimbenika uspješnog poučavanja/učenja motivacija učenika (motiv – svaki razlog čija je posljedica ponašanje; motivacija – preduvjet da bismo nešto učinili).

Opis druge metodološke pozadine eksperimentalnog istraživanja

Neke pretpostavke o programiranoj nastavi

Programirana nastava (matematike) definira se kao nastava u kojoj učenici samostalno provode nastavni plan i program koji određuje nastavni sadržaj i način učenja, uključujući postupke za sustavno informiranje o ostvarenim rezultatima.

Kada se izrađuje nastavni plan i program (za nastavu matematike), najprije se odabire nastavni sadržaj tako da se eliminira sve što je nevažno. Kao drugo, nastavni plan i program logički je strukturiran. Svaki je dio popraćen odgovarajućim zadatkom/zadatcima, a pronalaženje rješenja zahtijeva primjenu znanja i ovisi o stupnju njegove usvojenosti. Rješenja zadataka dana su u programu, pa učenici mogu usporediti svoja rješenja s programom, da bi saznali mogu li nastaviti s usvajanjem sljedećih dijelova nastavnog gradiva. Istraživanja su pokazala da je programirana nastava učinkovitija od uobičajene, konvencionalne nastave. Uočene su sljedeće prednosti:

– u programiranoj nastavi učenici su uvijek pravodobno obaviješteni o rezultatima svojeg rada
– programirana nastava potiče samostalnost učenika u procesu učenja
– programirana nastava odbacuje pasivno učenje i zamjenjuje ga učenjem koje se temelji na konstantnoj aktivnosti
– proces učenja u programiranoj nastavi uvelike je individualiziran
– programirana nastava kombinira grupni oblik nastave s pravom individualizacijom, što omogućava prilagodbu nastave pojedincima.

Programirana nastava u računalnom kabinetu

U današnje vrijeme u svijetu se poklanja iznimno mnogo pažnje izravnoj uporabi računala u učenju i poučavanju. Najbolja rješenja (modeli, nastavni planovi i programi, pristupi) u obrazovanju, učenju i poučavanju stvaraju nastavnici. Postoje razni pristupi uporabi informatike u obrazovanju, poučavanju i učenju. Jedan od pristupa temelji se na takozvanim sustavima inteligentnih mentora, što se opet temelji na biheviorističkom učenju.

Programirana nastava/učenje je primjer pojma učenja s rješenjima bihevioritičkog pristupa obrazovnom procesu. U takvoj nastavi mehanizam radi i osigurava kontinuiranu aktivnost učenika uz prisutnost izravne petlje povratne veze u sljedećem obliku:

PODRAŽAJ – REAKCIJA – PODRŠKA

U moderno vrijeme, kada je razvoj informacijske tehnologije golem, mediji su zauzeli važno mjesto u svakodnevnoj nastavi matematike (i nastavi općenito). Da bismo usustavili pojam funkcije, odabrali smo programiranu nastavu matematike koja se izvodi korištenjem računala u računalnom kabinetu (Cheng, 2000).

Nastavni proces i komunikacija između nastavnika i učenika, ali i računala i učenika, odvijaju se u računalnom kabinetu (Taylor, 1980). Nastavnik prati rad učenika na središnjem računalu (serveru), a svaki učenik radi na svojem računalu (klijent). Mape za svaki razred otvaraju se na serveru (npr. I1, I2, I3…), a svaka mapa sadrži imena učenika (Slika 3). Komunikacija se shematski može prikazati na sljedeći način:

Slika 3.

Detalji komunikacije su sljedeći:

1) Nastavnik odabire i/ili priprema programirani materijal (informacije, zadatke, testove).
2) Nastavnik šalje pripremljene programirane materijale sa središnjeg računala (servera) na računala učenika (klijente).
3) Učenici se upoznaju s informacijama i rješavaju zadane zadatke.
4) Nastavnik prati rad učenika.
5) Učenik uspoređuje svoje rješenje s rješenjem koje mu prikazuje računalo. Nastavnik kontrolira rad, pa ako učenik ne riješi točno matematički problem, nastavnik ga o tome obavijesti i pruži mu dodatne informacije.
6) Ako učenik naiđe na problem, nastavnik mu pošalje upute putem računala.
7) Proces je stalan za rješavanje svih zadataka. Važna je činjenica da je to individualan rad.
8) Učenici mogu rješavati zadatke koristeći GeoGebra programski paket.
9) Ako je komentar potreban većem broju učenika, nastavnik komentira tipične pogreške preko sustava ili verbalno, pred pločom, dok učenici vode bilješke.

