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ALI ALKHALDI
NORMAN JOHN WILDBERGER

The Parabola in Universal Hyperbolic Geometry I

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ABSTRACT

We introduce a novel definition of a parabola into the framework of universal hyperbolic geometry, show many analogs with the Euclidean theory, and also some remarkable new features. The main technique is to establish parabolic standard coordinates in which the parabola has the form $xz = y^2$. Highlights include the discovery of the twin parabola and the connection with sydpoints, many unexpected concurrences and collinearities, a construction for the evolute, and the determination of (up to) four points on the parabola whose normals meet.

Key words: universal hyperbolic geometry, parabola

MSC2010: 51M10, 14N99, 51E99

Parabola u univerzalnoj hiperboličkoj geometriji I

SAŽETAK

Uvodimo novu definiciju parabole u okvir univerzalne hiperboličke geometrije, pokazujemo mnoge analogone s euklidskom geometrijom, ali i neka izvanredna nova svojstva. Osnovna je tehnika uspostavljanje paraboličnih standardnih koordinata u kojima parabola ima jednadžbu oblika $xz = y^2$. Ističemo otkriće parabole blizanke, vezu sa sidtočkama, mnoge neočekivane konkurentnosti i kolinearnosti, konstrukciju evolute te određivanje (do najviše) četiriju točaka parabole u kojima normale parabole prolaze jednom točkom.

Ključne riječi: univerzalna hiperbolička geometrija, parabola

1 Introduction

This paper begins the study of the *parabola* in *universal hyperbolic geometry (UHG)*. The framework is that of [16], [17], [18], [19] and [20]; a completely algebraic and more general formulation of hyperbolic geometry which extends to general fields (not of characteristic two), and also unifies elliptic and hyperbolic geometries. We will see that this investigation opens up many new phenomenon, and hints again at the inexhaustible beauty of conic sections!

In Euclidean geometry, the parabola plays several distinguished roles. It is the graph resulting from a quadratic function $f(x) = a + bx + cx^2$, and so familiar as the second degree Taylor expansion of a general function. The parabola is also a conic section in the spirit of Apollonius, obtained by slicing a cone with a plane which is parallel to one of the generators of the cone. In affine geometry the parabola is the distinguished conic which is tangent to the line at infinity. In everyday life, the parabola occurs in reflecting mirrors and automobile head lamps, in satellite dishes and radio telescopes, and in the trajectories of comets.

Of course the ancient Greeks also studied the familiar metrical formulation of a parabola: it is the locus of a point which remains equidistant from a fixed point F, called the *focus*, and a fixed line f, called the *directrix*. (We have a good reason for using the same letters for both concepts, with only case separating them). Such a conic $\mathcal{P}$ has a line of symmetry: the *axis* a through F perpendicular to f. It also has a distinguished point V called the *vertex*, which is the only point of the parabola lying on the axis a, aside from the point at infinity. The vertex V is the midpoint between the focus F and the **base point** $B \equiv af$.

For such a classical parabola $\mathcal{P}$ hundreds of facts are known, see [1], [4], [5], [8], [10], [13], [14]; quite a few of them going back to Archimedes and Apollonius, others added in more recent centuries. Of particular importance are theorems that relate to an arbitrary point P on the conic and its tangent line p. In particular the construction of p itself is important: there are two common ways of doing this. One is to take the foot T of the altitude from P to the directrix f, and connect P to the midpoint M of $\overline{TF}$; so that $p = PM$. Another is to take the perpendicular line t to PF through F, and find its meet S with the directrix; this gives $p = PS$. The point S is equidistant from T and F, and the circle $\mathcal{S}$ with center S through F is tangent to both the lines PF and PT.

A related and useful fact is that a chord PN is a focal chord—meaning that it passes through F—precisely when the meet of the two tangents at P and N lies on the directrix f, and in this case the two tangents are perpendicular. These facts are illustrated in Figure 1. Another result, which figures often in calculus, is that if P and Q are arbitrary points on the parabola with Z the meet of their tangents p and q, and T,U and W are the feet of the altitudes from P,Q and Z to the directrix, then W is the midpoint of $\overline{TU}$.

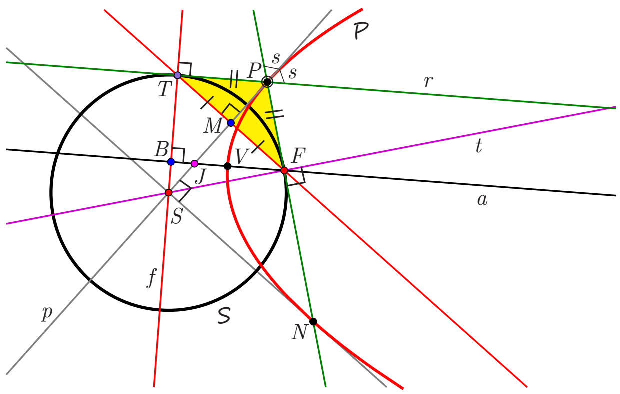

Figure 1: *The Euclidean Parabola*

So when we investigate hyperbolic geometry, some natural questions are: what is the analog of a parabola in this context, what properties of the Euclidean case carry over in this setting, and what additional properties might the hyperbolic parabola have that do not hold in the Euclidean case? These issues have been studied by several authors, such as [2], [15], [9].

In this paper we answer these questions in a new and more general way, using the wider framework of UHG, and allowing the beginnings of a much deeper investigation. There is a very natural analog of a parabola in this hyperbolic setting, and many, but certainly not all, properties of the Euclidean parabola hold or have reasonable analogs for it. But there are many interesting aspects which have no Euclidean counterpart, such as the existence of a dual or twin parabola, and an intimate connection with the theory of sydpoints, as laid out in [20].

The outline of the paper is as follows. We first give a very brief review of universal hyperbolic geometry, where the algebraic notions of *quadrance* and *spread* replace the more traditional transcendental measurements of *distance* and *angle*. We then define the parabola in the hyperbolic setting (we often refer simply to the *hyperbolic parabola*), give a dynamic geometry package construction for it, introduce some basic points associated to it, and use some of these and the Fundamental theorem of Projective Geometry to define *standard coordinates*, in which the parabola has the convenient equation $xz = y^2$. This allows a simple parametrization for the curve, as well as pleasant explicit formulas for many interesting points, lines, conics and higher degree curves associated to it.

In our study of the basic points and lines associated with the parabola $\mathcal{P}_0$, concrete and explicit formulae are key objectives, because they allow us a firm foundation for deeper investigations. The main thrust of the paper is then to show how the hyperbolic parabola shares many similarities with the Euclidean parabola. The highlights include the duality leading to the twin parabola, a straightedge construction of the evolute of the parabola, and a conic construction of four points on the parabola whose normals pass through a fixed point (in the Euclidean case there are at most three points with this property).

This paper is the first of a series on the hyperbolic parabola. In future papers we will show that there are many new and completely unexpected aspects of the hyperbolic parabola; it is a very rich topic indeed.

1.1 A brief review of universal hyperbolic geometry

We work over a fixed field, not of characteristic two, and give a formulation of universal hyperbolic geometry valid with a general symmetric bilinear form—this generality will be important for us when we introduce parabolic standard coordinates. This is only a quick introduction; the reader may consult [17], [18], [19], [20] for more details.

A **(projective) point** is a proportion $a = [x : y : z]$ in square brackets, or equivalently a projective row vector $a = \begin{bmatrix} x & y & z \end{bmatrix}$ (unchanged if multiplied by a non-zero number). A **(projective) line** is a proportion $L = \langle l : m : n \rangle$ in pointed brackets, or equivalently a projective column vector

$$L = \begin{bmatrix} l \\ m \\ n \end{bmatrix}.$$

The **incidence** between the point $a = [x : y : z]$ and the line $L = \langle l : m : n \rangle$ is given by the relation $aL = lx + my + nz = 0$. The **join** of points is defined by

$$\begin{aligned} a_1a_2 &\equiv [x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] \\ &\equiv \langle y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 \rangle \end{aligned} \quad (1)$$

while the **meet** L_1L_2 of lines $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is similarly defined by

$$\begin{aligned} L_1L_2 &\equiv \langle l_1 : m_1 : n_1 \rangle \times \langle l_2 : m_2 : n_2 \rangle \\ &\equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1]. \end{aligned} \quad (2)$$

Collinearity of three points a_1, a_2, a_3 will here be represented by the abbreviation $[[a_1a_2a_3]]$, and similarly the concurrency of three lines L_1, L_2, L_3 will be abbreviated $[[L_1L_2L_3]]$. These are determinantal conditions.

The metrical structure is given by a (non-degenerate) 3×3 projective symmetric matrix $\mathbf{C}$ and its adjugate $\mathbf{D}$ (where bold signifies a projective matrix- determined only up to a non-zero multiple). The points a_1 and a_2 are **perpendicular** precisely when $a_1\mathbf{C}a_2^T=0$, written $a_1\perp a_2$, while lines L_1 and L_2 are **perpendicular** precisely when $L_1^T\mathbf{D}L_2=0$, written $L_1\perp L_2$. The point a and the line L are **dual** precisely when $L=a^\perp\equiv\mathbf{C}a^T$, or equivalently $a=L^\perp\equiv L^T\mathbf{D}$, so that points are perpendicular precisely when one is incident with the dual of the other, and similarly for two lines. A point a is **null** precisely when it is perpendicular to itself, that is, when $a\mathbf{C}a^T=0$, while a line L is **null** precisely when it is perpendicular to itself, that is, when $L^T\mathbf{D}L=0$. The null points determine the **null conic**, sometimes also called the *absolute*.

Universal Hyperbolic geometry in the Cayley Klein model arises from the special case

$$\mathbf{C}=\mathbf{D}=\mathbf{J}\equiv\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}. \tag{3}$$

In this framework the point $a=[x:y:z]$ is null precisely when $x^2+y^2-z^2=0$, and dually the line $L=\langle l:m:n\rangle$ is null precisely when $l^2+m^2-n^2=0$. So we can picture the null circle in affine coordinates $X\equiv x/z$ and $Y\equiv y/z$ as the (blue) circle $X^2+Y^2=1$. The **quadrance** q between points and the **spread** S between lines are then given by essentially the same formulas:

$$\begin{aligned}&q([x_1:y_1:z_1],[x_2:y_2:z_2])\\&\quad=1-\frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}\\&S(\langle l_1:m_1:n_1\rangle,\langle l_2:m_2:m_2\rangle)\\&\quad=1-\frac{(l_1l_2+m_1m_2-n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}.\end{aligned} \tag{4}$$

The figures in this paper are generated in this model, with however the outside of the null circle playing just as big a role as the inside–this takes some getting used to for the classical hyperbolic geometer! In addition, it will be necessary for us to adopt a more general and flexible approach to deal with projective changes of coordinates, which will be needed to study the parabola in what we call standard coordinates.

So more generally, the bilinear forms determined by a general 3×3 projective symmetric matrix $\mathbf{C}$ and its adjugate $\mathbf{D}$ can be used to define the dual notions of (**projective**) **quadrance** $q(a_1,a_2)$ between points a_1 and a_2, and (**projective**) **spread** $S(L_1,L_2)$ between lines L_1 and L_2 as

$$q(a_1,a_2)\equiv 1-\frac{\left(a_1\mathbf{C}a_2^T\right)^2}{\left(a_1\mathbf{C}a_1^T\right)\left(a_2\mathbf{C}a_2^T\right)}\qquad\text{and}$$

$$S(L_1,L_2)\equiv 1-\frac{\left(L_1^T\mathbf{D}L_2\right)^2}{\left(L_1^T\mathbf{D}L_1\right)\left(L_2^T\mathbf{D}L_2\right)}. \tag{5}$$

While the numerators and denominators of these expressions depend on choices of representative vectors and matrices for $a_1,a_2,\mathbf{C},L_1,L_2$ and $\mathbf{D}$, (which are by definition defined only up to scalars), the overall expressions are well-defined projectively.

It follows that $q(a,a)=0$ and $S(L,L)=0$, while $q(a_1,a_2)=1$ precisely when $a_1\perp a_2$, and dually $S(L_1,L_2)=1$ precisely when $L_1\perp L_2$. Also quadrance and spread are naturally dual:

$$S\left(a_1^\perp,a_2^\perp\right)=q(a_1,a_2).$$

In [16], it was shown that both these metrical notions can also be reformulated projectively and rationally using suitable cross ratios (and no transcendental functions!) To connect with the more familiar distance between points $d(a_1,a_2)$, and angle between lines $\theta(L_1,L_2)$ in the Klein projective model: when we restrict to points and lines inside the null circle,

$$q(a_1,a_2)=-\sinh^2(d(a_1,a_2))\qquad\text{and}$$
$$S(L_1,L_2)=\sin^2(\theta(L_1,L_2)).$$

For a triangle $\overline{a_1a_2a_3}$ with associated trilateral $\overline{L_1L_2L_3}$, we define $q_1\equiv q(a_2,a_3)$, $q_2\equiv q(a_1,a_3)$ and $q_3\equiv q(a_1,a_2)$, and $S_1\equiv S(L_2,L_3)$, $S_2\equiv S(L_1,L_3)$ and $S_3\equiv S(L_1,L_2)$. The main trigonometric laws in the subject can be restated in terms of these quantities (see UHG I [17]).

2 The parabola and its construction

In this section we introduce definitions and some basic results for a *parabola* in *universal hyperbolic geometry*. We will work and illustrate the theory in the familiar Cayley-Klein setting with our null circle/absolute the unit circle in the plane. The situation is in some sense richer than in the Euclidean setting because of *duality*: whenever we define an important point x, its dual line $X=x^\perp$ is also likely to be important, and vice versa. We remind the reader that we will consistently employ small letters for points and capital letters for lines, with the convention that if x_i is a point, then $X_i=x_i^\perp$ is the corresponding dual line and conversely.

So what is a parabola in the hyperbolic setting? As already discussed in [9], the definition is not obvious: there are several different possible ways of trying to generalize the Euclidean theory. Recall that if a is a point and L is a line, then the quadrance $q(a,L)$ is defined to be the quadrance between a and the foot t of the altitude line from a to L.

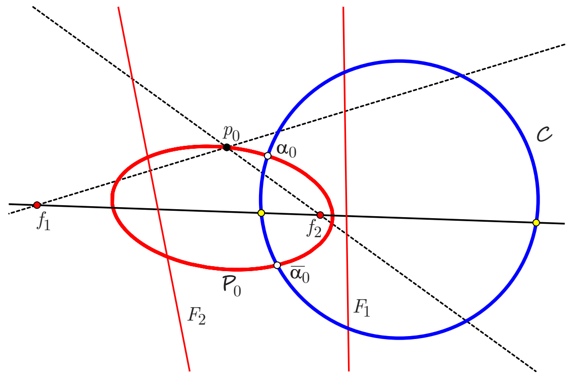

Figure 2: *A parabola $\mathcal{P}_0$ with foci f_1 and f_2*

Definition 1 *Suppose that f_1 and f_2 are two non-perpendicular points such that $f_1 f_2$ is a non-null line. The* ***parabola*** *$\mathcal{P}_0$ with* ***foci*** *f_1 and f_2 is the locus of a point p_0 satisfying*

$$q(f_1,p_0)+q(p_0,f_2)=1. \tag{6}$$

The lines $F_1 \equiv f_1^{\perp}$ and $F_2 \equiv f_2^{\perp}$ are the ***directrices*** *of the parabola $\mathcal{P}_0$.*

This definition is likely surprising to the classical geometer. In Euclidean geometry, such a relation defines a *circle*, so at this point it is not clear what justification we have for our definition of a parabola. The following connects our theory with the more traditional approach in [11] and [7].

Theorem 1 (Parabola focus directrix) *The point p_0 satisfies (6) precisely when either of the following hold:*

$$q(f_1,p_0)=q(p_0,F_2) \qquad \text{or} \qquad q(f_2,p_0)=q(p_0,F_1).$$

Proof. If $(f_1p_0)F_1 \equiv t_1$ and $(f_2p_0)F_2 \equiv t_2$ are the feet of the altitudes from a point p_0 on the parabola $\mathcal{P}_0$ with foci f_1 and f_2 to the directrices F_1 and F_2, then f_1 and t_1 are perpendicular points, as are f_2 and t_2. It follows that $q(f_1,p_0)+q(p_0,t_1)=1$ and $q(f_2,p_0)+q(p_0,t_2)=1$. But then (6) is equivalent to $q(f_1,p_0)=q(p_0,F_2)$ or to $q(f_2,p_0)=q(p_0,F_1)$. □

In this way we recover the ancient Greek metrical definition of the parabola, but we note now that there are *two* foci-directrix pairs: (f_1,F_2) and (f_2,F_1). This is a main feature of the hyperbolic theory of the parabola: a fundamental symmetry between the two foci-directrix pairs.
The reason for the index 0 on the point p_0 and the parabola $\mathcal{P}_0$ will become clearer when we introduce the twin parabola $\mathcal{P}^0$. We observe that the foci f_1 and f_2 do not lie on the parabola $\mathcal{P}_0$, since for example if f_1 lies on $\mathcal{P}_0$, then $q(f_1,f_1)+q(f_2,f_1)=1$, which would imply that $q(f_1,f_2)=1$, contradicting that the assumption of non-perpendicularity of f_1 and f_2. In Figure 2 we see an example of a parabola $\mathcal{P}_0$, in red, with foci f_1 and f_2, and directrices F_1 and F_2, also in red.

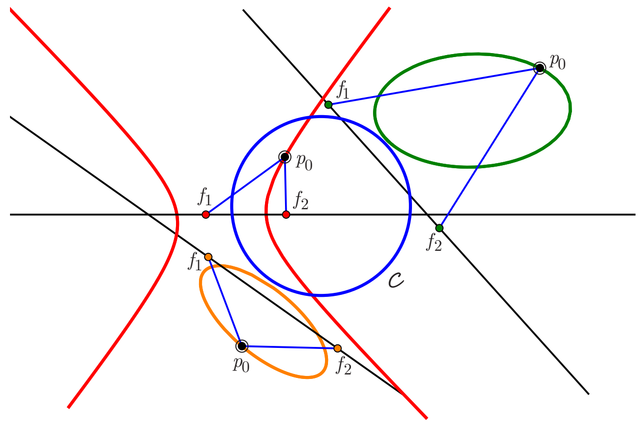

Figure 3: *Various examples of parabolas*

In Figure 3 we see some different examples of parabolas over the rational numbers, at least approximately. When the foci f_1 and f_2 are both interior points of the null circle $\mathcal{C}$, there is no point p satisfying the condition $q(p,f_1)+q(p,f_2)=1$, since the quadrance between any two interior points is always negative, and the quadrance between an interior point and an exterior point is greater than or equal to 1. This paper deals with non-empty parabolas, by extending the field if necessary, as we shall see.

Theorem 2 (Parabola conic) *The parabola $\mathcal{P}_0$ with foci f_1 and f_2 is a conic.*

Proof. Suppose that $f_1=[x_1:y_1:z_1]$ and $f_2=[x_2:y_2:z_2]$. Then the point $p=[x:y:z]$ lies on $\mathcal{P}_0$ precisely when

$$\begin{aligned}&\left(1-\frac{(xx_1+yy_1-zz_1)^2}{(x^2+y^2-z^2)\left(x_1^2+y_1^2-z_1^2\right)}\right)\\&\quad+\left(1-\frac{(xx_2+yy_2-zz_2)^2}{(x^2+y^2-z^2)\left(x_2^2+y_2^2-z_2^2\right)}\right)=1\end{aligned}$$

which yields the quadratic equation

$$\begin{aligned}&\left(x^2+y^2-z^2\right)\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\\&\quad=(xx_1+yy_1-zz_1)^2\left(x_2^2+y_2^2-z_2^2\right)\\&\qquad+(xx_2+yy_2-zz_2)^2\left(x_1^2+y_1^2-z_1^2\right).\end{aligned}$$ □

2.1 Basic definitions

We now define some basic points and lines associated to a parabola $\mathcal{P}_0$ with foci f_1 and f_2, and directrices $F_1 \equiv f_1^{\perp}$

and $F_2 \equiv f_2^{\perp}$. The **axis** of the parabola $\mathcal{P}_0$ is the line $A \equiv f_1 f_2$. The **axis point** of $\mathcal{P}_0$ is the dual point $a \equiv A^{\perp}$. By assumption the axis A is a non-null line, so that a does not lie on A.

