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Lamoenian Circles of the Collinear Arbelos

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ABSTRACT

We give an infinite sets of circles which generate Archimedean circles of a collinear arbelos.

Key words: arbelos, collinear arbelos, radical circle, Lamoenian circle

MSC2010: 51M04, 51M15, 51N20

Lamoenove kružnice kolinearnog arbelosa

SAŽETAK

Pokazujemo beskonačne skupove kružnica koje generiraju Arhimedove kružnice kolinearnog arbelosa.

Ključne riječi: arbelos, kolinearni arbelos, potencijalna kružnica, Lamoenova kružnica

1 Introduction

For a point O on the segment AB, let α, β and γ be circles with diameters AO, BO and AB respectively. Each of the areas surrounded by the three circles is called an arbelos. The radical axis of the circles α and β divides each of the arbeloi into two curvilinear triangles with congruent incircles (see the lower part of Figure 1). Circles congruent to those circles are said to be Archimedean.

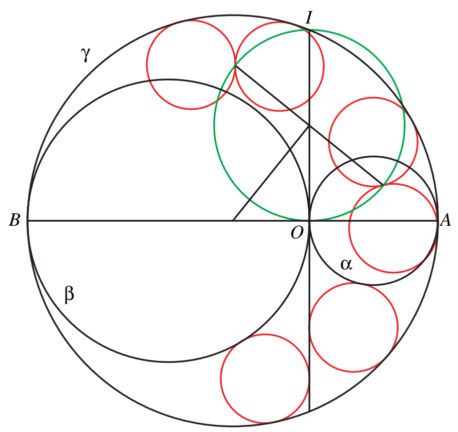

Figure 1: *A circle generating Archimedean circles with* γ

For a point T and a circle δ, if two congruent circles of radius r touching at T also touch δ at points different from T, we say T generates circles of radius r with δ, and the two circles are said to be generated by T with δ. If the generated circles are Archimedean, we say T generates Archimedean circles with δ. Frank Power seems to be the earliest discoverer of this kind Archimedean circles: The farthest points on α and β from AB generate Archimedean circles with γ [6].

Let I be one of the points of intersection of γ and the radical axis of α and β. Floor van Lamoen has found that the endpoints of the diameter of the circle with diameter IO perpendicular to the line joining the centers of this circle and γ generate Archimedean circles with γ [2] (see the upper part of Figure 1). We say a circle $\mathcal{C}$ generates circles of radius r with δ, if the endpoints of a diameter of $\mathcal{C}$ generate circles of radius r with δ. Circles generating Archimedean circles with γ are said to be Lamoenian. In this article we consider those circles in a general way.

2 The collinear arbelos

In this section we consider a generalized arbelos. For two points P and Q in the plane, (PQ) and $P(Q)$ denote the circle with diameter PQ and the circle with center P passing through Q respectively. For a circle δ, O_δ denotes its center. For two points P and Q on the line AB, let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. Let O be the point of intersection of AB and the radical axis of the circles α and β and let $u = |AB|$, $s = |AQ|/2$ and $t = |BP|/2$. Unless otherwise stated, we use a rectangular coordinate system with origin O such that the points A, B and P have coordinates $(a,0)$, $(b,0)$ and $(p,0)$ respectively with $a - b = u$. The configuration (α,β,γ) is called a collinear arbelos if the four points lie in the order (i) B, Q, P, A or (ii) B, P, Q, A, or (iii) P, B, A, Q. In each of the cases the configurations are explicitly denoted by $(BQPA)$, $(BPQA)$ and $(PBAQ)$ respectively. In the case $P = Q = O$, (α,β,γ) gives an ordinary arbelos, and (α,β,γ) is called a tangent arbelos. Archimedean circles of the ordinary arbelos are generalized to the collinear arbelos (α,β,γ) as circles of radius $st/(s+t)$, which we denote by r_{A} [3]. Circles of radius r_{A} are also called Archimedean circles of (α,β,γ). The radius is also expressed by

$$r_{\mathrm{A}} = \frac{|AO||BP|}{2u} = \frac{a|p-b|}{2u}. \tag{1}$$

3 Lamoenian circles of the collinear arbelos

A circle generating circles of radius r_{A} with γ is also said to be Lamoenian for the collinear arbelos (α,β,γ). In this section we give a condition that a circle is Lamoenian. For a circle δ of radius r and a point T, let us define

$$\mathrm{r}(T,\delta) = \frac{|r^2 - |TO_\delta|^2|}{2r},$$

which equals the radius of the generated circles by T with δ by the Pythagorean theorem.