U takvoj vrsti programirane nastave postoji kontinuirana petlja povratne veze između nastavnika i učenika, što je vrlo važno za provedbu modernoga metodičkog pristupa nastavi matematike, kojemu su u središtu zanimanja upravo individualizirani pristup i subjektivni doživljaj učenika.

Nastavnik matematike, koji vodi nastavni proces, ne bavi se pojedincem (učenikom), nego kolektivom (cijelim razredom). Putem računala učenici uvježbavaju učinkovitu komunikaciju, suradnju i timski rad. Grafičke mogućnosti računala pomažu tome da matematika „bude viđena", algebarski dio računalnog programa vodi računa o tome da se matematika radi, a programski jezik osigurava stvaranje matematike. Računalo je podražaj motivaciji, potiče učeničku kreativnost i učinkovit je alat u testiranju znanja učenika.

Opis treće metodološke pozadine eksperimentalnog istraživanja

Specifični sadržaji u eksperimentalnoj nastavi

Ovaj rad također uključuje poučavanje i testiranje sadržaja stupnja funkcije koji obuhvaća definiciju, domenu, kodomenu, neprekidnost i ekstremne vrijednosti, recipročne funkcije i grafički prikaz funkcije.

Istraživanje također obuhvaća i stupanj funkcije sljedećih oblika:

$$f(x)=\sqrt{x}\,,\ f(x)=\sqrt{x}-1\,,\ f(x)=\sqrt{x-1}\,,\ f(x)=\sqrt{x-k}\,,\quad k\in R$$

Ovaj dio istraživanja bio je proveden pred kraj školske godine 2010./2011., neposredno prije početka prijemnih ispita na fakultetima.

Uzorak korišten u eksperimentalnom istraživanju

Definiranjem predmeta eksperimentalnog istraživanja određeni su i ciljevi, zadatci i metodološka pozadina (varijable – eksperimentalni faktori), što je utjecalo na izbor modela istraživanja. Model glavnog eksperimenta je eksperiment s paralelnim grupama (Mužić, 1973). Sljedeće dvije paralelne grupe oblikovane su za eksperimentalno istraživanje u četvrtim razredima prirodoslovne i matematičke gimnazije:

- Eksperimentalna grupa (tri razreda u kojima su se nastavni sadržaji poučavali izvođenjem programirane nastave u računalnom kabinetu);
- Kontrolna grupa (tri razreda u kojima su se nastavni sadržaji poučavali primjenom klasičnih metoda).

Eksperimentalno istraživanje usmjereno je na jednolikost grupe ili ekvivalentnost, što se moralo ostvariti tako što su se školska pravila poštivala i uvažavala. Stoga su svi učenici iz razreda koji su činili uzorak bili uključeni u eksperiment, ali nisu morali biti uključeni u eksperimentalnu ili kontrolnu grupu. Zbog ekvivalentnosti grupe određeni broj učenika iz nekih razreda nije bio uključen u grupu. Tijekom eksperimenta, sa svim se učenicima postupalo na isti način, čak i tijekom testiranja. Međutim, kada su rezultati o utjecaju eksperimentalnog faktora bili sažimani, uzeli smo u obzir samo rezultate onih učenika koji su bili članovi eksperimentalne ili kontrolne grupe.

Odabrani uzorak bio je namjerno odabrani uzorak i sastojao se od razreda iz različitih škola u Novom Sadu.

Dva kriterija utjecala su na izbor razreda koji su činili uzorak:

- jednolikost nastavnika koji poučavaju matematiku u razredima odabranima za uzorak (jednolikost se mjerila prema stručnim kvalifikacijama, godinama radnog staža i rezultatima rada);
- jednolikost odabranih učenika prema spolu, uspješnosti u matematici, rezultatima testova o prethodno stečenom znanju i općem školskom uspjehu.

Smatra se da je jednolikost uzorka dobra s obzirom na definirane karakteristike.

Jedan od preduvjeta objektivnosti rezultata dobivenih eksperimentalnim istraživanjem je osigurati kontrolu i standardizirati eksperimentalne situacije. Eksperimentalno istraživanje organizirano je pod takvim uvjetima da je potpunu standardizaciju bilo teško postići. Da bi se osigurao viši stupanj standardizacije eksperimentalnih uvjeta, organizirana je pripremna faza u obliku seminara u sklopu svake etape eksperimenta (priprema za izvođenje programirane nastave u računalnom kabinetu).

Rezultati

Znanje (ishodi eksperimentalnog istraživanja) je bilo testirano zadatcima objektivnog tipa, T3. Testiranje je trajalo 35 minuta, a koristio se GeoGebra programski paket (Herceg i Herceg, 2007), na sljedeći način:

Tipični primjeri:

Test za provjeru usvojenoga znanja o pojmu funkcije.