If the axis A has null points, we shall call these the **axis null points** of $\mathcal{P}_0$, and denote them by η_1 and η_2, in no particular order. The axis point and line will generally be in black in our diagrams, while the axis null points will be in yellow.

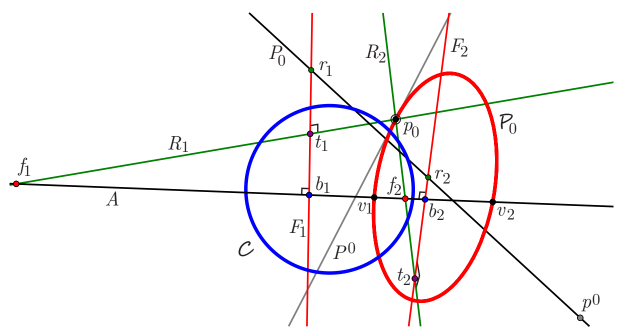

Figure 4: *Dual and tangent lines, twin point and focal lines*

Theorem 3 *The axis $A = f_1 f_2$ of a hyperbolic parabola $\mathcal{P}_0$ is a line of symmetry, and its dual point a is a center.*

Proof. We denote the reflection of an arbitrary point p_0 lying on $\mathcal{P}_0$ in the axis line A by $r_A(p_0) = \overline{p_0}$. Then we need to prove that $\overline{p_0}$ also lies on $\mathcal{P}_0$. Recall that the hyperbolic reflection in a line (or equivalently the reflection in the dual point of that line) is an isometry, so for any two points a and b,

$$q(a,b) = q(r_A(a), r_A(b)).$$

Thus, since f_1, f_2 are fixed by r_A (they lie on A),

$$\begin{aligned} 1 &= q(f_1, p_0) + q(f_2, p_0) \\ &= q(r_A(p_0), r_A(f_1)) + q(r_A(p_0), r_A(f_1)) \\ &= q(\overline{p_0}, f_1) + q(\overline{p_0}, f_2). \end{aligned}$$

This shows that $\overline{p_0}$ lies on the parabola $\mathcal{P}_0$. Since reflecting p_0 in A is the same as reflecting p_0 in a, the point a is also the center of the parabola. □

The **base points** of $\mathcal{P}_0$ are the points $b_1 \equiv AF_1$ and $b_2 \equiv AF_2$. The dual lines $B_1 \equiv af_1$ and $B_2 \equiv af_2$ are the **base lines** of $\mathcal{P}_0$. Both base points and base lines will be shown in blue in our diagrams.

The **vertices** v_1 and v_2 are the points, if they exist, where the parabola meets the axis; they are in no particular order. The duals of the vertices are the **vertex lines** $V_1 \equiv v_1^{\perp}$ and $V_2 \equiv v_2^{\perp}$. The vertices and vertex lines will be shown in black.

A generic point on $\mathcal{P}_0$ will be denoted p_0, and its **dual line** denoted P_0. Both are shown in black in our diagrams, with often a small circle drawn around p_0 to highlight it. The **tangent line** to $\mathcal{P}_0$ at p_0 will be denoted P^0, and its dual point p^0 will be called the **twin point** of p_0. Both p^0 and P^0 will be shown in grey.

The **focal lines of** p_0 are $R_1 \equiv p_0 f_1$ and $R_2 \equiv p_0 f_2$, and the **altitude base points of** p_0 are $t_1 \equiv R_1 F_1$ and $t_2 \equiv R_2 F_2$. The duals of the focal lines are the **focal points** $r_1 \equiv R_1^{\perp}$ and $r_2 \equiv R_2^{\perp}$ **of** p_0. The duals of the focal base points are the **altitude base lines** $T_1 \equiv t_1^{\perp}$ and $T_2 \equiv t_2^{\perp}$ **of** p_0. The focal lines and points will be shown in green in our diagrams. Figure 4 shows these various basic points and lines associated to the parabola $\mathcal{P}_0$.

2.2 Construction with a dynamic geometry program

It is helpful to have a construction of a hyperbolic parabola that can be used with a dynamic geometry package, such as Geometer's Sketchpad, GeoGebra, C.a.R., Cinderella, Cabri etc., used to create loci. For this it is helpful to refresh our minds about the construction of the Euclidean parabola, because a similar technique applies to construct a hyperbolic parabola. We also mention some related facts that will have analogs in the hyperbolic setting.

Firstly, we choose a point F (focus**)**, and a line f (directrix**)**, not passing through F. Draw the perpendicular line a (axis) to F through f. Using an arbitrary point T on the directrix f, construct the midpoint M of the side $\overline{TF}$, and draw the perpendicular line p to TF through M. Finally, the intersection of the altitude r to f through T and the line p is a point P on the parabola $\mathcal{P}$, which is then the locus of the point P as T moves on f, as in Figure 1.

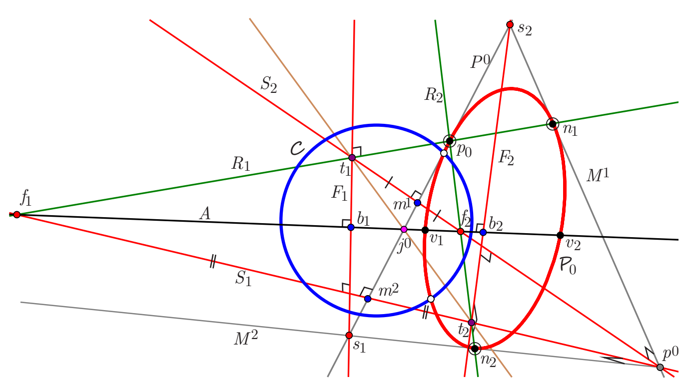

Figure 5: *Construction of a hyperbolic parabola $\mathcal{P}_0$*

To construct a hyperbolic parabola $\mathcal{P}_0$ from a pair of foci f_1 and f_2 with axis A, we proceed as in the Euclidean case, but we must be aware that the existence of midpoints is more subtle–they may not exist, and when they do, there are generally two of them! The situation is illustrated in Figure 5; choose a point t_1 on the directrix $F_1 \equiv f_1^{\perp}$ with the property that the side $\overline{t_1 f_2}$ has midpoints, call them m^1 and p^0, with corresponding midlines $M^1 = (m^1)^\perp$ and $P^0 = (p^0)^\perp$. One way of choosing such a point t_1 is to first choose an arbitrary point a_1 on F_1 and then reflect $b_1 \equiv F_1 A$ in a_1 to obtain t_1. In the triangle $\overline{b_1 t_1 f_2}$, two sides now have midpoints, so by Menelaus' theorem ([17]) the third side $\overline{t_1 f_2}$ will also have midpoints.

Now construct the meets $p_0 \equiv P^0 R_1$ and $n_1 \equiv M^1 R_1$, where $R_1 = t_1 f_1$. Then p_0 and n_1 will both be points on the parabola $\mathcal{P}_0$. The Figure also shows the symmetry available here: it is equally possible to choose a point t_2 on the other directrix $F_2 \equiv f_2^\perp$ with the property that the side $\overline{t_2 f_1}$ has midpoints, call them m^2 and p^0, with corresponding midlines $M^2 = (m^2)^\perp$ and $P^0 = (p^0)^\perp$. In that case the points $p_0 \equiv P^0 R_2$ and $n_2 \equiv M^2 R_2$, where $R_2 = t_2 f_2$, lie on the parabola $\mathcal{P}_0$. In Figure 5, the two points t_1 and t_2 are related by the fact that $t_1 t_2$ meets the axis A at the same point j^0 as does P^0; this accounts for the fact that $\overline{t_1 f_2}$ and $\overline{t_2 f_1}$ have a common midpoint p^0.

The justification for this construction will be given later, after we establish a suitable framework for coordinates and derive formulas for all the relevant points.

2.3 Dual conics and the connection with sydpoints

The theory of the hyperbolic parabola connects strongly with the *notion of sydpoints* as developed in [20].

The reason is that the sydpoints f^1 and f^2 of the side $\overline{f_1 f_2}$, should they exist (and our assumptions on our field will guarantee that they do) are naturally determined by the geometry of $\mathcal{P}_0$, and then they become the foci for the *twin parabola* $\mathcal{P}^0$ (in orange in our diagrams), which turns out to be the dual of the conic $\mathcal{P}_0$ with respect to the null circle $\mathcal{C}$. The sydpoint symmetry between the sides $\overline{f_1 f_2}$ and $\overline{f^1 f^2}$ is key to understanding many aspects of these conics. Although we will be studying the twin parabola more in the next paper in this series, it will be useful to be aware of it, as it explains some of our notational conventions.

In Figure 6, we see the parabola $\mathcal{P}_0$ with foci f_1, f_2 and a point p_0 on it, as well as the twin parabola $\mathcal{P}^0$ with foci f^1, f^2 and the twin point p^0 on it, which is the dual of the tangent P^0 to $\mathcal{P}_0$ at p_0. Reciprocally the dual of p_0 is the tangent to $\mathcal{P}^0$ at p^0. Note carefully that the tangents to both the parabola $\mathcal{P}_0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points α_0 and $\overline{\alpha_0}$, pass through the foci of the twin parabola $\mathcal{P}^0$. Dually, note that the tangents to both the parabola $\mathcal{P}^0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points δ_0 and $\overline{\delta_0}$, pass through the foci of $\mathcal{P}_0$. This Figure also shows the twin directrices F^1 and F^2, and the twin base points b^1 and b^2.

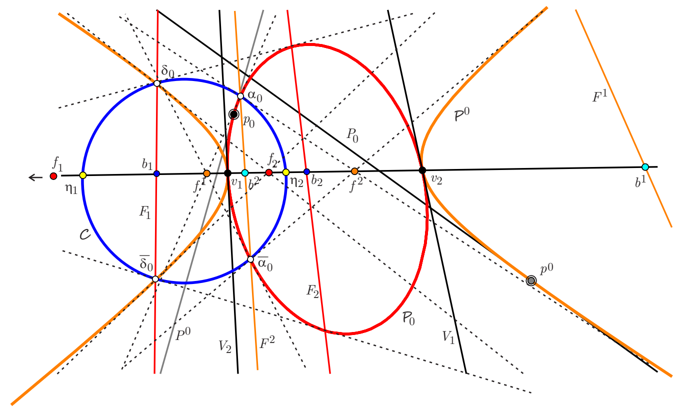

Figure 6: *The parabola $\mathcal{P}_0$ and its twin $\mathcal{P}^0$*

3 Standard Coordinates and duality

3.1 The four basis null points

In order to bring a systematic treatment to the study of the hyperbolic parabola $\mathcal{P}_0$, we need an appropriate coordinate system to bring $\mathcal{P}_0$ into as simple a form as possible. Although there is a great deal of choice for such an attempt, the one that we present here is the simplest and most elegant we could find; in it the beauty of the parabolic theory is reflected in an elegance and coherence in the corresponding formulae.

The key point is that aside from the two foci f_1 and f_2 which we used to define the parabola, there are four other points which naturally lie on the parabola and which can be used effectively as a basis for projective coordinates: the two vertices v_1 and v_2, together with two null points α_0 and $\overline{\alpha_0}$ which are symmetrically placed with respect to the axis.

We need to say some words about the existence of four such points. A priori there is no guarantee that the axis A meets the parabola; it will do so when the corresponding quadratic equation formed by meeting the line with the conic has a solution. The existence of the vertices is then an *assumption* that we may justify by adjoining an algebraic square root, if required, to our field.

We will use the four points v_1, v_2, α_0 and $\overline{\alpha_0}$, no three which are collinear, as a basis of a new projective coordinate system.

Theorem 4 (Parabola vertices) *If there is a non-null point v_1 lying both on the axis A and the parabola $\mathcal{P}_0$, then the perpendicular point $v_2 \equiv v_1^\perp A$ also lies on both the axis and the parabola, and these then are the only two points with this property.*

Proof. Suppose that v_1 lies on the axis $A \equiv f_1 f_2$ and on the parabola. Then if v_1 is not a null point,

$$q(f_1, v_1) + q(v_1, f_2) = 1.$$

Define $v_2 \equiv v_1^{\perp} A$, so that $q(v_1, v_2) = 1$. Now recall that if a, b and c are collinear points with $q(a,b) = 1$, then $q(a,c) + q(c,b) = 1$. So $q(v_1, f_1) + q(f_1, v_2) = 1$ and $q(v_1, f_2) + q(f_2, v_2) = 1$. Combining all three equations we see that $q(f_1, v_2) + q(v_2, f_2) = 1$, showing that v_2 also lies on the parabola. Since a line meets a conic at most at two points, there can be no other points on the axis and on $\mathcal{P}_0$. □

We can see from Figure 3 that a parabola need not necessarily meet its axis. However any given line will meet a given conic if we are allowed to augment the field to an appropriate quadratic extension. So by possibly extending our field, *we will henceforth assume that our parabola $\mathcal{P}_0$ meets the axis $A = f_1 f_2$*. By the above theorem, it then meets this axis in exactly two points, which we call the **vertices** of the parabola, and denote by v_1 and v_2.

What about the existence of null points on $\mathcal{P}_0$? The meet of any two conics might have from zero to four points.

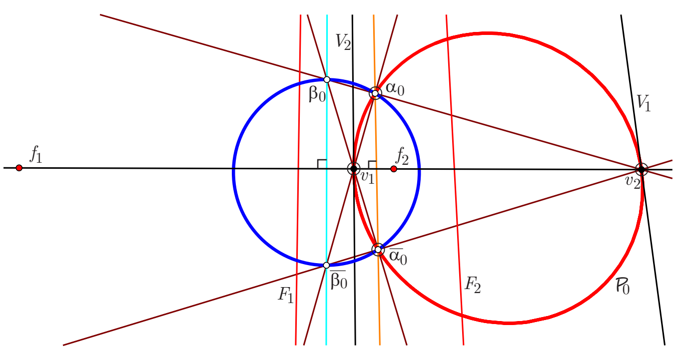

Figure 7: *The four basis points v_1, v_2, α_0 and $\overline{\alpha_0}$*

The parabola $\mathcal{P}_0$ with foci f_1 and f_2 need not meet the null conic $\mathcal{C}$. However for most examples, especially those of interest to a classical geometer working in the Klein model in the interior of the unit disk, we do have such an intersection—at least approximately over the rational numbers. So by possibly extending our field to a quartic extension, *we will henceforth assume that our parabola P_0 passes through at least one null point α_0*. By the assumption in the previous theorem such a null point α_0 cannot lie on the axis, so if we reflect it in the axis we get a second null point $\overline{\alpha_0} \equiv r_a(\alpha_0)$ which also lies on $\mathcal{P}_0$, since $\mathcal{P}_0$ is invariant under r_a. Clearly no three of the four **basis points** v_1, v_2, α_0 and $\overline{\alpha_0}$ are collinear, since they all lie on the parabola.

3.2 The Fundamental theorem and standard coordinates

We now invoke the *Fundamental Theorem of Projective Geometry*, which allows us to make a unique projective change of coordinates so that the four basis points become

$$v_1 = [0:0:1] \qquad v_2 = [1:0:0]$$
$$\alpha_0 = [1:1:1] \qquad \overline{\alpha_0} = [1:-1:1].$$

It follows that

$$A = v_1 v_2 = [0:0:1] \times [1:0:0] = \langle 0:1:0 \rangle.$$

These new coordinates will be called **standard coordinates** for the parabola $\mathcal{P}_0$, or **parabolic standard coordinates**. Note carefully that the introduction of such new coordinates will necessarily change the form of the quadrance and spread!

We now define, as in Figure 7, the points obtained by reflecting α_0 and $\overline{\alpha_0}$ in v_2: namely

$$\beta_0 \equiv r_{v_2}(\alpha_0) \quad \text{and} \quad \overline{\beta_0} \equiv r_{v_2}(\overline{\alpha_0}).$$

Because reflection is an isometry, these are also null points. Our notation with the overbar is something we will employ consistently: α_0 and $\overline{\alpha_0}$ are reflections in the point a, or equivalently in the dual line A, and so similarly for β_0 and $\overline{\beta_0}$.

Theorem 5 (β points) *We have $\beta_0 = (\alpha_0 v_2)(\overline{\alpha_0} v_1)$ and $\overline{\beta_0} = (\overline{\alpha_0} v_2)(\alpha_0 v_1)$. Furthermore in the new coordinate system $\beta_0 = [-1:1:1]$ and $\overline{\beta_0} = [-1:-1:1]$.*

Proof. The quadrangle of null points $\alpha_0 \overline{\alpha_0} \beta_0 \overline{\beta_0}$ has one diagonal point v_2, obviously from the definition of β_0 and $\overline{\beta_0}$. It has another diagonal point a, because both $\alpha_0 \overline{\alpha_0}$ and $\beta_0 \overline{\beta_0}$ pass it; the first by construction and the second because it is obtained from the first by reflection in v_2, which lies on $A = a^{\perp}$. So the third diagonal point is the dual of $a v_2$, which is v_1 by the previous theorem. It follows that $\beta_0 = (\alpha_0 v_2)(\overline{\alpha_0} v_1)$ and $\overline{\beta_0} = (\overline{\alpha_0} v_2)(\alpha_0 v_1)$. Now we can calculate that

$$\begin{aligned}\beta_0 &= ([1:1:1] \times [1:0:0]) \times ([1:-1:1] \times [0:0:1]) \\ &= \langle 0:1:-1 \rangle \times \langle 1:1:0 \rangle = [-1:1:1]\end{aligned}$$

$$\begin{aligned}\overline{\beta_0} &= ([1:-1:1] \times [1:0:0]) \times ([1:1:1] \times [0:0:1]) \\ &= \langle 0:1:1 \rangle \times \langle 1:-1:0 \rangle = [-1:-1:1].\end{aligned}$$ □

When we apply a general projective transformation of the projective plane to get the four points v_1, v_2, α_0 and $\overline{\alpha_0}$ into standard position, the metrical structure will change. While we started with the symmetric matrix J for the form, the new symmetric matrix is of the form $\mathbf{C} = MJM^T$ for some invertible matrix M. However this matrix $\mathbf{C}$ is not arbitrary; since we require that the four points lie on the parabola $\mathcal{P}_0$. We now arrive at the crucial result which sets up our coordinate system, and is the basis for all subsequent calculations. This is the fact that the new matrix $\mathbf{C}$, and its adjugate $\mathbf{D}$, have a particularly simple form, depending on a single parameter α which subsequently appears in almost all our formulas.