Theorem 1 *Let δ be a circle of radius r and let J, H be points with J lying on δ. The circle (HJ) generates circles of radius s with δ if and only if*

$$|HO_\delta|^2 = r(r \pm 4s). \tag{2}$$

In this event, the following statements are true.

(i) *If a points K lies on the circle $O_\delta(H)$, the circle (KJ) generates circles of radius s with δ.*

(ii) *The point $O_{(HJ)}$ lies on the circle of radius $r/2$ with center $O_{(HO_\delta)}$.*

Proof. Let $h = |HO_\delta|$ (see Figure 2). We use a rectangular coordinate system with origin O_δ such that the coordinates of H is $(h,0)$ in this proof. Let (f,g) be the coordinates of the point $O_{(HJ)}$, and let T be one of the endpoints of the diameter of (HJ) perpendicular to $O_\delta O_{(HJ)}$. Then $\overrightarrow{O_{(HJ)}T} = k(-g,f)$ and $\overrightarrow{O_\delta T} = (f-kg, g+kf)$ for a real number k. From $|O_{(HJ)}T| = |O_{(HJ)}H|$, $(-kg)^2 + (kf)^2 = (f-h)^2 + g^2$, which implies

$$k^2 = \frac{(f-h)^2 + g^2}{f^2+g^2}. \tag{3}$$

The circle (HJ) generates circles of radius s with δ if and only if

$$\mathrm{r}(T,\delta) = \frac{|r^2 - ((f-kg)^2 + (g+kf)^2)|}{2r} = s.$$

Since (3) holds, the last equation is equivalent to

$$\frac{1}{4}h^2 + \left(f - \frac{h}{2}\right)^2 + g^2 = \frac{1}{2}r(r \pm 2s),$$

where the plus (resp. minus) sigh should be taken when T lies outside (resp. inside) of δ. If (v,w) are the coordinates of the point J, $(v+h)/2 = f$ and $w/2 = g$. Therefore the last equation is equivalent to

$$\frac{1}{4}h^2 + \frac{1}{4}r^2 = \frac{1}{2}r(r \pm 2s),$$

which is also equivalent to (2). The part (i) obviously holds. The center of (HJ) is the image of J by the dilation with center H and scale factor $1/2$. This proves (ii). □

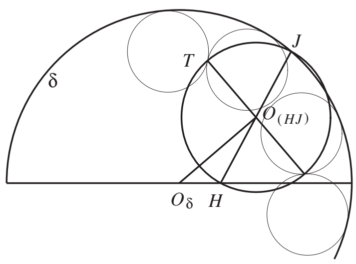

Figure 2

Let ε be the circle with center O_γ belonging to the pencil of circles determined by α and β for the collinear arbelos (α,β,γ). We call ε the radical circle of (α,β,γ). The circle is considered in [4] and [5] for $(BQPA)$ and $(BPQA)$. If α and β have a point in common, ε passes through the point. For $(BQPA)$ let V be the point of tangency of one of the tangents of α from O (see Figure 3). Then $|OV|^2 = ap$. If $|OO_\gamma|^2 > ap$, a tangent from O_γ to the circle $O(V)$ can be drawn. Then ε passes through the point of tangency. If $|OO_\gamma|^2 = ap$, ε is the point circle O_γ, which coincides with one of the limiting points of the pencil. If $|OO_\gamma|^2 < |ap|$, ε does not exist. Let e be the radius of ε. For $(BQPA)$, $e^2 = |OO_\gamma|^2 - ap$ by the Pythagorean theorem. For $(BPQA)$ and $(PBAQ)$, $e^2 = |OO_\gamma|^2 + |ap|$ (see Figure 4). In any case

$$e^2 = |OO_\gamma|^2 - ap. \tag{4}$$

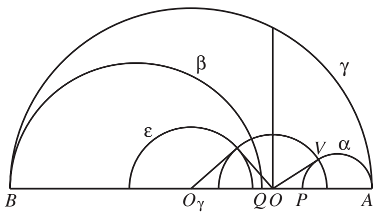

Figure 3: *The case $|O_\gamma O|^2 > |ap|$ for $(BQPA)$*

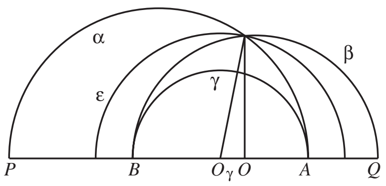

Figure 4: $(PBAQ)$

Theorem 2 *For a collinear arbelos* (α, β, γ) *with radical circle* ε*, if points* J *and* H *lie on* γ *and* ε *respectively, then the circle* (HJ) *is Lamoenian.*