Upitnik

Upitnik je dan trima grupama srednjoškolskih učenika.

Prije upitnika učenici su već rješavali nešto više zadataka nego što im je dano u upitniku.

Upitnik je imao sljedeći oblik:

1. Zadana je funkcija $f(x)=\sqrt{x}$.
   a) Odredi domenu i područje funkcije.
   b) Odredi neprekidnost i ekstremne vrijednosti funkcije.
   c) Odredi recipročnu funkciju zadane funkcije.
   d) Nacrtaj graf funkcije.
2. Zadana je funkcija $f(x)=\sqrt{x}-1$.
   a) Odredi domenu i područje funkcije.
   b) Odredi neprekidnost i ekstremne vrijednosti funkcije.
   c) Odredi recipročnu funkciju zadane funkcije.
   d) Nacrtaj graf funkcije.
3. Zadana je funkcija $f(x)=\sqrt{x-1}$.
   a) Odredi domenu i područje funkcije.
   b) Odredi neprekidnost i ekstremne vrijednosti funkcije.
   c) Odredi recipročnu funkciju zadane funkcije.
   d) Nacrtaj graf funkcije.
4. Zadana je funkcija $f(x)=\sqrt{x-k}, \quad k \in R$.
   a) Odredi domenu i područje funkcije.
   b) Odredi neprekidnost i ekstremne vrijednosti funkcije.
   c) Odredi recipročnu funkciju zadane funkcije.
   d) Nacrtaj graf funkcije.

Rješenja danih primjera koja je dao GeoGebra programski paket.

Slika 4., 5., 6. i 7.

Rasprava

Testiranje (T3) je obuhvatilo 126 učenika iz četvrtog razreda prirodoslovne i matematičke gimnazije, tj. 63 učenika iz eksperimentalne i 63 učenika iz kontrolne grupe.

Nakon analize dobivenih rezultata uočeno je da su rezultati testa eksperimentalne grupe bili bolji od rezultata kontrolne grupe (kao što se može vidjeti iz postotaka: 41%, 11%, 49%, 29%), što je prikazano u Tablici 1, Slika 8.

Dobiveni rezultati razlikuju se od zadatka do zadatka zbog različitog stupnja težine zadataka za učenike, kao i zbog razlika u organizaciji provođenja eksperimentalne nastave u grupama.

Stoga:

1) U eksperimentalnoj grupi poučavanje/učenje nastavnih sadržaja bilo je provedeno nakon pripremne faze u kojoj su učenici bili potpuno osposobljeni za poučavanje/učenje u računalnom kabinetu, što im je pružilo dodatnu motivaciju, a time i bolje rezultate.
2) U kontrolnoj grupi programirani eksperimentalni nastavni sadržaji bili su provedeni na tradicionalni način, bez pripremne faze, što je rezultiralo nedostatkom dodatne motivacije i puno lošijim rezultatima.

Slika 8.

Zaključak

U procesu eksperimentalnog istraživanja o dijelu stupnja funkcije potvrdili smo pretpostavku da su obrazovni ishodi programirane nastave u računalnom kabinetu znatno bolji od obrazovnih ishoda tradicionalne nastave matematike unutar namjernog uzorka učenika četvrtog razreda gimnazije.

Razmatrajući prikazane rezultate dobivene eksperimentalnim istraživanjem, možemo odrediti doprinos u sklopu odgovora na ova dva pitanja:

1) Koji su stvarni doprinosi provedenog eksperimentalnog istraživanja?
2) Kakvo se drugo eksperimentalno istraživanje nameće u sklopu poučavanja i učenja matematike?

Odgovori:

1) U odgovoru na prvo pitanje naglasak je na sljedećem:
   a) Najvažniji doprinos ovoga rada je konačno kreiranje obrazovnog prostora koje se prikazuje kao uređeni skup od šest elemenata (B, L, M, P, S, Z), prema kartezijanskoj metodi, tako što se uspostavlja petlja povratne veze između šest elemenata.
   b) Drugi doprinos je metodologija primijenjena u provedenom istraživanju i evaluacija rezultata.

2) U odgovoru na drugo pitanje naglasak je na sljedećem:
Daljnje istraživanje predlaže se pod sljedećim uvjetima:
- automatizirana frontalna nastava (koju izvodi računalo)
- programirana nastava uz uporabu GeoGebra programskog paketa
- dodatno istraživanje moglo bi biti o daljnjoj modifikaciji obrazovnog prostora nastave matematike tako što bi se njezine komponente podijelile na manje dijelove.