Theorem 6 (Parabola standard coordinates) *The symmetric bilinear form in standard coordinates is given by $v_1 \odot v_2 = v_1 \mathbf{C} v_2^T$ where*

$$\mathbf{C} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and}$$

$$\mathbf{D} = \operatorname{adj}(\mathbf{C}) = \begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2\left(1-\alpha^2\right) \end{bmatrix} \tag{7}$$

for some number α. In terms of α, the parabola $\mathcal{P}_0$ has equation $xz - y^2 = 0$ and its foci are

$$f_1 = [\alpha+1 : 0 : \alpha(\alpha-1)] \quad \text{and} \quad f_2 = [1-\alpha : 0 : \alpha(\alpha+1)].$$

Proof. Suppose that our new bilinear form in standard coordinates is given by $v_1 \odot v_2 = v_1 \mathbf{C} v_2^T$ where

$$\mathbf{C} = \begin{bmatrix} a & d & f \\ d & b & g \\ f & g & c \end{bmatrix} \qquad \text{and}$$

$$\mathbf{D} = \operatorname{adj}(\mathbf{C}) = \begin{bmatrix} bc-g^2 & fg-cd & dg-bf \\ fg-cd & ac-f^2 & df-ag \\ dg-bf & df-ag & ab-d^2 \end{bmatrix}.$$

The fact that the four points $\alpha_0 = [1:1:1]$, $\overline{\alpha_0} = [1:-1:1]$, $\beta_0 = [-1:1:1]$ and $\overline{\beta_0} = [-1:-1:1]$ must all be null points means

$$\alpha_0 \mathbf{C} \alpha_0^T = \overline{\alpha_0}\mathbf{C}\left(\overline{\alpha_0}\right)^T = \beta_0 \mathbf{C} \beta_0^T = \overline{\beta_0}\mathbf{C}\left(\overline{\beta_0}\right)^T = 0.$$

These conditions lead to the following linear system of equations involving the entries of $\mathbf{C}$:

$$a+b+c+2d+2f+2g = 0$$
$$a+b+c-2d+2f-2g = 0$$
$$a+b+c-2d-2f+2g = 0$$
$$a+b+c+2d-2f-2g = 0.$$

From this we deduce that $d = f = g = 0$, and $a = -(b+c)$. So the matrices have the form, up to scaling, of:

$$\mathbf{C} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and}$$

$$\mathbf{D} = \begin{bmatrix} a-1 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a(a-1) \end{bmatrix}.$$

But there is also the condition that $\mathcal{P}_0$ is a parabola with foci f_1 and f_2, passing through all four basis points $v_1 = [0:0:1], v_2 = [1:0:0], \alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$. Since the foci lie on the axis $A = v_1 v_2$, we can write $f_1 = [m_1 : 0 : 1]$ and $f_2 = [m_2 : 0 : 1]$ for some m_1, m_2. Then recall that the quadrance and spread are determined by the projective matrices $\mathbf{C}$ and $\mathbf{D}$ by the rules (5).
We then compute

$$\begin{aligned} q(f_1, f_2) &= 1 - \frac{\left(am_1m_2 - 1\right)^2}{\left(am_1^2 - 1\right)\left(am_2^2 - 1\right)} \\ &= -a\frac{\left(m_1 - m_2\right)^2}{\left(am_2^2 - 1\right)\left(am_1^2 - 1\right)}. \end{aligned}$$

Since f_1 and f_2 are by assumption not perpendicular,

$$am_1m_2 - 1 \neq 0. \tag{8}$$

Also v_1 and v_2 lie on $\mathcal{P}_0$, so that

$$\begin{aligned} &q([m_1:0:1],[0:0:1]) + q([m_2:0:1],[0:0:1]) - 1 \\ &\quad = \frac{\left(am_1m_2-1\right)\left(am_1m_2+1\right)}{\left(am_2^2-1\right)\left(am_1^2-1\right)} = 0 \qquad \text{and} \\ &q([m_1:0:1],[1:0:0]) + q([m_2:0:1],[1:0:0]) - 1 \\ &\quad = -\frac{\left(am_1m_2-1\right)\left(am_1m_2+1\right)}{\left(am_1^2-1\right)\left(am_2^2-1\right)} = 0. \end{aligned}$$

Both these conditions, given (8), are equivalent to the relation

$$am_1m_2 + 1 = 0 \tag{9}$$

which we henceforth assume, implying that we may write

$$m_1 = m \quad \text{and} \quad m_2 = -\frac{1}{am}$$

for some non-zero number m.
In addition we must ensure that α_0 and $\overline{\alpha_0}$ lie on $\mathcal{P}_0$, but since these are both null points, the quadrances $q(f_1, \alpha_0)$ and $q(f_2, \alpha_0)$ etc. are undefined, and we must rather work with the general equation of the parabola. This is

$$\begin{aligned} &q([m:0:1],[x:y:z]) + q\left(\left[-\frac{1}{am}:0:1\right],[x:y:z]\right) - 1 \\ &\quad = \frac{4amxz - y^2(a-1)\left(am^2-1\right)}{\left(am^2-1\right)\left(ax^2 - ay^2 + y^2 - z^2\right)} = 0 \end{aligned}$$

which shows the equation of the parabola to be

$$4amxz - y^2(a-1)\left(am^2-1\right) = 0. \tag{10}$$

Now the condition that $\alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$ lie on $\mathcal{P}_0$ is that

$$\begin{aligned} 4am - (a-1)\left(am^2-1\right) &= a(1-a)m^2 + 4am + (a-1) \\ &= 0. \end{aligned} \tag{11}$$

Given that we started out with the existence of f_1 and f_2 assumed, we see that the discriminant of this quadratic equation

$$(4a)^2 - 4a(1-a)(a-1) = 4a(a+1)^2$$

must be a square. But this occurs precisely when a is a square, say

$$a = \alpha^2.$$

In this case the quadratic equation (11) has the form $\alpha^2(1-\alpha^2)m^2 + 4\alpha^2 m + (\alpha^2 - 1) = 0$ with solutions

$$m = m_1 = \frac{1+\alpha}{\alpha(\alpha-1)} \quad \text{and} \quad m_2 = \frac{1-\alpha}{\alpha(\alpha+1)}.$$

Combining these with (10), the identity

$$4\alpha^2 \frac{(\alpha+1)}{\alpha(\alpha-1)} xz - y^2(\alpha^2-1)\left(\alpha^2\left(\frac{1+\alpha}{\alpha(\alpha-1)}\right)^2 - 1\right) = \frac{4(xz-y^2)\alpha(\alpha+1)}{\alpha-1} = 0$$

shows that the equation of the parabola pleasantly simplifies to be

$$xz - y^2 = 0. \tag{12}$$

The foci may now be expressed as

$$f_1 = [m_1 : 0 : 1] = [\alpha+1 : 0 : \alpha(\alpha-1)] \quad \text{and}$$
$$f_2 = [m_2 : 0 : 1] = [1-\alpha : 0 : \alpha(\alpha+1)]. \qquad \square$$

Notice that

$$\det \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \alpha^2(\alpha-1)(\alpha+1) \neq 0$$

so $\alpha \neq 0, \pm 1$, since $\mathbf{C}$ is an invertible projective matrix.
The following Figure shows a view in the standard coordinate plane, where $[x:y:1]$ is represented by the affine point $[x,y]$. This corresponds roughly to a value of $\alpha = 0.3$. While it is both interesting and instructive to see different views of such a standard coordinate plane, this is somewhat unfamiliar to the classical geometer, so we will stick mostly to the Universal Hyperbolic Geometry model for our diagrams, where the unit circle always appears in blue as the unit circle $x^2 + y^2 = 1$.

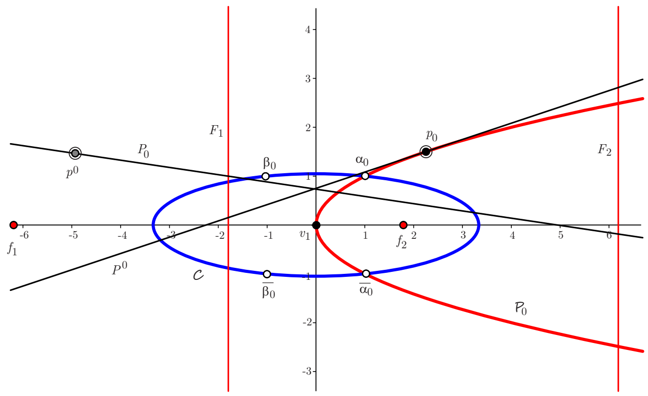

Figure 8: *A standard coordinate view of a parabola*

Theorem 7 (Parabola quadrance) *The* ***quadrance*** *of the parabola is*

$$q_{\mathcal{P}_0} \equiv q(f_1, f_2) = \frac{(\alpha^2+1)^2}{4\alpha^2}.$$

Proof. We compute that

$$q_{\mathcal{P}_0} = q([\alpha+1 : 0 : \alpha(\alpha-1)], [1-\alpha : 0 : \alpha(\alpha+1)])$$
$$= \frac{1}{4\alpha^2}(\alpha-1)^2(\alpha+1)^2 + 1 = \frac{(\alpha^2+1)^2}{4\alpha^2}. \qquad \square$$

We note that $q_{\mathcal{P}_0}$ is a square. This is a reflection of the fact that the assumption of the existence of vertices implies that the sides $\overline{f_1 b_2}$ and $\overline{f_2 b_1}$ have midpoints, see the Midpoint theorem [17].
The condition for points and lines to be null, in other words the equation for the null circle, is the following in standard coordinates.

Theorem 8 (Null points/ lines) *The point $p = [x:y:z]$ in standard coordinates is a null point precisely when*

$$\alpha^2 x^2 + (1-\alpha^2)y^2 - z^2 = 0.$$

The line $L = \langle l:m:n \rangle$ is a null line precisely when

$$(1-\alpha^2)l^2 + \alpha^2 m^2 + \alpha^2(\alpha^2-1)n^2 = 0.$$

Proof. These follow by using (7) to expand the respective conditions

$$[x:y:z]\,\mathbf{C}\,[x:y:z]^T = 0 \quad \text{and}$$
$$\langle l:m:n \rangle^T \mathbf{D} \langle l:m:n \rangle = 0. \qquad \square$$

3.3 Quadrance and spread in standard coordinates

We can now give explicit formulas for quadrance and spread in standard coordinates.

Theorem 9 (Quadrance formula) *The quadrance between the points $p_1 = [x_1 : y_1 : z_1]$ and $p_2 = [x_2 : y_2 : z_2]$ in parabolic standard coordinates is*

$$q(p_1,p_2) = -\frac{(x_1y_2-x_2y_1)^2\alpha^4+\left((x_1z_2-x_2z_1)^2-(y_1z_2-y_2z_1)^2-(x_1y_2-x_2y_1)^2\right)\alpha^2+(y_1z_2-y_2z_1)^2}{\left(\alpha^2x_1^2-y_1^2(\alpha^2-1)-z_1^2\right)\left(\alpha^2x_2^2-y_2^2(\alpha^2-1)-z_2^2\right)}.$$

Proof. From (4) and formula (7) for $\mathbf{C}$,

$$[x_1,y_1,z_1]\,\mathbf{C}\,[x_2,y_2,z_2]^T = \alpha^2x_1x_2 - y_1y_2\left(\alpha^2-1\right) - z_1z_2.$$

The formula follows using an identity calculation. □

Theorem 10 (Spread formula) *The spread between $L_1 = \langle l_1 : m_1 : n_1\rangle$ and $L_2 = \langle l_2 : m_2 : n_2\rangle$ is*

$$S(L_1,L_2) = \frac{\left((l_1n_2-l_2n_1)^2-(m_1n_2-m_2n_1)^2\right)\alpha^2+\left((l_1m_2-l_2m_1)^2-(l_2n_1-l_1n_2)^2\right)}{\left(l_1^2(\alpha^2-1)-\alpha^2m_1^2-\alpha^2n_1^2(\alpha^2-1)\right)\left(l_2^2(\alpha^2-1)-\alpha^2m_2^2-\alpha^2n_2^2(\alpha^2-1)\right)}.$$

Proof. From (4) and

$$\begin{aligned}&[l_1,m_1,n_1]\,\mathbf{D}\,[l_2,m_2,n_2]^T\\ &\quad= l_1l_2\left(\alpha^2-1\right)-\alpha^2m_1m_2-\alpha^2\left(\alpha^2-1\right)n_1n_2\end{aligned}$$

the formula follows using an identity calculation. □

Theorem 11 (Axis reflection) *The reflection r_a in the point a has the form*

$$r_a([x:y:z]) = [x:-y:z].$$

Proof. We use the usual formula for reflection in a vector:

$$r_v(u) = 2\frac{(u\cdot v)\,v}{v\cdot v} - u = 2\frac{\left(uCv^T\right)v}{vCv^T} - u.$$

With the matrix C above, and working with regular vectors, we get

$$\begin{aligned}r_{[0,1,0]}([x,y,z]) &= 2\frac{[0,1,0]\,C\,[x,y,z]^T}{[0,1,0]\,C\,[0,1,0]^T}[0,1,0]-[x,y,z]\\ &= [-x,y,-z] = [x,-y,z].\end{aligned}$$

□

3.4 Duality with respect to a conic and parametrizations

Let's recall some basic facts from the general theory of points and tangents to a projective conic. Suppose that a general conic $\mathcal{C}$ is given by the projective symmetric 3×3 matrix $\mathbf{A}$, with adjugate $\mathbf{B}$, so that a general point $p = [x:y:z]$ lies on $\mathcal{C}$ precisely when $p\mathbf{A}p^T = 0$. The tangent line P to a point p lying on $\mathcal{C}$ is $P = p^\perp \equiv \mathbf{A}p^T$. Dually, the point at which a tangent line L meets the conic is $l = L^\perp \equiv L^T\mathbf{B}$. While a point p on the conic satisfies the equation $p\mathbf{A}p^T = 0$, a line L on the conic (that is, a tangent line to the conic at some point) satisfies the dual equation $L^T\mathbf{B}L = 0$ (where we regard lines as projective column vectors).

More generally, we can regard the projective matrix $\mathbf{A}$ as determining a projective bilinear form, which is equivalent to a duality between points and lines. For a general point p, not necessarily lying on $\mathcal{C}$, its **dual** with respect to $\mathcal{C}$ is the line $p^\perp = \mathbf{A}p^T$, while for a general point L, its **dual** with respect to $\mathcal{C}$ is the point $L^\perp = L^T\mathbf{B}$. These are inverse procedures.

These notions of course go back to Apollonius, and it could be argued that this duality between points and lines is the essential feature or characteristic of a conic. But this modern formulation in the language of linear algebra and matrices makes many of its aspects much easier to understand, see [3], [12].

In this work, the main example of duality is with respect to the null circle $\mathcal{C}$, for which we will stick with the notation that if x_j is a point, then $X_j = \mathbf{C}x_j^T$ refers to the dual line and conversely. However the secondary duality with respect to the parabola $\mathcal{P}_0$ will also be involved, as we now see.

The equation (12) for the parabola $\mathcal{P}_0$ in standard coordinates, namely $p(x,y,z) = xz - y^2 = 0$, can be expressed in homogeneous matrix form as $p\mathbf{A}p^T = 0$ or

$$\begin{bmatrix}x & y & z\end{bmatrix}\mathbf{A}\begin{bmatrix}x & y & z\end{bmatrix}^T = 0$$

where

$$\mathbf{A} = \begin{bmatrix}0 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 0\end{bmatrix} \quad\text{and}\quad \operatorname{adj}(\mathbf{A}) \equiv \mathbf{B} = \begin{bmatrix}0 & 0 & 2\\ 0 & -1 & 0\\ 2 & 0 & 0\end{bmatrix}.$$

Theorem 12 (Parabola parametrization) *The parabola $\mathcal{P}_0$ is parametrized, using an affine parameter t, by $p_0 = \left[t^2 : t : 1\right] \equiv p(t)$ or by using a projective parameter $[t:r]$ as $p_0 = \left[t^2 : tr : r^2\right] \equiv p(t:r)$. The tangent line P^0 to the parabola at $p_0 = \left[t^2 : t : 1\right]$ is $P^0 = \left\langle 1 : -2t : t^2\right\rangle \equiv P(t)$ or projectively the tangent to $p_0 = \left[t^2 : tr : r^2\right]$ is $P^0 = \left\langle r^2 : -2rt : t^2\right\rangle \equiv P(t:r)$. A line $L = \langle l : m : n\rangle$ is tangent to the parabola precisely when $m^2 = 4nl$.*

Proof. The simple form of the equation $xz = y^2$ makes the parametrization immediate. The formula for the tangent line is a direct application of the discussion above, so that

$$\begin{aligned}P^0 \equiv \mathbf{A}p_0^T &= \begin{bmatrix}0 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}t^2 & t & 1\end{bmatrix}^T = \begin{bmatrix}1\\ -2t\\ t^2\end{bmatrix}\\ &= \left\langle 1 : -2t : t^2\right\rangle\end{aligned}$$

or using projective parameters

$$P^0 \equiv \mathbf{A}p_0^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^2 & tr & r^2 \end{bmatrix}^T = \begin{bmatrix} r^2 \\ -2rt \\ t^2 \end{bmatrix}$$
$$= \left\langle r^2 : -2rt : t^2 \right\rangle.$$

The relation $m^2 = 4nl$ is exactly satisfied by those lines of this form. □

Theorem 13 (Tangent meets) *If $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ are two distinct points on $\mathcal{P}_0$, then their tangents P^0 and Q^0 meet at the **polar point** $z \equiv P^0Q^0 = [2tu : t+u : 2]$ while $Z \equiv p_0q_0 = \langle 1 : -(t+v) : tv \rangle$.*

Proof. We compute that

$$z \equiv P^0Q^0 = \left\langle 1 : -2t : t^2 \right\rangle \times \left\langle 1 : -2u : u^2 \right\rangle = [2tu : t+u : 2]$$

and

$$p_0q_0 = \left[t^2 : t : 1\right] \times \left[v^2 : v : 1\right] = \langle 1 : -(t+v) : tv \rangle. \qquad \square$$

The projective parametrization of $\mathcal{P}_0$ has the advantage that it includes the important point at infinity $p(1:0) = [1:0:0] = v_2$. We can recover the affine parametrization by setting $r = 1$, and we can go from the affine to the projective parametrization by replacing t with t/r and clearing denominators. In practice we will generally use the affine parametrization, since it is requires only one variable, not two. The existence of this simple parametrization will be extremely useful for us: giving us the same amount of control over the hyperbolic parabola as we have over the much simpler Euclidean parabola (which of course can be positioned to have exactly the same equation!)

Theorem 14 *The dual of the point $p_0 = \left[t^2 : t : 1\right]$ on $\mathcal{P}_0$ is $P_0 = \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right) : -1 \right\rangle$. The dual of the tangent line $P^0 = \left\langle 1 : -2t : t^2 \right\rangle$ is $p^0 = \left[\alpha^2 - 1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]$.*

Proof. We compute that

$$P_0 = \mathbf{C}p_0^T = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} t^2 & t & 1 \end{bmatrix}^T$$
$$= \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right) : -1 \right\rangle$$

and

$$p^0 = \left(P^0\right)^T \mathbf{D} = \begin{bmatrix} 1 & -2t & t^2 \end{bmatrix} \begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2\left(1-\alpha^2\right) \end{bmatrix}$$
$$= \left[\alpha^2 - 1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]. \qquad \square$$

We will say that p^0 is the **twin point** to p_0. Later we will see that the locus of p^0 is also a parabola, whose foci f^1 and f^2 are the sydpoints of $\overline{f_1 f_2}$.