Proof. For $(BPQA)$ and $(BQPA)$, $r_{\mathrm{A}} = a(p-b)/(2u)$ by (1). Therefore by (4),

$$\frac{u}{2}\left(\frac{u}{2} - 4r_{\mathrm{A}}\right) = \frac{(a-b)^2}{4} - a(p-b) = \frac{(a+b)^2}{4} - ap = e^2.$$

Similarly for $(PBAQ)$, we get

$$\frac{u}{2}\left(\frac{u}{2} + 4r_{\mathrm{A}}\right) = e^2.$$

Hence the theorem is proved by Theorem 1. □

4 Quartet of circles

In this section we show that a Lamoenian circle given by Theorem 2 is a member of a set of four Lamoenian circles. All the suffixes are reduced modulo 4 in this section. Let J_0 be a point on a circle δ, and let H be a point which does not lie on δ (see Figures 5, 6). Let R_0R_1 be the diameter of the circle (HJ_0) perpendicular to the line $O_\delta O_{(HJ_0)}$ and let R_0 and R_1 generate circles of radius s with δ. Let J_1 be the point of intersection of the line J_0R_1 and δ, and let R_2 be the point such that $HR_1J_1R_2$ is a rectangle. Then the circle (HJ_1) also generates circles of radius s with δ by Theorem 1. While R_1 generates circles of radius s with δ. Therefore R_2 also generates circles of radius s with δ. Similarly we construct the points J_2 and J_3 on δ and the points R_3 and R_4 such that J_2 and J_3 lie on the lines J_1R_2 and J_2R_3 respectively and $HR_2J_2R_3$ and $HR_3J_3R_4$ are rectangles. Then R_3 generates circles of radius s with δ and R_4 coincides with R_0. Now we get the points J_i on δ and R_i $(i = 0,1,2,3)$ such that R_iR_{i+1} is the diameter of (HJ_i), R_i generates circles of radius s with δ, $J_0J_1J_2J_3$ is a rectangle, R_i lies on the line J_iJ_{i-1}. The four circles (HJ_i) $(i = 0,1,2,3)$ are called a quartet on δ, and H and $J_0J_1J_2J_3$ are called the base point and the rectangle of the quartet respectively.

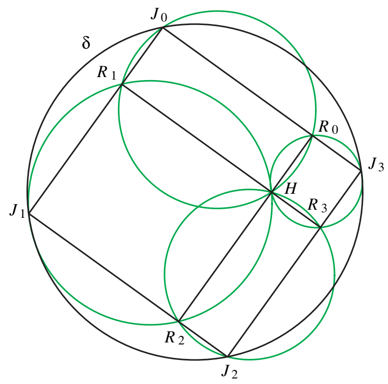

Figure 5: *H lies inside of* δ

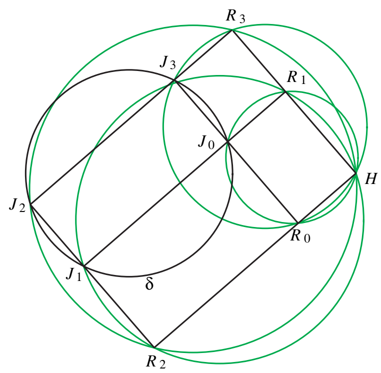

Figure 6: *H lies outside of* δ

By the definition of R_i, R_0, R_2, H are collinear, also R_1, R_3, H are collinear, and the two lines are perpendicular. Let $l_i = |HR_i|$. Then $|HJ_0|^2 + |HJ_2|^2 = l_0^2 + l_1^2 + l_2^2 + l_3^2 = |HJ_1|^2 + |HJ_3|^2$. Therefore $|HJ_0|^2 + |HJ_2|^2 = |HJ_1|^2 + |HJ_3|^2$ holds.