Theorem 15 (Focus directrix polarity) *The focus f_1 is the pole of the directrix F_2 with respect to the parabola $\mathcal{P}_0$, and similarly the focus f_2 is the pole of the directrix F_1.*

Proof. We check that

$$F_2^T\mathbf{B} = \left[\alpha(\alpha-1) \quad 0 \quad \alpha+1\right]\mathbf{B} = [\alpha+1 : 0 : \alpha(\alpha-1)] = f_1$$

or

$$\mathbf{A}f_1^T = \mathbf{A}\left[\alpha+1 \quad 0 \quad \alpha(\alpha-1)\right]^T = \langle \alpha(\alpha-1) : 0 : \alpha+1 \rangle = F_2.$$

Similarly,

$$F_1^T\mathbf{B} = \left[\alpha(\alpha+1) \quad 0 \quad 1-\alpha\right]\mathbf{B} = [-(\alpha-1) : 0 : \alpha(\alpha+1)] = f_2$$

or

$$\mathbf{A}f_2^T = \mathbf{A}\left[1-\alpha \quad 0 \quad \alpha(\alpha+1)\right]^T = \langle \alpha(\alpha+1) : 0 : 1-\alpha \rangle = F_1.$$

□

In order for the parabola $y^2 = xz$ to have a null point $p(t)$, the parameter t must satisfy $\left[t^2 : t : 1\right]\mathbf{C}\left[t^2 : t : 1\right]^T = 0$, which yields $\left(t^2-1\right)\left(t^2\alpha^2+1\right) = 0$. Over the rational field, the values $t = \pm 1$ agree with the null points $\alpha_0 = [1:1:1]$ and $\overline{\alpha_0} = [1:-1:1]$ with which we begun our work.

However, there are also another two solutions which are invisible over the rational field, but exist in an extension field obtained by adjoining a square root i of -1. These points are $\zeta_1 \equiv \left[1 : i\alpha : -\alpha^2\right]$ and $\zeta_2 \equiv \left[1 : -i\alpha : -\alpha^2\right]$. In this paper we will not mention these points too much.

3.5 Formulas for directrices, vertex lines, base points and base lines

We can now augment our formulas using standard coordinates. The directrices are

$$F_1 \equiv f_1^\perp = \mathbf{C}\,[\alpha+1 : 0 : \alpha(\alpha-1)]^T = \langle \alpha(\alpha+1) : 0 : 1-\alpha \rangle$$
$$F_2 \equiv f_2^\perp = \mathbf{C}\,[1-\alpha : 0 : \alpha(\alpha+1)]^T = \langle \alpha(\alpha-1) : 0 : 1+\alpha \rangle.$$

The base points are the meets of the directrices and the axis line. They are

$$b_1 \equiv F_1A = \left\langle \alpha^2(\alpha+1) : 0 : \alpha(1-\alpha) \right\rangle \times \langle 0 : 1 : 0 \rangle$$
$$= [\alpha-1 : 0 : \alpha(\alpha+1)]$$
$$b_2 \equiv F_2A = \left\langle \alpha^2(\alpha-1) : 0 : \alpha(1+\alpha) \right\rangle \times \langle 0 : 1 : 0 \rangle$$
$$= [\alpha+1 : 0 : \alpha(1-\alpha)].$$

The duals are the **base lines** B_1, B_2, which are the altitudes to the axis A through the foci f_1, f_2 of the parabola:

$$B_1 \equiv b_1^\perp = \mathbf{C}\,[(\alpha-1) : 0 : \alpha(\alpha+1)]$$
$$= \langle -\alpha(\alpha-1) : 0 : \alpha+1 \rangle$$
$$B_2 \equiv b_2^\perp = \mathbf{C}\,[(\alpha+1) : 0 : \alpha(1-\alpha)]$$
$$= \langle \alpha(\alpha+1) : 0 : \alpha-1 \rangle.$$

The **vertex lines** V_1, V_2 are the altitudes to the axis A through the vertices v_1, v_2 of the parabola**:**

$$V_1 \equiv v_1^{\perp} = \mathbf{C}[0:0:1] = [0:0:1] \qquad \text{and}$$
$$V_2 \equiv v_2^{\perp} = \mathbf{C}[1:0:0] = [1:0:0].$$

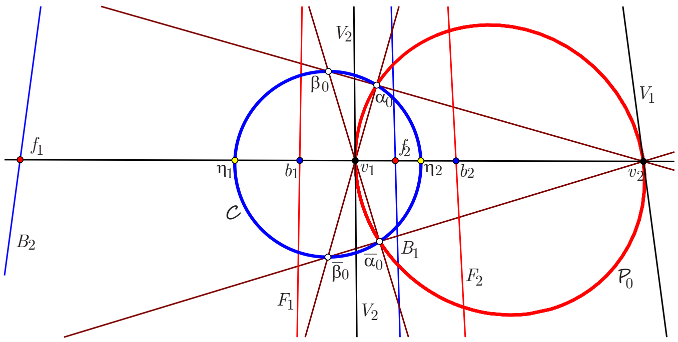

Figure 9: *Some basic points associated to a parabola* $\mathcal{P}_0$

3.6 The j, h and d points and lines

We define the **axis null points** to be the meets of the axis A and the null conic C. These points exist under our assumptions, and are

$$\eta_1 \equiv AC = [1:0:\alpha] \quad \text{and} \quad \eta_2 = AC = [-1:0:\alpha].$$

We now introduce some other secondary points and lines associated to a generic point p_0 on the parabola $\mathcal{P}_0$. The reflection of $p_0 = \left[t^2:t:1\right]$ in the axis is the **opposite point**

$$\overline{p_0} = r_a(p_0) = \left[t^2:-t:1\right].$$

Clearly $\overline{p_0}$ also lies on the parabola.

The meet of the dual line P_0 with the axis A is the j**-point**

$$j_0 \equiv P_0A = \left\langle t^2\alpha^2 : t\left(1-\alpha^2\right):-1\right\rangle \times \langle 0:1:0\rangle = \left[1:0:t^2\alpha^2\right]$$

with dual the J**-line**

$$J_0 = ap_0 = [0:1:0] \times \left[t^2:t:1\right] = \left\langle 1:0:-t^2\right\rangle.$$

By duality J_0 is the altitude from p_0 to the axis, and so also $J_0 = p_0\overline{p_0}$. The meet of the J-line with the axis is the foot of this altitude; it is the h**-point**

$$h_0 \equiv AJ_0 = \langle 0:1:0\rangle \times \left\langle 1:0:-t^2\right\rangle = \left[t^2:0:1\right]$$

and its dual is the H-line

$$H_0 \equiv h_0^{\perp} = aj_0 = [0:1:0] \times \left[1:0:t^2\alpha^2\right] = \left\langle t^2\alpha^2:0:-1\right\rangle.$$

The meet of the tangent line P^0 with the axis is the **twin** j**-point**

$$j^0 \equiv P^0A = \left\langle 1:-2t:t^2\right\rangle \times \langle 0:1:0\rangle = \left[-t^2:0:1\right]$$

with dual the **twin** J**-line**

$$J^0 = ap^0 = [0:1:0] \times \left[\alpha^2-1:2t\alpha^2:-t^2\alpha^2\left(\alpha^2-1\right)\right] = \left\langle t^2\alpha^2:0:1\right\rangle.$$

The meet of the twin J-line with the axis is the **twin** h**-point**

$$h^0 \equiv AJ^0 = \langle 0:1:0\rangle \times \left\langle t^2\alpha^2:0:1\right\rangle = \left[-1:0:t^2\alpha^2\right]$$

and its dual is the **twin** H**-line**

$$H^0 \equiv \left(h^0\right)^{\perp} = aj^0 = [0:1:0] \times \left[-t^2:0:1\right] = \left\langle 1:0:t^2\right\rangle.$$

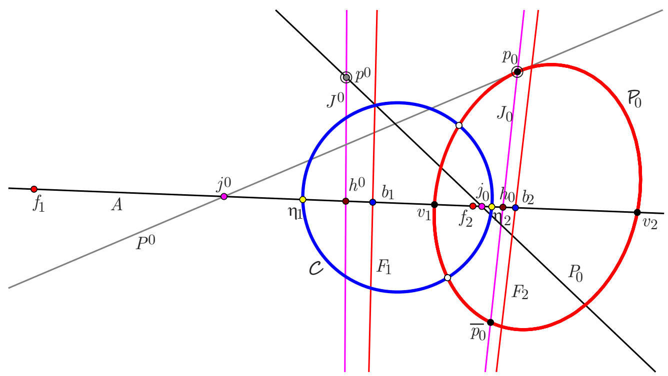

Figure 10: *The j and h points and lines*

Theorem 16 (Null tangent) *The tangent P^0 to the parabola $\mathcal{P}_0$ at p_0 is a null line precisely when p_0 lies on a directrix, and in this case the twin point p^0 is a null point lying on the other directrix, j_0 coincides with a focus, and j^0 with the other focus.*

Proof. If the tangent $P^0 = \left\langle 1:-2t:t^2\right\rangle$ at $p_0 = \left[t^2:t:1\right]$ is a null line, then by the Null points/lines theorem

$$\left(1-\alpha^2\right) + 4\alpha^2t^2 + \alpha^2\left(\alpha^2-1\right)t^4 = 0.$$

This factors as

$$\left(\alpha(\alpha+1)t^2 - (\alpha-1)\right)\left(\alpha(\alpha-1)t^2 + (\alpha+1)\right) = 0$$

so that

$$t^2 = \frac{\alpha-1}{\alpha(\alpha+1)} \qquad \text{or} \qquad t^2 = -\frac{\alpha+1}{\alpha(\alpha-1)}. \tag{13}$$

Now $p_0 = \left[t^2:t:1\right]$ is on the directrix F_1 or F_2, precisely when

$$\left[t^2:t:1\right]\left[\alpha^2(\alpha+1):0:\alpha(1-\alpha)\right]^T = 0 \qquad \text{or}$$
$$\left[t^2:t:1\right]\left[\alpha^2(\alpha-1):0:\alpha(1+\alpha)\right]^T = 0$$

and similarly, the point $p^0 = \left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]$ is on the directrix F_1 or F_2, precisely when

$$\left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]\left[\alpha^2\left(\alpha+1\right) : 0 : \alpha\left(1-\alpha\right)\right]^T = 0$$

or

$$\left[\alpha^2-1 : 2t\alpha^2 : -t^2\alpha^2\left(\alpha^2-1\right)\right]\left[\alpha^2\left(\alpha-1\right) : 0 : \alpha\left(1+\alpha\right)\right]^T = 0.$$

These conditions are exactly the same as (13). Using (13) we get either $j_0 = \left[1:0:t^2\alpha^2\right] = \left[\alpha+1:0:\alpha\left(\alpha-1\right)\right] = f_1$ and $j^0 = \left[-t^2:0:1\right] = \left[1-\alpha:0:\alpha\left(\alpha+1\right)\right] = f_2$ or $j_0 = \left[1-\alpha:0:\alpha\left(\alpha+1\right)\right] = f_2$ and $j^0 = \left[\alpha+1:0:\alpha\left(\alpha-1\right)\right] = f_1$. $\square$

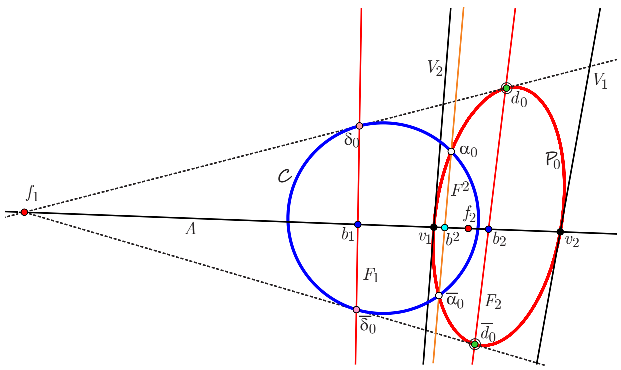

Figure 11: *Null tangents and* d_0, $\overline{d_0}$ *points*

We introduce the points d_0 and $\overline{d_0}$ to be the meets of the directrix F_2 with the parabola $\mathcal{P}_0$, should they exist, and the corresponding twin null points δ_0 and $\overline{\delta_0}$ lying on the directrix F_1. These are important canonical points associated with the parabola. Since their existence requires solutions to (13), and so a number τ satisfying $\tau^2 = \alpha\left(\alpha^2-1\right)$, we may write

$$d_0 = F_2\mathcal{P}_0 = \left[\alpha-1 : \tau : \alpha\left(\alpha+1\right)\right]$$
$$\overline{d_0} = F_2\mathcal{P}_0 = \left[\alpha-1 : -\tau : \alpha\left(\alpha+1\right)\right]$$

and

$$d^0 \equiv \delta_0 = \left[\left(\alpha-1\right)^2\left(\alpha+1\right) : -2\alpha i\tau : \alpha\left(\alpha+1\right)^2\left(\alpha-1\right)\right]$$
$$\overline{d_0} = \overline{\delta_0} = \left[\left(\alpha-1\right)^2\left(\alpha+1\right) : 2\alpha i\tau : \alpha\left(\alpha+1\right)^2\left(\alpha-1\right)\right]$$

where $\left(i\tau\right)^2 = -\alpha\left(\alpha^2-1\right)$.

In Figure 11, notice that the lines $f_1\delta_0$ and $f_1\overline{\delta_0}$ are joint tangents to both $\mathcal{C}$ and the parabola $\mathcal{P}_0$, touching $\mathcal{P}_0$ at the points d_0 and $\overline{d_0}$.

3.7 The sydpoints of a parabola

It is a remarkable fact that the theory of sydpoints that we developed in [20] plays a crucial role in the theory of the parabola. Define the lines

$$F^2 \equiv \alpha_0\overline{\alpha_0} = [1:1:1]\times[1:-1:1] = \langle 1:0:-1\rangle$$
$$B^1 \equiv \beta_0\overline{\beta_0} = [-1:1:1]\times[-1:-1:1] = \langle 1:0:1\rangle$$

with corresponding axis meets

$$b^2 \equiv F^2A = \langle 1:0:-1\rangle\times\langle 0:1:0\rangle = [1:0:1]$$
$$f^1 \equiv B^1A = \langle 1:0:1\rangle\times\langle 0:1:0\rangle = [-1:0:1].$$

The duals of these points and lines are

$$f^2 \equiv \left(F^2\right)^{\perp} = \begin{bmatrix}1 & 0 & -1\end{bmatrix}\mathbf{D} = \left[1:0:\alpha^2\right]$$
$$b^1 \equiv \left(B^1\right)^{\perp} = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\mathbf{D} = \left[1:0:-\alpha^2\right]$$
$$B^2 \equiv \left(b^2\right)^{\perp} = \mathbf{C}\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T = \left\langle -\alpha^2:0:1\right\rangle$$
$$F^1 \equiv \left(f^1\right)^{\perp} = \mathbf{C}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T = \left\langle \alpha^2:0:1\right\rangle.$$

The points f^1 and f^2 are the **twin foci**, or **t-foci** for short**,** of the parabola $\mathcal{P}_0$. They will play a major role in the theory. The dual lines of f^1 and f^2, namely F^1 and F^2 respectively, are the **t-directrices of** $\mathcal{P}_0$. The meets of the t-directrices and the axis A are $F^1A \equiv b^1$ and $F^2A \equiv b^2$ respectively; these are the **t-base points** of $\mathcal{P}_0$. The dual lines of b^1 and b^2, namely B^1 and B^2 respectively, are the **t-base lines** of $\mathcal{P}$. These are all shown in Figure 12.

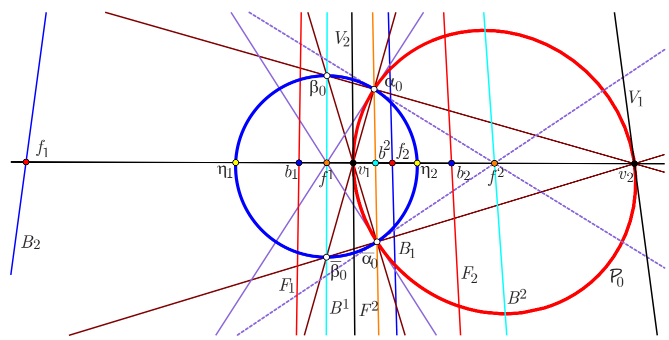

Figure 12: *Sydpoints and the twin foci* f^1 *and* f^2 *of* $\mathcal{P}_0$

Theorem 17 (Parabola sydpoints) *The points* f^1 *and* f^2 *are the sydpoints of the side* $\overline{f_1f_2}$.

Proof. We calculate that

$$q\left(f_1,f^1\right) = q\left(\left[\alpha+1:0:\alpha\left(\alpha-1\right)\right],\left[1:0:-1\right]\right)$$
$$= 1+\frac{\left(\alpha\left(\alpha-1\right)+\alpha^2\left(\alpha+1\right)\right)^2}{4\alpha^3-4\alpha^5} = -\frac{\left(\alpha^2+1\right)^2}{4\alpha\left(\alpha^2-1\right)}$$
$$q\left(f_2,f^1\right) = q\left(\left[1-\alpha:0:\alpha\left(\alpha+1\right)\right],\left[1:0:-1\right]\right)$$
$$= 1-\frac{\left(\alpha\left(\alpha+1\right)-\alpha^2\left(\alpha-1\right)\right)^2}{4\alpha^3-4\alpha^5} = \frac{\left(\alpha^2+1\right)^2}{4\alpha\left(\alpha^2-1\right)}$$

$$q\left(f_1,f^2\right)=q\left(\left[\alpha+1:0:\alpha(\alpha-1)\right],\left[1:0:\alpha^2\right]\right)$$
$$=1-\frac{\left(\alpha^2(\alpha+1)-\alpha^3(\alpha-1)\right)^2}{4\alpha^5-4\alpha^7}=\frac{1}{4}\frac{\left(\alpha^2+1\right)^2}{\alpha\left(\alpha^2-1\right)}$$
$$q\left(f_2,f^2\right)=q\left(\left[1-\alpha:0:\alpha(\alpha+1)\right],\left[1:0:\alpha^2\right]\right)$$
$$=1+\frac{\left(\alpha^2(\alpha-1)+\alpha^3(\alpha+1)\right)^2}{4\alpha^5-4\alpha^7}=-\frac{1}{4}\frac{\left(\alpha^2+1\right)^2}{\alpha\left(\alpha^2-1\right)}.$$

Clearly $q\left(f_1,f^1\right)=-q\left(f_2,f^1\right)$ and $q\left(f_1,f^2\right)=-q\left(f_2,f^2\right)$ so f^1 and f^2 are the sydpoints of the side $\overline{f_1f_2}$. $\square$

Theorem 18 (Parabola null tangents) *The tangents to the null circle at α_0 and $\overline{\alpha_0}$ meet at f^2. The tangents to $\mathcal{P}_0$ at α_0 and $\overline{\alpha_0}$ meet at f^1.*

Proof. The tangents to the null circle at α_0 and $\overline{\alpha_0}$ are the dual lines

$$\alpha_0^{\perp}=C[1:1:1]^T=\left\langle\alpha^2:1-\alpha^2:-1\right\rangle\qquad\text{and}$$
$$\left(\overline{\alpha_0}\right)^{\perp}=C[1:-1:1]^T=\left\langle\alpha^2:\alpha^2-1:-1\right\rangle$$

and these meet at

$$\alpha_0^{\perp}\left(\overline{\alpha_0}\right)^{\perp}=\left\langle\alpha^2:1-\alpha^2:-1\right\rangle\times\left\langle\alpha^2:\alpha^2-1:-1\right\rangle$$
$$=\left[1:0:\alpha^2\right]=f^2.$$

The tangents to the parabola $\mathcal{P}_0$ at α_0 and $\overline{\alpha_0}$ are the lines

$$\mathbf{A}\begin{bmatrix}1&1&1\end{bmatrix}^T=\langle 1:-2:1\rangle\quad\text{and}\quad\mathbf{A}\begin{bmatrix}1&-1&1\end{bmatrix}^T=\langle 1:2:1\rangle$$

and these meet at
$\langle 1:-2:1\rangle\times\langle 1:2:1\rangle=[-1:0:1]=f^1$. $\square$

3.8 A rational parabola

In this section we show the existence of a two-parameter family of rational hyperbolic parabolas, and give the associated transformations to parabolic standard coordinates.
The conic $\mathcal{P}_0$ with equation

$$\left(t_1^2t_2^2-1\right)x^2+2\left(t_1^2t_2^2+1\right)x+\left(t_1^2-t_2^2\right)y^2+\left(t_1^2t_2^2-1\right)=0$$

meets the null circle at the null points $\alpha_0=\left[1-t_1^2:2t_1:t_1^2+1\right]$ and $\overline{\alpha_0}=\left[1-t_1^2:-2t_1:t_1^2+1\right]$. This is a parabola with foci
$f_1=\left[t_1+t_2-t_1t_2^2+t_1^2t_2:0:t_1+t_2+t_1t_2^2-t_1^2t_2\right]$ and $f_2=\left[t_1-t_2-t_1t_2^2-t_1^2t_2:0:t_1-t_2+t_1t_2^2+t_1^2t_2\right]$, axis $A=\langle 0:1:0\rangle$, and t-foci $f^1=\left[t_1^2+1:0:-\left(t_1^2-1\right)\right]$ and $f^2=\left[t_2^2-1:0:-\left(t_2^2+1\right)\right]$. The null points $\beta_0,\overline{\beta_0}$ are $\beta_0=\left[1-t_2^2:2t_2:t_2^2+1\right]$ and $\overline{\beta_0}=\left[1-t_2^2:-2t_2:t_2^2+1\right]$, and the vertices are $v_1=\left[t_1t_2-1:0:-(t_1t_2+1)\right]$ and $v_2=\left[t_1t_2+1:0:-(t_1t_2-1)\right]$. Note that

$$q\left(f^1,f^2\right)-1$$
$$=q\left(\left[t_1^2+1:0:-\left(t_1^2-1\right)\right],\left[t_2^2-1:0:-\left(t_2^2+1\right)\right]\right)-1$$
$$=\frac{1}{4}\left(t_1-t_2\right)^2\frac{\left(t_1+t_2\right)^2}{t_1^2t_2^2}$$

is a square.
We are now interested in sending these points $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ to the points $[1:1:1],[1:-1:1],[-1:1:1],[-1:-1:1]$ respectively, using a projective transformation. Firstly, we send $[1:1:1],[1:0:0],[0:1:0],[0:0:1]$ to $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ respectively by the linear transformation $T_1(v)=vN$ where N is

$$N=\begin{bmatrix}-t_2\left(t_1^2-1\right)&-2t_1t_2&t_2\left(t_1^2+1\right)\\-t_1\left(t_2^2-1\right)&2t_1t_2&t_1\left(t_2^2+1\right)\\t_1\left(t_2^2-1\right)&2t_1t_2&-t_1\left(t_2^2+1\right)\end{bmatrix}.$$

Its inverse sends $\alpha_0,\overline{\alpha_0},\beta_0,\overline{\beta_0}$ back to $[1:1:1]$, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ by $T(v)=vR$ where R is the adjugate of N:

$$R=\begin{bmatrix}-2t_1\left(t_2^2+1\right)&(t_1-t_2)(t_1t_2-1)&-(t_1t_2+1)(t_1+t_2)\\0&t_1^2-t_2^2&t_1^2-t_2^2\\-2t_1\left(t_2^2-1\right)&(t_1-t_2)(t_1t_2+1)&-(t_1t_2-1)(t_1+t_2)\end{bmatrix}.$$

Secondly, the linear transformation $T_2(v)=vM$, where M is

$$M=\begin{bmatrix}1&-1&1\\-1&1&1\\1&1&-1\end{bmatrix},$$

sends $[1:1:1]$, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ to $[1:1:1]$, $[-1:1:1]$, $[-1:1:1]$, $[-1:-1:1]$ respectively. Thus, the required transformation is $T(v)=v(RM)$ where RM is

$$\begin{bmatrix}-(t_1t_2+1)(t_1+t_2)&0&(t_1-t_2)(t_1t_2-1)\\0&(t_1-t_2)(t_1+t_2)&0\\-(t_1t_2-1)(t_1+t_2)&0&(t_1-t_2)(t_1t_2+1)\end{bmatrix}.$$

After applying this linear transformation, the matrix J is replaced by

$$\mathbf{C}=(RM)^{-1}J\left((RM)^{-1}\right)^T$$
$$=\begin{bmatrix}t_1t_2\left(t_1-t_2\right)^2&0&0\\0&4t_1^2t_2^2&0\\0&0&-t_1t_2\left(t_1+t_2\right)^2\end{bmatrix}\qquad\text{and}$$
$$\mathbf{D}=(RM)^TJ(RM)$$
$$=\begin{bmatrix}4t_1t_2\left(t_1+t_2\right)^2&0&0\\0&\left(t_1^2-t_2^2\right)^2&0\\0&0&-4t_1t_2\left(t_1-t_2\right)^2\end{bmatrix}.$$

and we get $\alpha = \frac{t_1 - t_2}{t_1 + t_2}$. In this new coordinate system, the parabola is $y^2 = xz$ with foci $f_1 = [t_1 (t_1 + t_2) : 0 : -t_2 (t_1 - t_2)]$ and $f_2 = [t_2 (t_1 + t_2) : 0 : t_1 (t_1 - t_2)]$.

Example 1 *If $t_1 = 1/2$ and $t_2 = 3$ then the parabola $\mathcal{P}_0$ has equation $26x + 5x^2 - 35y^2 + 5 = 0$ which meets the null circle at the null points $\alpha_0 = [3:4:5]$ and $\overline{\alpha_0} = [3:-4:5]$; has axis $A = \langle 0:1:0 \rangle$, foci $f_1 = [-1:0:29]$ and $f_2 = [-31:0:11]$, vertices $v_1 = [1:0:-5]$ and $v_2 = [5:0:-1]$, t-foci $f^1 = [5:0:3]$ and $f^2 = [4:0:-5]$, and $\beta_0 = [-4:3:5]$ and $\overline{\beta_0} = [4:3:-5]$.*

3.9 Focal and base lines

We now define some other fundamental points and lines associated with a point $p_0 \equiv \left[t^2 : t : 1\right]$ on the parabola $\mathcal{P}_0$. It will be convenient to introduce the quantities

$$\Delta_1 (t) \equiv \alpha + 1 + t^2\alpha - t^2\alpha^2$$
$$\Delta_2 (t) \equiv \alpha - 1 + t^2\alpha + t^2\alpha^2$$
$$\Delta_3 (t) \equiv \alpha + 1 - t^2\alpha + t^2\alpha^2$$
$$\Delta_4 (t) \equiv \alpha - 1 - t^2\alpha^2 - t^2\alpha$$

which depends on t, and so on p_0, and which will appear in many formulas to follow. Notice that

$$\Delta_1^2 - \Delta_2^2 = -4\alpha \left(t^2\alpha^2 - 1\right)\left(t^2 + 1\right), \quad \Delta_1^2 - \Delta_3^2 = -4\alpha t^2 \left(\alpha^2 - 1\right)$$
$$\Delta_1^2 - \Delta_4^2 = -4\alpha \left(t^4\alpha^2 - 1\right), \quad \Delta_2^2 - \Delta_3^2 = 4\alpha \left(t^4\alpha^2 - 1\right)$$
$$\Delta_2^2 - \Delta_4^2 = 4t^2\alpha \left(\alpha^2 - 1\right), \quad \Delta_3^2 - \Delta_4^2 = -4\alpha \left(t^2 - 1\right)\left(t^2\alpha^2 + 1\right).$$

The **focal lines** R_1, R_2 and the dual **focal line points** r_1, r_2 are defined by, and calculated as:

$$\begin{aligned} R_1 &\equiv f_1 p_0 = [\alpha + 1 : 0 : \alpha (\alpha - 1)] \times \left[t^2 : t : 1\right] \\ &= \langle t\alpha (\alpha - 1) : \Delta_1 : -t (\alpha + 1) \rangle \\ R_2 &\equiv f_2 p_0 = [1 - \alpha : 0 : \alpha (\alpha + 1)] \times \left[t^2 : t : 1\right] \\ &= \langle t\alpha (\alpha + 1) : -\Delta_2 : t (\alpha - 1) \rangle \\ r_1 &\equiv R_1^\perp = F_1 P_0 \\ &= \left[t (\alpha - 1)^2 (\alpha + 1) : -\alpha\Delta_1 : t\alpha (\alpha - 1) (\alpha + 1)^2\right] \\ r_2 &\equiv R_2^\perp = F_2 P_0 \\ &= \left[t (\alpha - 1) (\alpha + 1)^2 : \alpha\Delta_2 : -t\alpha (\alpha - 1)^2 (\alpha + 1)\right]. \end{aligned}$$

Since R_1, R_2 and P^0 are concurrent at p_0, dually we see that r_1, r_2 and p^0 are collinear on P_0.
The **altitude base points** t_1 and t_2 and the dual **altitude base lines** T_1, T_2 are defined by, and calculated as:

$$\begin{aligned} t_1 &\equiv F_1 R_1 = \left[(\alpha - 1)\Delta_1 : 4t\alpha^2 : \alpha (\alpha + 1) \Delta_1\right] \\ t_2 &\equiv F_2 R_2 = \left[(\alpha + 1)\Delta_2 : 4t\alpha^2 : -\alpha (\alpha - 1) \Delta_2\right] \\ T_1 &\equiv t_1^\perp = f_1 r_1 = \left\langle \alpha (\alpha - 1) \Delta_1 : -4t\alpha \left(\alpha^2 - 1\right) : -(\alpha + 1)\Delta_1 \right\rangle \\ T_2 &\equiv t_2^\perp = f_2 r_2 = \left\langle \alpha (\alpha + 1) \Delta_2 : -4t\alpha \left(\alpha^2 - 1\right) : (\alpha - 1)\Delta_2 \right\rangle. \end{aligned}$$

The focal lines R_1 and R_2 also meet the directrices at the **second altitude base points** u_1, u_2, with dual lines U_1, U_2:

$$\begin{aligned} u_1 &\equiv R_2 F_1 = \left[(\alpha - 1)\Delta_2 : 2t\alpha \left(\alpha^2 - 1\right) : \alpha (\alpha + 1) \Delta_2\right] \\ u_2 &\equiv R_1 F_2 = \left[-(\alpha + 1)\Delta_1 : 2t\alpha \left(\alpha^2 - 1\right) : \alpha (\alpha - 1) \Delta_1\right] \\ U_1 &\equiv u_1^\perp = \left\langle \alpha (\alpha + 1) \Delta_1 : 2t \left(\alpha^2 - 1\right)^2 : (\alpha - 1) \Delta_1 \right\rangle \\ U_2 &\equiv u_2^\perp = \left\langle -\alpha (\alpha - 1) \Delta_2 : 2t \left(\alpha^2 - 1\right)^2 : (\alpha + 1) \Delta_2 \right\rangle. \end{aligned}$$

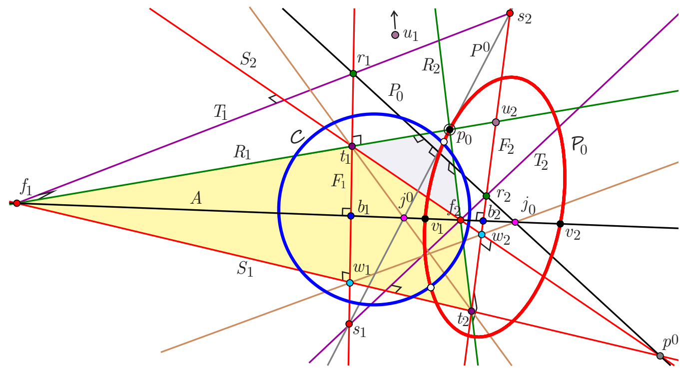

Figure 13: *The r, s, t and w points of p_0 on $\mathcal{P}_0$*

The **t-base lines** S_1, S_2 and their duals the **t-base points** s_1, s_2 are defined by, and calculated as:

$$\begin{aligned} S_1 &\equiv f_1 t_2 = \left\langle -2t\alpha^2 (\alpha - 1) : \left(\alpha^2 - 1\right) \Delta_2 : 2t\alpha (\alpha + 1) \right\rangle \\ S_2 &\equiv f_2 t_1 = \left\langle 2t\alpha^2 (\alpha + 1) : -\left(\alpha^2 - 1\right) \Delta_1 : 2t\alpha (\alpha - 1) \right\rangle \\ s_1 &\equiv S_1^\perp = F_1 T_2 = [2t (\alpha - 1) : \Delta_2 : 2t\alpha (\alpha + 1)] \\ s_2 &\equiv S_2^\perp = F_2 T_1 = [2t (\alpha + 1) : \Delta_1 : -2t\alpha (\alpha - 1)]. \end{aligned}$$

Theorem 19 (T-base) *Both s_1 and s_2 lie on the tangent P^0. Dually the lines S_1 and S_2 meet at p^0.*

Proof. We verify that s_1 and s_2 lie on the tangent $P^0 = \left\langle 1 : -2t : t^2 \right\rangle$ by computing

$$[2t (\alpha - 1) : \Delta_2 : 2t\alpha (\alpha + 1)] \left[1 : -2t : t^2\right]^T = 0$$
$$[2t (\alpha + 1) : \Delta_1 : -2t\alpha (\alpha - 1)] \left[1 : -2t : t^2\right]^T = 0.$$

The statement that S_1 and S_2 meet at p^0 follows from duality. □

The *w*-**points** w_1 and w_2, and their duals W_1 and W_2, are defined and computed as:

$$\begin{aligned} w_1 &\equiv F_1 S_1 = \left[\left(\alpha^2 - 1\right)(\alpha - 1)\Delta_2 : -8t\alpha^3 : \alpha \left(\alpha^2 - 1\right)(\alpha + 1)\Delta_2\right] \\ w_2 &\equiv F_2 S_2 = \left[\left(\alpha^2 - 1\right)(\alpha + 1)\Delta_1 : 8t\alpha^3 : -\alpha \left(\alpha^2 - 1\right)(\alpha - 1)\Delta_1\right] \\ W_1 &\equiv f_1 s_1 = \left(\alpha (\alpha - 1) \Delta_2 : 8t\alpha^2 : -(\alpha + 1) \Delta_2\right) \\ W_2 &\equiv f_2 s_2 = \left(\alpha (\alpha + 1) \Delta_1 : -8t\alpha^2 : (\alpha - 1) \Delta_1\right). \end{aligned}$$

Theorem 20 (J points collinearities) *We have collinearities* $\left[\left[t_1 t_2 j^0\right]\right]$, $[[j_0 u_1 u_2]]$ *and* $[[w_1 w_2 j_0]]$.

Proof. Using the various formulas above, we compute

$$\det\begin{pmatrix}(\alpha-1)\Delta_1 & 4t\alpha^2 & \alpha(\alpha+1)\Delta_1\\ (\alpha+1)\Delta_2 & 4t\alpha^2 & -\alpha(\alpha-1)\Delta_2\\ -t^2 & 0 & 1\end{pmatrix}=0,$$

$$\det\begin{pmatrix}1 & 0 & t^2\alpha^2\\ -(\alpha+1)\Delta_1 & 2t\alpha\left(\alpha^2-1\right) & \alpha(\alpha-1)\Delta_1\\ (\alpha-1)\Delta_2 & 2t\alpha\left(\alpha^2-1\right) & \alpha(\alpha+1)\Delta_2\end{pmatrix}=0$$

and

$$\det\begin{pmatrix}\left(\alpha^2-1\right)(\alpha-1)\Delta_2 & -8t\alpha^3 & \alpha\left(\alpha^2-1\right)(\alpha+1)\Delta_2\\ \left(\alpha^2-1\right)(\alpha+1)\Delta_1 & 8t\alpha^3 & -\alpha\left(\alpha^2-1\right)(\alpha-1)\Delta_1\\ 1 & 0 & t^2\alpha^2\end{pmatrix}$$
$$=0.$$ □

Theorem 21 (Null focal line) *The focal line R_1 of a point p_0 on the parabola $\mathcal{P}_0$ is a null line precisely when $\Delta_3=0$. Similarly, the focal line R_2 is a null line precisely when $\Delta_4=0$.*

Proof. By the Null points/lines theorem, the focal line $R_1=\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle$ of $p_0=\left[t^2:t:1\right]$ is a null line precisely when

$$\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle\,\mathbf{D}\,\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle^T=0$$

or

$$-\alpha^2\left(\alpha+t^2\alpha^2-t^2\alpha+1\right)^2=0.$$

Since $\alpha\neq 0$, this is equivalent to $\Delta_3=0$. Similarly $R_2=\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle$ is a null line precisely when

$$-\alpha^2\left(-\alpha+t^2\alpha^2+t^2\alpha+1\right)^2=0$$

or $\Delta_4=0$. □

4 Parallels between the Euclidean and hyperbolic parabolas

4.1 Some congruent triangles

Recall that the focal line $R_1\equiv p_0 f_1$ meets the directrix F_1 in the point t_1. We will assume that the focal lines R_1 and R_2 are non-null line so that we have $\Delta_3\neq 0$ and $\Delta_4\neq 0$.