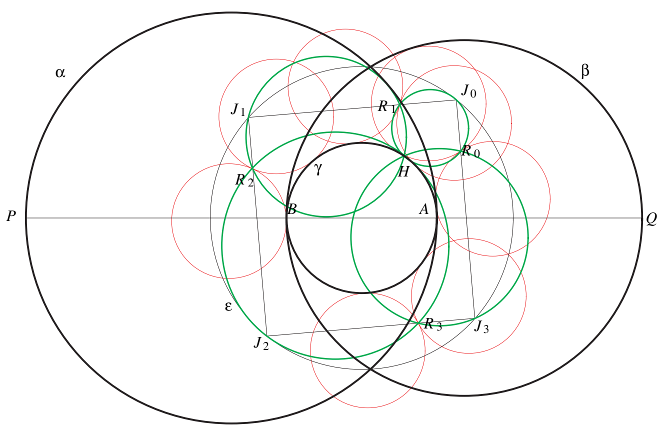

Figure 7: *A quartet of Lamoenian circles on* ε *for* $(PBAQ)$

For a collinear arbelos (α,β,γ) with radical circle ε, if the two points H and J_0 lie on ε and γ respectively, we can construct a quartet (HJ_i) $(i=0,1,2,3)$ on γ consisting of Lamoenian circles by Theorem 2. Also if H and J_0 lie on γ and ε respectively, we can construct a quartet (HJ_i) $(i=0,1,2,3)$ on ε consisting of Lamoenian circles (see Figure 7).

Theorem 3 *For a quartet (HJ_i) $(i=0,1,2,3)$ on a circle δ, the rectangle is a square if and only if (HJ_i) touches δ for some i. In this event, (HJ_{i+2}) also touches δ, and (HJ_{i-1}) and (HJ_{i+1}) are congruent and intersect at O_δ.*

Proof: If (HJ_0) touches δ, $R_0J_0R_1$ is an isosceles right triangle, since $|O_\delta R_0|=|O_\delta R_1|$. This implies that $J_3J_0J_1$ is also an isosceles right triangle, i.e., $J_0J_1J_2J_3$ is a square. Conversely let us assume $J_0J_1J_2J_3$ is a square. We assume that (HJ_i) does not touch δ for $i=0,1,2,3$. The sides or the extended sides of the square and the circle $O_\delta(R_0)$ intersect at eight points, four of which are R_0, R_1, R_2, R_3. If $|J_iR_i|=|J_iR_{i+1}|$, (HJ_i) touches δ. Therefore $|J_iR_i|\neq|J_iR_{i+1}|$ for $i=0,1,2,3$. This can happen only when R_1, R_2, R_3, R_4 lie inside of δ (see Figures 8 and 9). Hence $|J_0R_0|=|J_1R_1|=|J_2R_2|=|J_3R_3|\neq|J_0R_1|=|J_1R_2|=|J_2R_3|=|J_3R_0|$. Therefore the four rectangles $HR_iJ_iR_{i+1}$ $(i=0,1,2,3)$ are congruent. Then they must be squares, since H is their common vertex. But this implies $|J_iR_i|=|J_iR_{i+1}|$, a contradiction. Hence (HJ_i) touches δ for some i. Then H lies on J_iJ_{i+2}. Therefore (HJ_{i+2}) also touches δ. While $J_{i-1}J_{i+1}$ and HO_δ are perpendicular and intersect at O_δ. Therefore (HJ_{i-1}) and (HJ_{i+1}) are congruent and pass through O_δ. □

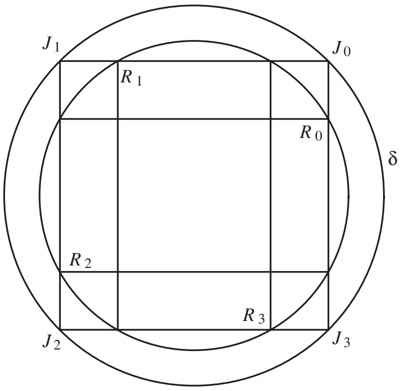

Figure 8

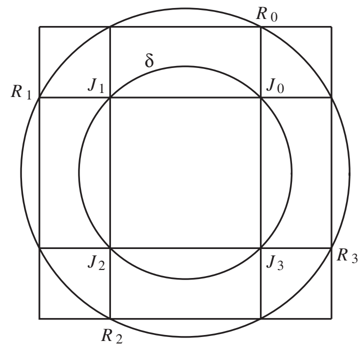

Figure 9

5 Special cases

We conclude this article by considering the tangent arbelos (α,β,γ) with $O=P=Q$. Since $\varepsilon=O_\gamma(O)$, Power's result mentioned in the introduction is restated as both α and β are Lamoenian. Figure 10 shows a quartet on γ with base point O with $J_0=A$, in which α and β are members of the quartet. Figure 11 shows a quartet on ε with base point A with $J_0=O$. In this figure α and the reflected image of β in O_γ are members of the quartet. In each of the cases, the rectangle is a square.