Theorem 22 (Congruent triangles) *Suppose that the tangent P^0 to $\mathcal{P}_0$ at p_0 meets $S_2=t_1 f_2$ at the point m^1. Then the triangles $\overline{p_0 t_1 m^1}$ and $\overline{p_0 f_2 m^1}$ are congruent. In particular i) $q(p_0,t_1)=q(p_0,f_2)$; ii) $q\left(t_1,m^1\right)=q\left(m^1,f_2\right)$; iii) $S_2\perp P^0$; iv) the tangent P^0 is a bisector of the vertex $\overline{R_1R_2}$; v) $S(S_2,R_1)=S(S_2,R_2)$; and vi) the tangent P^0 is a midline of the side $\overline{t_1 f_2}$. The same statements are true by f_1-f_2 symmetry if we interchange the indices 1 and 2.*

Proof. i) The first statement $q(p_0,t_1)=q(p_0,f_2)$ comes from the definition of the parabola $\mathcal{P}_0$, and we can also calculate quadrances to obtain

$$\begin{aligned}q(p_0,t_1)&=q\left(\left[t^2:t:1\right],\left[(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1\right]\right)\\&=\frac{\Delta_4^2}{\Delta_4^2-\Delta_3^2}=q\left(\left[t^2:t:1\right],[1-\alpha:0:\alpha(\alpha+1)]\right)\\&=q(p_0,f_2).\end{aligned}$$

ii) Calculate

$$\begin{aligned}m^1&=P^0S_2\\&=\left\langle 1:-2t:t^2\right\rangle\times\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle\\&=\left[t^2(\alpha-1)^2\Delta_4:-2t\alpha\Delta_4:-(\alpha+1)^2\Delta_4\right]\\&=\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right].\end{aligned}$$

Here we have used the fact that the focal line R_2 is non-null so that Δ_4 is nonzero. Thus

$$\begin{aligned}&q\left(t_1,m^1\right)=\\&q\left(\left[(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1\right],\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right]\right)\\&=-\frac{1}{4}\frac{\left(\alpha^2-1\right)\Delta_4}{\alpha\Delta_3}\\&=q\left(\left[-t^2(\alpha-1)^2:2t\alpha:(\alpha+1)^2\right],[1-\alpha:0:\alpha(\alpha+1)]\right)\\&=q\left(m^1,f_2\right).\end{aligned}$$

iii) Since the tangent line P^0 passes through s_2, which is the dual of the line $S_2=t_1 f_2$, the tangent P^0 is perpendicular to the line S_2; and we can also check that

$$\left\langle 1:-2t:t^2\right\rangle\mathbf{D}\left\langle 2t\alpha^2(\alpha+1):\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle^T=0.$$

iv) The tangent P^0 is a bisector of the vertex $\overline{R_1R_2}$ since

$$\begin{aligned}S\left(R_1,P^0\right)&=S\left(\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle,\left\langle 1:-2t:t^2\right\rangle\right)\\&=\frac{\left(\alpha^2-1\right)\left(\Delta_3^2-\Delta_4^2\right)}{4\alpha\Delta_4\Delta_3}\\&=S\left(\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle,\left\langle 1:-2t:t^2\right\rangle\right)\\&=S\left(R_2,P^0\right).\end{aligned}$$

v) Now calculate the spreads

$$\begin{aligned}S(S_2,R_1)&=S\left(\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle,\right.\\&\qquad\left.\langle t\alpha(\alpha-1):\Delta_1:-t(\alpha+1)\rangle\right)\\&=\frac{4t^2\alpha\left(\alpha^2+1\right)^2}{16t^2\alpha^3-\Delta_1^2\left(\alpha^2-1\right)}\\&=S\left(\left\langle 2t\alpha^2(\alpha+1):-\left(\alpha^2-1\right)\Delta_1:2t\alpha(\alpha-1)\right\rangle,\right.\\&\qquad\left.\langle t\alpha(\alpha+1):-\Delta_2:t(\alpha-1)\rangle\right)\\&=S(S_2,R_2).\end{aligned}$$

vi) It is obvious that the tangent P^0 is a midline of the side $\overline{t_1 f_2}$, since P^0 is perpendicular to the line $S_2 = t_1 f_2$ through the point m^1 which is, from ii), a midpoint of $\overline{t_1 f_2}$. The symmetry between f_1 and f_2 in the definition of the parabola $\mathcal{P}_0$ ensures that all these statements hold also if we interchange the indices $1, 2$. $\square$

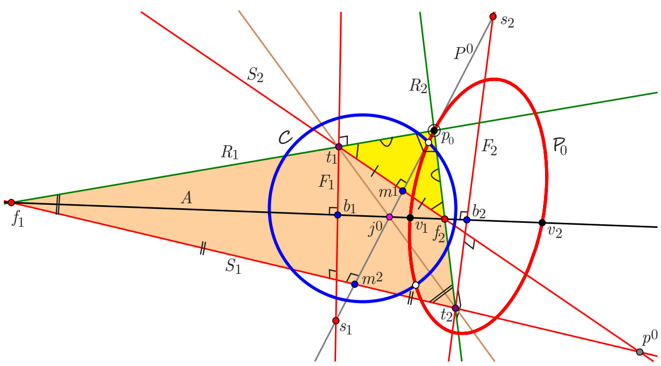

Figure 14: *Two pairs of congruent triangles*

In Figure14 we see also the point $m^2 = P_0 S_1$ and the congruent triangles $\overline{p_0 t_2 m^2}$ and $\overline{p_0 f_1 m^2}$. We call m^1 and m^2 the **t-perpendicular points** of p_0. Note that the theorem allows us a simple construction of the tangent P^0 at p_0: drop the perpendicular to the line $t_1 f_2$.

Corollary 1 *We have i) the triangles $\overline{m^1 t_1 j^0}$ and $\overline{m^1 f_2 j^0}$ are congruent, and ii) the triangles $\overline{p_0 t_1 j^0}$ and $\overline{p_0 f_2 j^0}$ are congruent. The same statements are true by $f_1 - f_2$ symmetry if we interchange the indices 1 and 2.*

Proof. The triangles $\overline{m^1 f_2 j^0}$ and $\overline{m^1 t_1 j^0}$ are right triangles since P^0 is perpendicular to S_2; we also have $q(t_1, m^1) = q(m^1, f_2)$ and $\overline{m^1 j^0}$ is a common side.
i) We calculate the quadrances

$$
\begin{aligned}
q(t_1, j^0) &= q\left(\left[(\alpha-1)\Delta_1 : 4t\alpha^2 : \alpha(\alpha+1)\Delta_1\right], \left[-t^2 : 0 : 1\right]\right) \\
&= \frac{\Delta_4^2}{\Delta_4^2 - \Delta_1^2} \\
&= q\left([1-\alpha : 0 : \alpha(\alpha+1)], \left[-t^2 : 0 : 1\right]\right) = q\left(j^0, f_2\right).
\end{aligned}
$$

and spreads

$$
\begin{aligned}
S\left(t_1 m^1, t_1 j^0\right) &= \frac{q\left(m^1, j^0\right)}{q\left(t_1, j^0\right)} = \frac{16t^2\alpha^3}{16t^2\alpha^3 - \Delta_1^2\left(\alpha^2-1\right)} \\
&= \frac{q\left(m^1, j^0\right)}{q\left(j^0, f_2\right)} = S\left(f_2 j^0, f_2 m^1\right).
\end{aligned}
$$

$$
\begin{aligned}
S\left(j^0 m^1, j^0 t_1\right) &= \frac{q\left(m^1, t_1\right)}{q\left(t_1, j^0\right)} = \frac{\left(\alpha^2-1\right)\Delta_1^2}{16t^2\alpha^3 - \Delta_1^2\left(\alpha^2-1\right)} \\
&= \frac{q\left(m^1, f_2\right)}{q\left(j^0, f_2\right)} = S\left(j^0 m^1, j^0 f_2\right).
\end{aligned}
$$

Therefore, the triangles $\overline{m^1 t_1 j^0}$ and $\overline{m^1 f_2 j^0}$ are congruent.
ii) The triangles $\overline{p_0 f_2 j^0}$ and $\overline{p_0 t_1 j^0}$ have one common side $\overline{p_0 j^0}$. Using the Spread law and the congruences above,

$$
\begin{aligned}
S\left(t_1 p_0, t_1 j^0\right) &= \frac{S\left(p_0 t_1, p_0 j^0\right) q\left(p_0, j^0\right)}{q\left(t_1, j^0\right)} \\
&= \frac{S\left(p_0 f_2, p_0 j^0\right) q\left(p_0, j^0\right)}{q\left(f_2, j^0\right)} = \frac{\Delta_1^2 - \Delta_3^2}{\Delta_4^2} \\
&= S\left(f_2 p_0, f_2 j^0\right).
\end{aligned}
$$

Therefore, the triangles $\overline{p_0 f_2 j^0}$ and $\overline{p_0 t_1 j^0}$ are congruent. $\square$

Theorem 23 (Tangent base symmetry) *Let $j^0 \equiv AP^0$ be the meet of the axis A and the tangent P^0, and h_0 the base of the altitude from p_0 to A. Then i) $q\left(b_1, j^0\right) = q\left(f_2, h_0\right)$ and ii) $q\left(v_1, j^0\right) = q\left(v_1, h_0\right)$. The same statements are true if we interchange the indices 1 and 2 by $f_1 - f_2$ symmetry.*

Proof. i) We calculate the quadrances

$$
\begin{aligned}
q\left(b_1, j^0\right) &= q\left(\left[\alpha(\alpha-1) : 0 : \alpha^2(\alpha+1)\right], \left[-t^2 : 0 : 1\right]\right) \\
&= \frac{\Delta_2^2}{\Delta_2^2 - \Delta_3^2} = q\left([1-\alpha : 0 : \alpha(\alpha+1)], \left[t^2 : 0 : 1\right]\right) \\
&= q\left(f_2, h_0\right).
\end{aligned}
$$

ii) Similarly, we calculate the quadrances

$$
\begin{aligned}
q\left(v_1, j^0\right) &= q\left([0:0:1], \left[-t^2 : 0 : 1\right]\right) = \frac{t^4\alpha^2}{t^4\alpha^2 - 1} \\
&= q\left([0:0:1], \left[t^2 : 0 : 1\right]\right) = q\left(v_1, h_0\right).
\end{aligned}
$$

$\square$

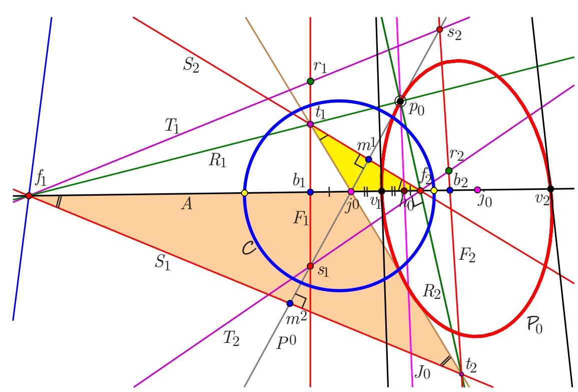

Figure 15: *The j^0 and h_0 points*

Theorem 24 (Two chord midpoints) *Let $p_0 \equiv p(t)$, $q_0 \equiv p(u)$ be two points on a hyperbolic parabola $\mathcal{P}_0$, with $\overline{p_0}$ the opposite point of p_0 with respect to the axis A. Suppose that the chords $\overline{p_0 q_0}$ and $\overline{q_0 \overline{p_0}}$ meet A at x and y respectively. Then the vertices v_1, v_2 of $\mathcal{P}_0$ are the midpoints of the side $\overline{xy}$.*

Proof. Suppose $p_0 = \left[t^2 : t : 1\right]$ and $q_0 = \left[u^2 : u : 1\right]$. The line $p_0q_0 = \langle 1 : -(t+u) : tu \rangle$ meets the axis $A = [0:1:0]$ at $x = [-tu : 0 : 1]$. The chord $\overline{p_0}q_0 = \langle 1 : t-u : -tu \rangle$ intersects $A = \langle 0:1:0 \rangle$ at $y = [tu : 0 : 1]$. Thus

$$q(v_1,x) = q([0:0:1],[-tu:0:1]) = \frac{\alpha^2 t^2 u^2}{(t^2u^2\alpha^2-1)}$$
$$= q([0:0:1],[tu:0:1]) = q(v_1,y)$$

which shows v_1 is a midpoint of the side $\overline{xy}$. The other midpoint will be perpendicular to v_1, which must be v_2 without calculation. □

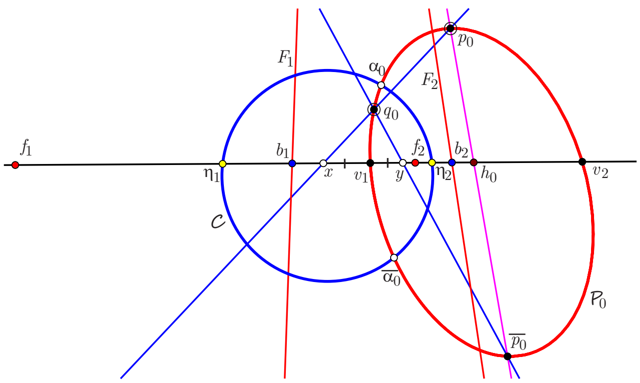

Figure 16: *Two chord midpoints*

4.2 The optical property

Recall the famous *optical property* of a parabola $\mathcal{P}$ in Euclidean geometry: if P is a point lying on $\mathcal{P}$, and light emanates from the focus F heading towards the point P, then the light will be reflected to be parallel to the axis. An analogous result in the hyperbolic case is the statement iv) of the Congruent triangles theorem: that the tangent line P_0 to a point p_0 is a biline (bisector) of the vertex $\overline{R_1R_2}$. So reflecting the focal line $R_1 \equiv f_1p_0$ in the tangent P^0 results in the other focal line R_2, which is perpendicular to the directrix F_2.

Recall from [16] that in Universal Hyperbolic Geometry there is an important notion of parallelism, which is quite different from the usage in classical hyperbolic geometry. We say rather generally that the **parallel line** P **through a point** a **to a line** L is the line through a perpendicular to the altitude from a to L.

Now recall that given a point p_0 on the hyperbolic parabola $\mathcal{P}_0$, the perpendicular to the axis A through p_0 is $J_0 \equiv (j_0)^{\perp} = ap_0 = \left\langle 1:0:-t^2 \right\rangle$ with dual point $j_0 = P_0A = \left[1:0:t^2\alpha^2\right]$. Therefore, the parallel line to the axis A through the point p_0 is

$$L_0 = j_0p_0 = \left\langle -t^3\alpha^2 : t^4\alpha^2 - 1 : t \right\rangle.$$

Here then is another analog of the optical property, dealing with the relationship between two spreads formed by the tangent line P_0. Recall that the quadrance of the parabola was defined as $q_{\mathcal{P}_0} \equiv q(f_1, f_2)$.

Theorem 25 (Parallel line spread relation) *Let p_0 be a point on the hyperbolic parabola $\mathcal{P}_0$. If $\widehat{T}$ is the spread between the tangent line P^0 at p_0 and the parallel line L_0 to the axis through p_0, and $\widehat{S}$ is the common spread between the tangent P^0 and the lines R_1 and R_2, then*

$$\frac{(\widehat{S}-\widehat{T})\widehat{S}}{1-\widehat{T}} = 1 - q_{\mathcal{P}_0}.$$

Proof. Using the Spread formula, we compute that

$$\widehat{S} = S\left(R_1, P^0\right) = \frac{\left(\alpha^2-1\right)\left(\Delta_3^2-\Delta_4^2\right)}{4\alpha\Delta_4\Delta_3}$$

and

$$\widehat{T} = S\left(L_0, P^0\right) = \frac{-\left(\alpha^2-1\right)\left(t^4\alpha^2+1\right)^2}{\left(t^4\alpha^2-1\right)\Delta_3\Delta_4}.$$

So now

$$\frac{(\widehat{S}-\widehat{T})\widehat{S}}{1-\widehat{T}} = -\frac{1}{4}\frac{\left(\alpha^2-1\right)^2}{\alpha^2} = 1 - q_{\mathcal{P}_0}.$$ □

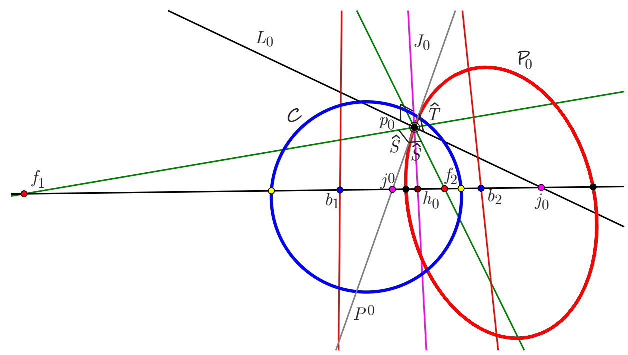

Figure 17: *The parallel line spread relation*

Note that $1 - q_{\mathcal{P}_0} = q(b_1, f_2)$ since b_1 and f_1 are perpendicular points. So in the limiting Euclidean case when b_1 is very close to f_2, it follows that $\widehat{S}$ is very close to $\widehat{T}$.

4.3 The s points and $\mathcal{S}$ circles

Recall that $s_1 \equiv F_1P^0$ and $s_2 \equiv F_2P^0$.

Theorem 26 (The $\mathcal{S}_1$ and $\mathcal{S}_2$ circles) *The circle $\mathcal{S}_1$ with center s_1 passing through f_2 also passes through t_1, and is tangent at these points to R_2 and R_1 respectively. In particular i) $q(s_1,t_1) = q(s_1, f_2)$; ii) $R_1 \perp F_1$; iii) $R_2 \perp T_2$ and iv) $S(s_1t_1,t_1f_2) = S(s_1f_2, f_2t_1)$. The same statements are true if we interchange the indices 1 and 2, giving also a circle $\mathcal{S}_2$ with center s_2.*

Proof. i) Calculate

$$q(s_1,t_1) = q\left([2t(\alpha-1):\Delta_2:2t\alpha(\alpha+1)],\left[(\alpha-1)\Delta_1:4t\alpha^2:\alpha(\alpha+1)\Delta_1\right]\right) = \frac{(\alpha^2-1)\Delta_4^2}{16t^2\alpha^3+\Delta_2^2(\alpha^2-1)} = q(s_1,f_2).$$

ii) The line $R_1 = f_1p_0$ is clearly perpendicular to the directrix F_1 since it passes through the focus $f_1 = F_1^{\perp}$.

iii) Since $t_2 = R_2F_2$, $S_1 = f_1t_2$, the lines R_2, F_2 and S_1 are concurrent at t_2, so the line $T_2 = t_2^{\perp}$ passes through r_2, f_2 and s_1. Therefore T_2 is perpendicular to the line R_2.

iv) Calculate

$$S(t_1s_1,t_1f_2) = S(\langle\alpha(\alpha+1):0:1-\alpha\rangle,\left\langle 2t\alpha^2(\alpha+1):-(\alpha^2-1)\Delta_1:2t\alpha(\alpha-1)\right\rangle) = \frac{(\alpha^2-1)\Delta_3^2}{16t^2\alpha^3-\Delta_1^2(\alpha^2-1)} = S(f_2s_1,f_2t_1).$$

□

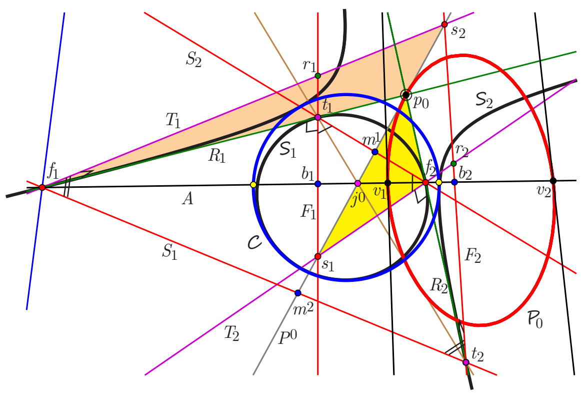

Figure 18: *The $\mathcal{S}_1$ and $\mathcal{S}_2$ circles*

In particular, property iii) provides us with an important alternate construction of the tangent P^0 to the parabola $\mathcal{P}_0$ at p_0 : namely we construct the altitude T_2 to p_0f_2 through f_2, and obtain $s_1 = F_1T_2$, giving $P^0 = p_0s_1$ (or similarly construct p_0s_2). In Figure 18 we see the circles $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that $\mathcal{S}_2$ looks like a hyperbola tangent to the null circle, in fact it is tangent at exactly the points where S_2 meets the null circle $\mathcal{C}$ – see the discussion in [18].

4.4 Focal chords and conjugates

A chord $\overline{p_0q_0}$ is a **focal chord** precisely when p_0q_0 passes through a focus. Such chords play an important role both in the Euclidean and the hyperbolic theory.