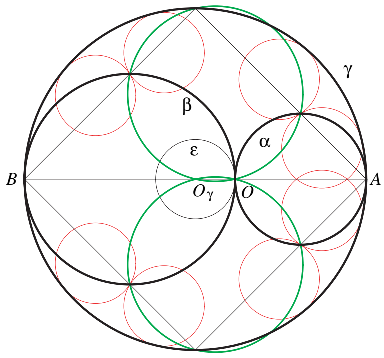

Figure 10: *A quartet on γ with base point O*

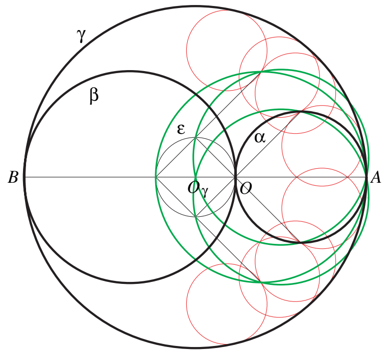

Figure 11: *A quartet on ε with base point A*

Let $\mathcal{L}$ be the radical axis of α and β. Quang Tuan Bui has found that the points of intersection of the circles (AO_β) and (BO_α) lie on $\mathcal{L}$ and generate Archimedean circles with γ for the tangent arbelos (α, β, γ) [1]. Let R_1 be one of the points of intersection, and let the line parallel to AB passing through R_1 intersect γ at a point K, where K lies on the same side of $\mathcal{L}$ as A. Figure 12 shows a quartet on γ with base point O with $J_0 = K$. In this figure R_0 and R_2 lie on AB while R_3 lies on $\mathcal{L}$. Figure 13 shows a quartet on ε with base point K with $J_0 = O$. In this figure, R_1J_0 touches ε at O. Therefore $J_1 = J_0 = O$, i.e., the rectangle degenerates into a segment, and the quartet consists of two different Lamoenian circles.

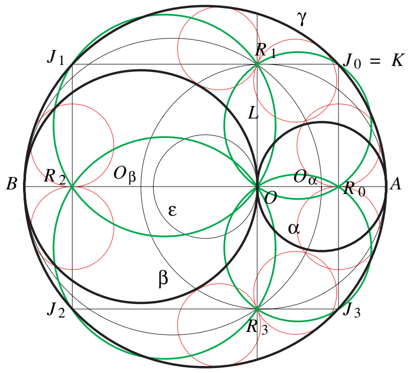

Figure 12: *A quartet on γ with base point O*

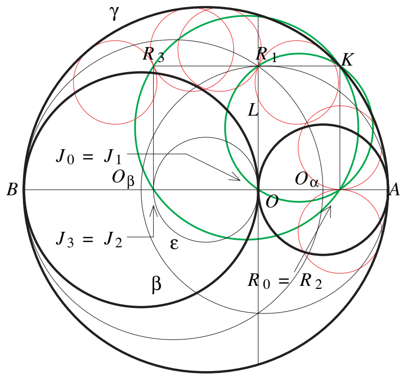

Figure 13: *A quartet on ε with base point K*

References

[1] Q. T. Bui, The arbelos and nine-point circles, *Forum Geom.* **7** (2007), 115–120.

[2] F. van Lamoen, Some Powerian pairs in the arbelos, *Forum Geom.* **7** (2007), 111-113.

[3] H. Okumura, Ubiquitous Archimedean circles of the collinear arbelos, *KoG* **16** (2012), 17–20.

[4] H. Okumura and M. Watanabe, Generalized arbelos in aliquot part: non-intersecting case, *J. Geom. Graph.* **13** (2009), No.1, 41–57.

[5] H. Okumura and M. Watanabe, Generalized arbelos in aliquot part: intersecting case, *J. Geom. Graph.* **12** (2008), No.1, 53–62.

[6] F. Power, Some more Archimedean circles in the Arbelos, *Forum Geom.* **5** (2005), 133–134.

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