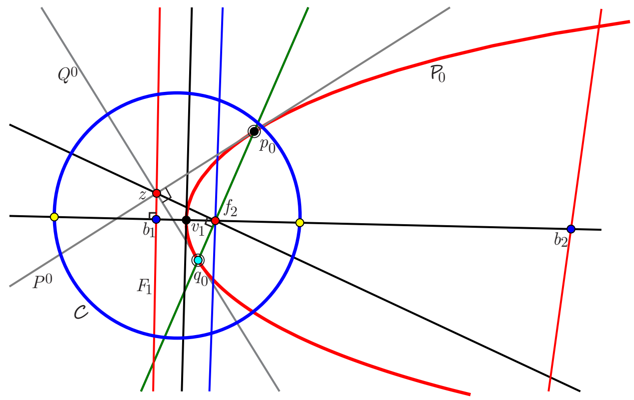

Figure 19: *A focal chord $\overline{p_0q_0}$ with polar point z on directrix*

Theorem 27 (Focal tangents perpendicularity) *If $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ are two points on $\mathcal{P}_0$ then $\overline{p_0q_0}$ is a focal chord precisely when the respective tangents P^0 and Q^0 are perpendicular; and precisely when the polar point $z \equiv P^0Q^0$ lies on a directrix.*

Proof. Suppose $p_0 = \left[t^2:t:1\right]$ and $q_0 = \left[u^2:u:1\right]$ lie on $\mathcal{P}_0$. Then $p_0q_0 = \langle 1:-(t+u):tu\rangle$ is a focal line precisely when it passes through either f_1 of f_2, in other words precisely when

$$(1:-(t+u):tu)[\alpha+1:0:\alpha(\alpha-1)]^T = \alpha+1+tu\alpha(\alpha-1)=0 \qquad \text{or}$$
$$(1:-(t+u):tu)[1-\alpha:0:\alpha(\alpha+1)]^T = -\alpha+1+tu\alpha(\alpha+1)=0.$$

On the other hand the tangents $P^0 = \left\langle 1:-2t:t^2\right\rangle$ and $Q^0 = \left\langle 1:-2u:u^2\right\rangle$ are perpendicular precisely when

$$0 = \left\langle 1:-2t:t^2\right\rangle \mathbf{D} \left\langle 1:-2u:u^2\right\rangle^T = \alpha^2-4tu\alpha^2-t^2u^2\alpha^2\left(\alpha^2-1\right)-1 = (\alpha+1+tu\alpha(\alpha-1))(\alpha-1-tu\alpha(\alpha+1)).$$

Thus the two conditions are equivalent.
As in the Tangent meets theorem, the tangents P^0 and Q^0 meet at $z = [2tu:t+u:2]$. This point lies on $F_1 = \langle\alpha(\alpha+1):0:1-\alpha\rangle$ or $F_2 = \langle\alpha(\alpha-1):0:\alpha+1\rangle$ precisely when

$$[2tu:t+u:2](\alpha(\alpha+1):0:1-\alpha)^T = 2\left(-\alpha+tu\alpha+tu\alpha^2+1\right)=0 \qquad \text{or}$$
$$[2tu:t+u:2](\alpha(\alpha-1):0:\alpha+1)^T = 2\left(\alpha-tu\alpha+tu\alpha^2+1\right)=0.$$

Since we work over a field not of characteristic two, the conditions are equivalent to the previous ones. □

Given a point p_0 on the parabola $\mathcal{P}_0$, we define the **conjugate points** n_1, n_2 as the second meets of the focal lines R_1 and R_2 with the parabola $\mathcal{P}_0$ respectively. Since one meet is known, solving the quadratic equations is straightforward and yields

$$n_1 = \left[(\alpha+1)^2 : t\alpha\left(1-\alpha^2\right) : t^2\alpha^2(\alpha-1)^2\right]$$
$$n_2 = \left[(\alpha-1)^2 : t\alpha\left(\alpha^2-1\right) : t^2\alpha^2(\alpha+1)^2\right]. \qquad (14)$$

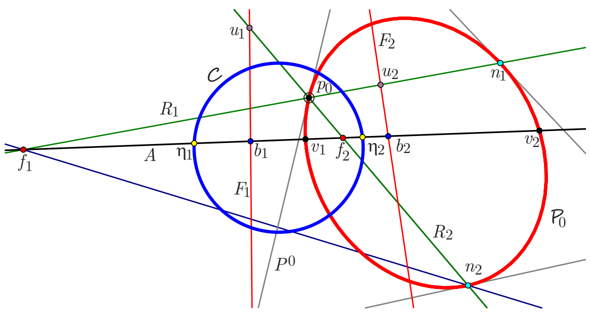

Figure 20: *Focal conjugates n_1 and n_2*

The dual lines are the **conjugate lines;**

$$N_1 \equiv n_1^{\perp} = \left\langle \alpha(\alpha+1)^2 : t\left(\alpha^2-1\right)^2 : -t^2\alpha(\alpha-1)^2\right\rangle$$
$$N_2 \equiv n_2^{\perp} = \left\langle -\alpha(\alpha-1)^2 : t\left(\alpha^2-1\right)^2 : t^2\alpha(\alpha+1)^2\right\rangle.$$

Theorem 28 (Conjugate points parameter) *Let $p_0 \equiv p(t)$ be a point on the parabola $\mathcal{P}_0$, then the point $p(u)$ is the conjugate point n_1 of p_0 with respect to the focus f_1 precisely when $u = -\frac{\alpha+1}{\alpha t(\alpha-1)}$, while $p(u)$ is the conjugate point n_2 of p_0 with respect to the focus f_2 precisely when $u = \frac{\alpha-1}{\alpha t(\alpha+1)}$.*

Proof. Let $p_0 = \left[t^2 : t : 1\right]$ and $p(u) = \left[u^2 : u : 1\right]$ lie on $\mathcal{P}_0$. Then, the line $p_0q_0 = \langle 1 : -(t+u) : tu\rangle$ is a focal line with respect to the focus f_1 when it passes through the focus f_1 and then we have

$$[1 : -(t+u) : tu]\,[\alpha+1 : 0 : \alpha(\alpha-1)]^T = 0 \qquad \text{so that}$$
$$\alpha - tu\alpha + tu\alpha^2 + 1 = 0.$$

This gives the condition $u = -\frac{\alpha+1}{\alpha t(\alpha-1)}$. Similarly, the other direction is straightforward.
When the line $p_0q_0 = \langle 1 : -(t+u) : tu\rangle$ is a focal line with respect to the focus f_2, then the focal line passes through the focus f_2 and we have

$$[1 : -(t+u) : tu]\,[1-\alpha : 0 : \alpha(\alpha+1)]^T = 0 \qquad \text{so that}$$
$$-\alpha + tu\alpha + tu\alpha^2 + 1 = 0.$$

This gives the condition $u = \frac{\alpha-1}{\alpha t(\alpha+1)}$. Similarly, the other direction is straightforward. $\square$

4.5 Quadrance cross ratios

Theorem 29 (Quadrance cross ratio) *Suppose that a, b, c, d are a harmonic range of points on a line L in UHG. Then*

$$\frac{q(a,c)}{q(a,d)} = \frac{q(b,c)}{q(b,d)}.$$

Proof. We know from projective geometry that a harmonic range of points a, b, c, d in the projective space can be realized as $[v], [u], [\alpha v + \beta u], [\alpha v - \beta u]$ for two vectors v and u and two scalars α and β. Then using the short hand notation $v^2 \equiv v \cdot v$ and $uv = u \cdot v$, we calculate that

$$q([v],[\alpha v+\beta u]) = 1 - \frac{(v\cdot(\alpha v+\beta u))^2}{(v\cdot v)\,((\alpha v+\beta u)\cdot(\alpha v+\beta u))}$$
$$= \frac{v^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)-\left(\alpha v^2+\beta uv\right)^2}{v^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)}$$
$$= \frac{\beta^2\left(u^2v^2-(uv)^2\right)}{v^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)}$$

and similarly

$$q([u],[\alpha v+\beta u]) = 1 - \frac{(u\cdot(\alpha v+\beta u))^2}{(u\cdot u)\,((\alpha v+\beta u)\cdot(\alpha v+\beta u))}$$
$$= \frac{u^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)-\left(\alpha\,(uv)+\beta u^2\right)^2}{u^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)}$$
$$= \frac{\alpha^2\left(u^2v^2-(uv)^2\right)}{u^2\left(\alpha^2v^2+2\alpha\beta\,(uv)+\beta^2u^2\right)}.$$

It follows that

$$\frac{q(a,c)}{q(b,c)} = \frac{q([v],[\alpha v+\beta u])}{q([u],[\alpha v+\beta u])} = \frac{\beta^2u^2}{\alpha^2v^2}.$$

But this quantity is then unchanged if we replaced α with $-\alpha$, or β with $-\beta$. $\square$

Theorem 30 (Conjugate cross ratios) *Let p_0 be a point on the parabola $\mathcal{P}_0$, with n_1 and n_2 the focal conjugates and u_1 and u_2 the meets of R_2 and R_1 with the directrices F_1 and F_2 respectively. Then*

$$\frac{q(p_0,f_1)}{q(f_1,n_1)} = \frac{q(p_0,u_2)}{q(u_2,n_1)} \quad \text{and} \quad \frac{q(p_0,f_2)}{q(f_2,n_2)} = \frac{q(p_0,u_1)}{q(u_1,n_2)}.$$

Proof. From the Focus directrix polarity theorem, we know that f_2 and F_1 are a pole-polar pair with respect to the parabola $\mathcal{P}_0$. Hence $f_1, u_2; p_0, n_1$ is a harmonic range. From the previous theorem, that implies that

$$\frac{q(p_0,f_1)}{q(f_1,n_1)} = \frac{q(p_0,u_2)}{q(u_2,n_1)}.$$

The other relation follows similarly since $f_2, u_1; p_0, n_2$ is also a harmonic range of points. $\square$

4.6 Spreads related to chords of a parabola

Theorem 31 (Polar point spreads) *If the tangents P^0 and Q^0 at the points $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ lying on the parabola $\mathcal{P}_0$ meet at the polar point z, then $S(f_1p_0, f_1z) = S(f_1q_0, f_1z)$ and $S(f_2p_0, f_2z) = S(f_2q_0, f_2z)$.*

Proof. Suppose that $p_0 \equiv \left[t^2 : t : 1\right]$ and $q_0 \equiv \left[u^2 : u : 1\right]$ are on the parabola $\mathcal{P}_0$. Then $z = [2tu : t+u : 2]$ and we calculate

$$S(f_1p_0, f_1z) = \frac{\alpha(t-u)^2\left(\alpha^2-1\right)}{\left(\alpha+\alpha^2t^2-\alpha t^2+1\right)\left(\alpha+\alpha^2u^2-\alpha u^2+1\right)} = \frac{\alpha(t-u)^2\left(\alpha^2-1\right)}{\Delta_3(t)\Delta_3(u)} = S(f_1q_0, f_1z)$$

and

$$S(f_2p_0, f_2z) = \frac{\alpha\left(\alpha^2-1\right)(t-u)^2}{\left(\alpha-u^2\alpha^2-u^2\alpha-1\right)\left(-\alpha+t^2\alpha^2+t^2\alpha+1\right)} = \frac{-\alpha\left(\alpha^2-1\right)(t-u)^2}{\Delta_4(t)\Delta_4(u)} = S(f_2q_0, f_2z).$$

□

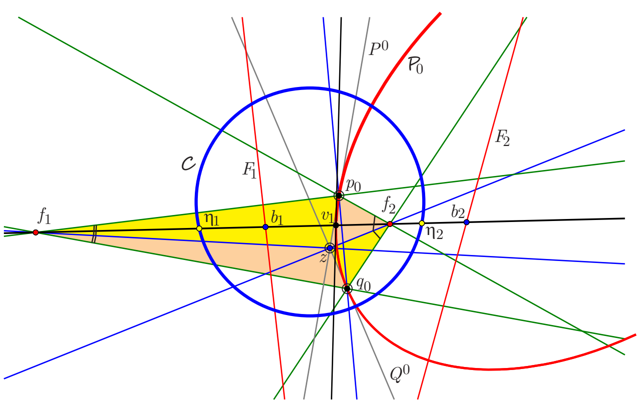

Figure 21: *The polar point z of the chord $\overline{p_0q_0}$*

Theorem 32 (Chord directrix meets) *Let $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ be two points on a parabola $\mathcal{P}_0$. Let z be the polar point of the chord $\overline{p_0q_0}$, and $x_1 \equiv F_1(p_0q_0)$ and $x_2 \equiv F_2(p_0q_0)$. Then i) $f_1z \perp f_1x_2$, ii) $f_2z \perp f_2x_1$ and iii) $S(x_1z, zf_2) = S(x_2z, zf_1)$.*

Proof. We suppose as usual that $p_0 = \left[t^2 : t : 1\right]$ and $q_0 = \left[u^2 : u : 1\right]$. Then
i) We compute that

$$\begin{aligned} x_2 &\equiv F_2(p_0q_0) \\ &= \left\langle \alpha^2(\alpha-1) : 0 : \alpha(1+\alpha)\right\rangle \times \langle 1 : -(t+u) : tu\rangle \\ &= \left[(\alpha+1)(t+u) : \alpha+tu\alpha-tu\alpha^2+1 : -\alpha(\alpha-1)(t+u)\right]. \end{aligned}$$

Also

$$\begin{aligned} f_1z &= \left\langle -\alpha(\alpha-1)(t+v) : 2\left(-\alpha-tv\alpha+tv\alpha^2-1\right)\right. \\ &\qquad \left. : (\alpha+1)(t+v)\right\rangle \\ f_1x_2 &= \left\langle \alpha(\alpha-1)\left(-\alpha-tv\alpha+tv\alpha^2-1\right) : 2\alpha\left(\alpha^2-1\right)(t+v)\right. \\ &\qquad \left. : (\alpha+1)\left(\alpha+tv\alpha-tv\alpha^2+1\right)\right\rangle \end{aligned}$$

and so we may verify that

$$\begin{aligned} 0 = &\left\langle -\alpha(\alpha-1)(t+v) : 2\left(-\alpha-tv\alpha+tv\alpha^2-1\right)\right. \\ &\quad \left. : (\alpha+1)(t+v)\right\rangle D\times \\ &\left\langle \alpha(\alpha-1)\left(-\alpha-tv\alpha+tv\alpha^2-1\right) : 2\alpha\left(\alpha^2-1\right)(t+v)\right. \\ &\quad \left. : (\alpha+1)\left(\alpha+tv\alpha-tv\alpha^2+1\right)\right\rangle^T. \end{aligned}$$

Thus

$$S(f_1z, f_1x_2) = 1.$$

ii) Similarly

$$\begin{aligned} x_1 &\equiv F_1(p_0q_0) = \langle \alpha(\alpha+1) : 0 : 1-\alpha\rangle \times \langle 1 : -(t+u) : tu\rangle \\ &= \left[(\alpha-1)(t+u) : \alpha+tu\alpha+tu\alpha^2-1 : \alpha(\alpha+1)(t+u)\right] \end{aligned}$$

and the lines

$$\begin{aligned} f_2z &= \left\langle -\alpha(\alpha+1)(t+u) : 2\left(\alpha+tu\alpha+tu\alpha^2-1\right)\right. \\ &\qquad \left. : -(\alpha-1)(t+u)\right\rangle \\ f_2x_1 &= \left\langle \alpha(\alpha+1)\left(\alpha+tu\alpha+tu\alpha^2-1\right) : -2\alpha\left(\alpha^2-1\right)(t+u)\right. \\ &\qquad \left. : (\alpha-1)\left(\alpha+tu\alpha+tu\alpha^2-1\right)\right\rangle \end{aligned}$$

are perpendicular, so that

$$S(f_2z, f_2x_1) = 1.$$

iii) Another calculation shows that

$$S(x_1z, zf_2) = \frac{1}{4}\frac{\left(\alpha^2-1\right)\left((2tu)^2-(t+u)^2\right)\alpha^2+(t+u)^2-4}{\left(t^2u^2\right)\alpha^4+\left((t+u)^2-\left(t^2u^2+1\right)\right)\alpha^2+1} = S(x_2z, zf_1).$$

□

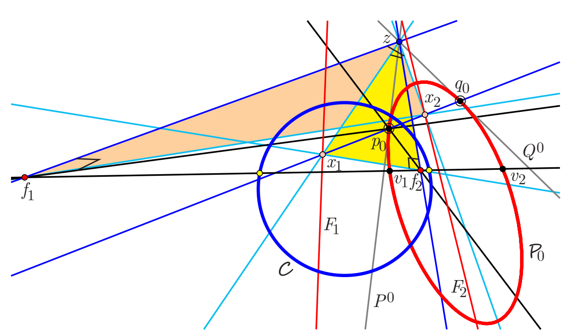

Figure 22: *Chord directrix meets x_1 and x_2*

In Figure 22 we see the two triangles $\overline{f_1zx_2}$ and $\overline{f_2zx_1}$, which are both right triangles sharing a common spread.

Theorem 33 (Tangent directrix meets) *If the two tangents P^0 and Q^0 to a parabola $\mathcal{P}_0$ at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ respectively meet the directrix F_1 at s_1 and s_1' respectively, and meet F_2 at s_2 and s_2' respectively, then $S(f_1p_0, f_1q_0) = S(f_1s_2, f_1s_2')$ and $S(f_2p_0, f_2q_0) = S(f_2s_1, f_2s_1')$.*

Proof. Suppose that $p_0 \equiv \left[t^2 : t : 1\right]$ and $q_0 \equiv \left[u^2 : u : 1\right]$ are on the parabola $\mathcal{P}_0$. Then

$$S(f_1p_0, f_1q_0) = \frac{4\alpha\left(\alpha^2-1\right)(t-u)^2\left(\alpha+\alpha^2tu-\alpha tu+1\right)^2}{\Delta_3^2(t)\,\Delta_3^2(u)} = S\left(f_1s_2, f_1s_2'\right).$$

Also, we have that

$$S(f_2p_0, f_2q_0) = \frac{-4\alpha\left(\alpha^2-1\right)(t-u)^2\left(-\alpha+tu\alpha+tu\alpha^2+1\right)^2}{\Delta_4^2(t)\,\Delta_4^2(u)} = S\left(f_2s_1, f_2s_1'\right). \qquad \square$$

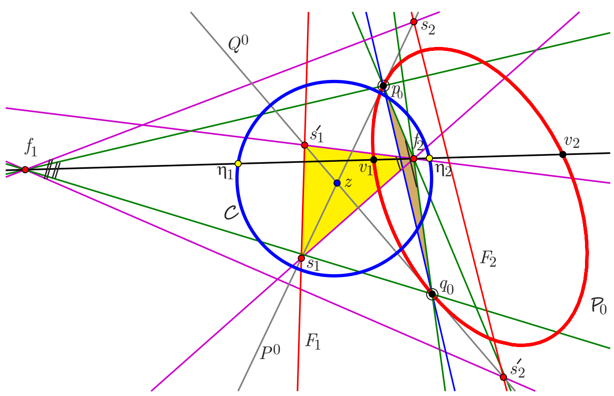

Figure 23: *Tangent directrix meets s_1 and s_2*

Recall that in universal hyperbolic geometry, a triangle may have four circumcircles.

Theorem 34 (Two tangents circumcircle) *Suppose that the two points $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on a parabola $\mathcal{P}_0$ have respective altitude base points t_1, t_2 and t_1', t_2' on F_1, F_2 respectively, and that their tangents meet at the polar point z. Then z is a circumcenter of both the triangles $\overline{t_1 f_2 t_1'}$ and $\overline{t_2 f_1 t_2'}$. In particular $q(t_1, z) = q(t_1', z) = q(z, f_2)$ and $q(t_2, z) = q(t_2', z) = q(z, f_1)$.*

Proof. Suppose that $p_0 \equiv \left[t^2 : t : 1\right]$ and $q_0 \equiv \left[u^2 : u : 1\right]$ are on the parabola $\mathcal{P}_0$, then,

$$q(z, f_2) = q([2tu : t+u : 2], [1-\alpha : 0 : \alpha(\alpha+1)]) = \frac{\Delta_4(t)\,\Delta_4(u)}{\alpha\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)} = q(z, t_1) = q(z, t_1').$$

Hence z is a circumcenter of the triangle $\overline{t_1 f_2 t_1'}$. Similarly, z is the circumcenter of the triangle $\overline{t_2 f_1 t_2'}$ since

$$q(z, f_1) = q([2tu : t+u : 2], [\alpha+1 : 0 : \alpha(\alpha-1)]) = -\frac{\Delta_3(t)\,\Delta_3(u)}{\alpha\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)} = q(z, t_2) = q(z, t_2'). \qquad \square$$

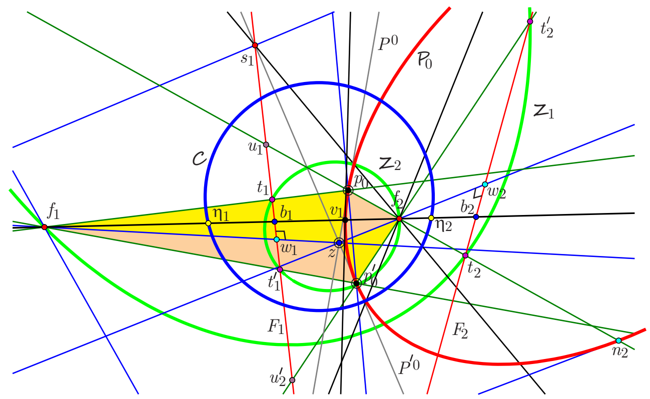

Figure 24: *Two points and polar circles*

In Figure 24 we see the polar point of $\overline{p_0 p_0'}$ together with the two **polar circles** centered at z through the foci.

Corollary 2 *If the tangents at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on $\mathcal{P}_0$ meet at z then the line f_1z is a midline of the side $\overline{t_1t_1'}$ and similarly f_2z is a midline of the side $\overline{t_2t_2'}$.*

Proof. This follows immediately from the previous theorem, since f_1z is the altitude from z to the directrix F_1, so it bisects the chord $\overline{t_1t_1'}$. $\square$

Theorem 35 (Opposite triangle spreads) *If the tangents at $p_0 \equiv p(t)$ and $q_0 \equiv p(u)$ on $\mathcal{P}_0$ meet at z, then $S(zp_0, zf_1) = S(zq_0, zf_2)$ and $S(zp_0, zf_2) = S(zq_0, zf_1)$.*

Proof. Using the Spread formula, we obtain

$$S(zp_0, zf_2) = -\frac{\left(\alpha^2-1\right)\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)}{4\Delta_4(u)\,\Delta_3(t)} = S(zq_0, zf_1)$$

and

$$S(zp_0, zf_1) = -\frac{\left(\alpha^2-1\right)\left(\left(4t^2u^2-(t+u)^2\right)\alpha^2+(t+u)^2-4\right)}{4\Delta_3(u)\,\Delta_4(t)} = S(zq_0, zf_2). \qquad \square$$

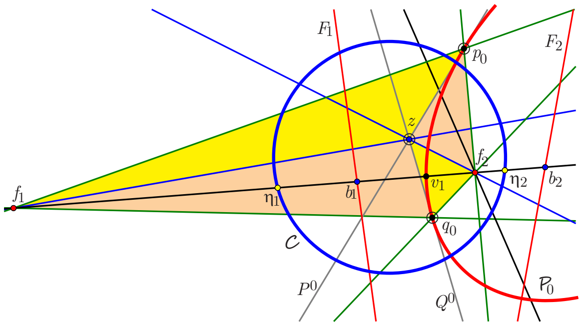

Figure 25: *Opposite triangle spreads*

5 Normals to the parabola $\mathcal{P}_0$

In the Euclidean case, it is well known that the evolute of the parabola, which is defined as the locus of the center of curvature of the curve—namely the meet of adjacent normals, as Huygens or Newton would have said—is a *semi-cubical parabola*. For the curve $y = x^2$, shown in Figure 26, the evolute has equation

$$\left(y-\frac{1}{2}\right)^3 = \frac{27}{16}x^2.$$

This formula suggests that there is no Euclidean ruler and compass construction for the center of curvature C_0 of the parabola for a general point P_0 on it. We will see that in the hyperbolic case, the situation is in some ways simpler, and indeed we will show how to give a straightedge construction for the center of curvature!

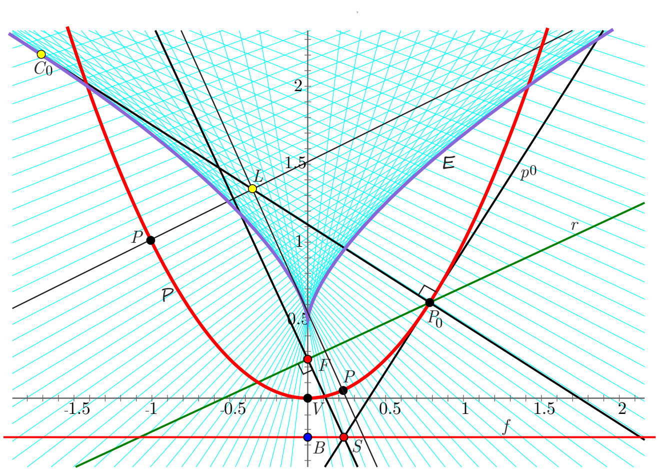

Figure 26: *Evolute of a Euclidean parabola*

In Figure 26 we see a point P_0 on the Euclidean parabola, with its tangent p^0, obtained by finding the meet S of the directrix f with the altitude to the focal line $r = FP_0$ through the focus F. The center of curvature is the point C_0 on the evolute $\mathcal{E}$. The figure shows also that for points L above the evolute, there are three normals that meet there; we exhibit also the other two points marked P whose normals also pass through L. Below the evolute only one normal passes through any fixed point.

For a point p_0 on the hyperbolic parabola $\mathcal{P}_0$, the altitude line P to the tangent P^0 through p_0 is called the **normal line** at p_0.

Since the dual of P^0 is the twin point p^0, we see that

$$\begin{aligned} P &\equiv p_0p^0 = \left[t^2:t:1\right]\times\left[\alpha^2-1:2t\alpha^2:-t^2\alpha^2\left(\alpha^2-1\right)\right] \\ &= \left\langle -t\alpha^2\left(t^2\alpha^2-t^2+2\right):\left(\alpha^2-1\right)\left(t^4\alpha^2+1\right)\right. \\ &\qquad \left. :t\left(2t^2\alpha^2-\alpha^2+1\right)\right\rangle. \end{aligned} \tag{15}$$

By symmetry, this means that P is both the normal line to the parabola $\mathcal{P}_0$ at p_0 as well as the normal line to the twin parabola $\mathcal{P}^0$ at p^0.

The meet of P and the axis A is the point

$$\begin{aligned} n &\equiv PA \\ &= \left\langle -t\alpha^2\left(t^2\alpha^2-t^2+2\right):\left(\alpha^2-1\right)\left(t^4\alpha^2+1\right)\right. \\ &\qquad \left. :t\left(2t^2\alpha^2-\alpha^2+1\right)\right\rangle\times\langle 0:1:0\rangle \\ &= \left[t\left(2t^2\alpha^2-\alpha^2+1\right):0:t\alpha^2\left(t^2\alpha^2-t^2+2\right)\right] \\ &= \left[2t^2\alpha^2-\alpha^2+1:0:\alpha^2\left(t^2\alpha^2-t^2+2\right)\right] \end{aligned}$$

provided that $t \neq 0$. Since the normal P of is perpendicular to the tangent P^0, and since P^0 is a biline of the vertex $\overline{R_1R_2}$, the normal P is the other biline for the vertex $\overline{R_1R_2}$. In fact we may calculate that

$$S(R_1,P) = S(P,R_2) = \frac{t^2\left(\alpha^2+1\right)^2}{-\Delta_3\Delta_4}.$$

5.1 Conjugate normals and conics

Recall that the conjugate points n_1, n_2 of p_0 are the second meets of the focal lines $R_1 \equiv f_1p_0$ and $R_2 \equiv f_2p_0$ with the parabola $\mathcal{P}_0$ respectively. They are given in (14). The normal lines to $\mathcal{P}_0$ at the conjugate points n_1 and n_2 can then be computed using the formula (15):

$$\begin{aligned} P_1 &\equiv \left\langle t\alpha(\alpha-1)\left(2\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)\right. \\ &\qquad :\alpha^2(\alpha-1)^4t^4+(\alpha+1)^4 \\ &\qquad \left. :-t\alpha(\alpha+1)\left(2(\alpha+1)-(\alpha-1)^3t^2\right)\right\rangle \\ P_2 &\equiv \left\langle -t\alpha(\alpha+1)\left(2\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)\right. \\ &\qquad :\alpha^2(\alpha+1)^4t^4+(\alpha-1)^4 \\ &\qquad \left. :t\alpha(\alpha-1)\left(2(\alpha-1)-(\alpha+1)^3t^2\right)\right\rangle. \end{aligned}$$

We will call these the **conjugate normal lines** of p_0.

Theorem 36 (Conjugate normal conics) *There are two conics $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following properties. Let h_1 be the meet of the normal P and the conjugate normal P_1 of a point p_0 on $\mathcal{P}_0$. Then h_1 lies on $\mathcal{H}_1$, which passes through f_2 and is tangent to B_1 there. Similarly if h_2 is the meet of P and P_2 at p_0, then h_2 lies on $\mathcal{H}_2$, which passes through f_1 and is tangent to B_2 there. Furthermore we have collinearities $[[f_1s_2h_2]]$ as well as $[[f_2s_1h_1]]$. In addition $\mathcal{H}_1$ passes through the points d_0 and $\overline{d_0}$.*

Proof. The conjugate normal P_1 will meet the normal P at

$$\begin{aligned} h_1 &\equiv PP_1 = \\ &\Big[-\alpha^2(\alpha-1)^3t^4+4\alpha^2(\alpha+1)t^2-(\alpha-1)(\alpha+1)^2 : t\alpha\left(\alpha^2+1\right)\Delta_1 \\ &: \alpha\left(\alpha^2(\alpha+1)(\alpha-1)^2t^4+4\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)\Big] \end{aligned}$$

A computation shows this point always lies on the conic $\mathcal{H}_1$ with equation

$$\begin{aligned} &\alpha^2\left(\alpha^2-1\right)\left(1+4\alpha+\alpha^2\right)x^2 \\ &+2\alpha\left(1-2\alpha-\alpha^2\right)\left(1+2\alpha-\alpha^2\right)xz \\ &+32\alpha^3y^2+\left(\alpha^2-1\right)\left(1-4\alpha+\alpha^2\right)z^2=0. \end{aligned}$$

The conjugate normal P_2 will meet the normal P at

$$\begin{aligned} h_2 &\equiv PP_2 = \\ &\Big[\alpha^2(\alpha+1)^3t^4-4\alpha^2(\alpha-1)t^2+(\alpha+1)(\alpha-1)^2 \\ &: t\alpha\left(\alpha^2+1\right)\Delta_2 \\ &: \alpha\left(\alpha^2(\alpha-1)(\alpha+1)^2t^4+4\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)\Big]. \end{aligned}$$

This point always lies on the conic $\mathcal{H}_2$ with equation

$$\begin{aligned} &\alpha^2\left(\alpha^2-1\right)\left(1-4\alpha+\alpha^2\right)x^2 \\ &-2\alpha\left(2\alpha+\alpha^2-1\right)\left(-2\alpha+\alpha^2-1\right)xz \\ &-32\alpha^3y^2+\left(\alpha^2-1\right)\left(1+4\alpha+\alpha^2\right)z^2=0. \end{aligned}$$

The collinearity $[[f_1s_1h_2]]$ is established by checking that the determinant formed by the respective vectors is indeed 0 (it is!), and similarly for the collinearity $[[f_2s_2h_1]]$. We can also check (with a computer package) that both of the points d_0 and $\overline{d_0}$ identically satisfy the equation of $\mathcal{H}_1$. $\square$

The normal P at p_0 meets the parabola $\mathcal{P}_0$ again at a second point

$$\begin{aligned} p_0' = \Big[&\left(2t^2\alpha^2-\alpha^2+1\right)^2 : t\alpha^2\left(-t^2\alpha^2+t^2-2\right)\left(2t^2\alpha^2-\alpha^2+1\right) \\ &: t^2\alpha^4\left(t^2\alpha^2-t^2+2\right)^2\Big] \end{aligned}$$

and similarly the conjugate normals P_1,P_2 at n_1,n_2 meet $\mathcal{P}_0$ respectively also at

$$\begin{aligned} n_1' = \Big[&t^2\alpha^2\left(\left(\alpha-1\right)^3t^2-2\left(\alpha+1\right)\right)^2 \\ &: t\alpha\left(2(\alpha+1)-(\alpha-1)^3t^2\right)\left(2\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right) \\ &: \left(2\alpha^2(\alpha-1)t^2+(\alpha+1)^3\right)^2\Big] \end{aligned}$$

$$\begin{aligned} n_2' = \Big[&t^2\alpha^2\left(\left(\alpha+1\right)^3t^2+2\left(1-\alpha\right)\right)^2 \\ &: \left(2\alpha^2(\alpha+1)t^2+(\alpha-1)^3\right)\left(\alpha\left(\alpha+1\right)^3t^3-2\alpha\left(\alpha-1\right)t\right) \\ &: \left(2\alpha^2\left(\alpha+1\right)t^2+\left(\alpha-1\right)^3\right)^2\Big]. \end{aligned}$$

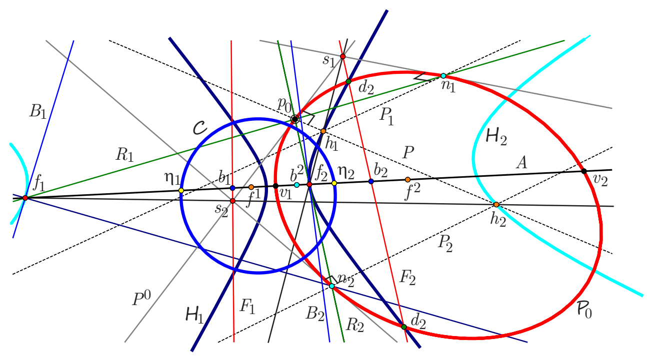

Figure 27: *Conjugate normal meets h_1 and h_2 and conics*

Theorem 37 (Normal conjugate colliearities) *Let p_0', n_1' and n_2' be the second meets of the normals and conjugate normals P, P_1 and P_2 of p_0 with the parabola $\mathcal{P}_0$ respectively, and t_1, t_2 the altitude base points of p_0. Then we have collinearities $[[p_0'n_1't_1]]$ and $[[p_0'n_2't_2]]$.*

Proof. Since the forms of all the points involved are known, it is straightforward (with a computer package) to verify that the corresponding determinants for both collinearities do evaluate identically to 0. $\square$

These collinearities are illustrated in Figure 28.

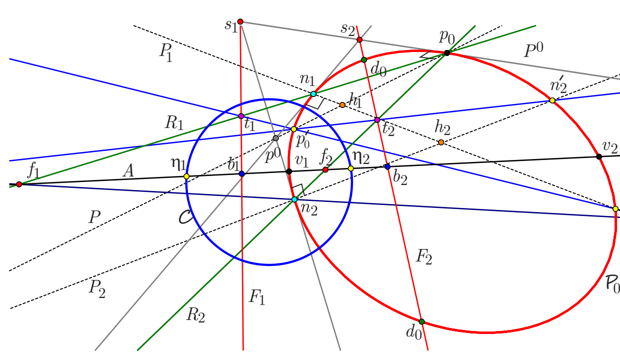

Figure 28: *Normal conjugate collinearities*

5.2 Four points with concurrent normals

In the Euclidean case, finding the three points P on the parabola whose normals pass through a given point L above the evolute is not straightforward [8]. We will show that in the hyperbolic case there is an interesting conic, related to the elementary symmetric functions of four variables t_1, t_2, t_3, t_4, that allows us to find *four* such points.

Theorem 38 (Four parabola normals) *If l is a point in the hyperbolic plane, then there are at most four points p on the parabola $\mathcal{P}_0$ whose normals pass through l.*

Proof. We know that the normal to $p_0 = \left[t^2 : t : 1\right]$ is the line

$$\begin{aligned}P = \langle t\alpha^2\left(-t^2\alpha^2+t^2-2\right) : \left(\alpha^2-1\right)\left(t^4\alpha^2+1\right) \\ : t\left(2t^2\alpha^2-\alpha^2+1\right)\rangle.\end{aligned}$$

If P passes through a point $l = [x_0 : y_0 : z_0]$, then $lP = 0$, which after rearranging is the equation

$$\begin{aligned}&\alpha^2\left(\alpha^2-1\right)y_0t^4+\alpha^2\left(\left(1-\alpha^2\right)x_0+2z_0\right)t^3 \\ &\quad+\left(\left(1-\alpha^2\right)z_0-2\alpha^2x_0\right)t+\left(\alpha^2-1\right)y_0 = 0. \qquad (16)\end{aligned}$$

This is a polynomial of degree four in t, so it has at most four solutions. □

Theorem 39 (Quadratic normal meets) *Suppose $p_0 = p(t)$ and $q_0 = p(u)$ are two points on the parabola, whose respective normals P and Q meet at a point l, and suppose $\alpha^2+1 \neq 0$. Then there are $0, 1$ or 2 other points on the parabola whose normals pass through l precisely when $\nabla = \left(t^2u^2\alpha^2+1\right)^2 - 4tu\alpha^2(t+u)^2$ is not a square, is zero, or is a non-zero square respectively.*

Proof. The meet of the two normals is

$$\begin{aligned}&l \equiv PQ = \\ &\left[\left(\alpha^2-1\right)\left(\left(tu\left(2t^2u^2-tu-t^2-u^2\right)\right)\alpha^4+\left((tu-2)\left(tu+t^2+u^2\right)+1\right)\alpha^2-1\right)\right. \\ &: -tu\alpha^2\left(\alpha^2+1\right)^2(t+u) \\ &\left.: \alpha^2\left(\alpha^2-1\right)\left(t^3u^3\alpha^4+\left((2tu-1)\left(tu+t^2+u^2\right)-t^3u^3\right)\alpha^2+\left(t^2+tu+u^2-2\right)\right)\right]\end{aligned}$$

and we need to check when a third point $r_0 \equiv p(v)$ on $\mathcal{P}_0$ has a normal R also passing through l. This is equivalent to $lR = 0$ which yields, after remarkable simplification,

$$\begin{aligned}&-\alpha^2\left(\alpha^2-1\right)\left(\alpha^2+1\right)^2(u-v)(t-v) \\ &\quad\cdot\left(t+u+v+tu^2v^2\alpha^2+t^2uv^2\alpha^2+t^2u^2v\alpha^2\right) = 0.\end{aligned}$$

Since $\alpha \neq 0, \pm 1$ and u, t, v are disjoint, this condition reduces to the quadratic equation $tu\alpha^2(t+u)v^2 + \left(t^2u^2\alpha^2+1\right)v + (t+u) = 0$ in v with discriminant

$$\nabla = \left(t^2u^2\alpha^2+1\right)^2 - 4tu\alpha^2(t+u)^2. \qquad □$$

The question of the existence of four points on the parabola $\mathcal{P}_0$ with a common normal point is closely related to an interesting conic associated to four points on the parabola; namely the conic $\mathcal{A}$ through those four points and the axis point a, which has independent interest due to its form. We call this conic $\mathcal{A}(p_1, p_2, p_3, p_4)$ the **four-point conic** through p_1, p_2, p_3 and p_4.

Theorem 40 (Four point conic) *For any four points $p_1 \equiv p(t)$, $p_2 \equiv p(u)$, $p_3 \equiv p(v)$ and $p_4 \equiv p(w)$ lying on $\mathcal{P}_0$, the four-point conic $\mathcal{A}(p_1, p_2, p_3, p_4)$ has equation*

$$\begin{aligned}0 =& x^2-(t+u+v+w)xy+(tu+tv+tw+uv+uw+vw)xz \\ &-(tuv+tuw+tvw+uvw)yz+tuvwz^2. \qquad (17)\end{aligned}$$

Proof. We use a standard technique for computing a conic through five given points: by taking a combination of the degenerate line products formed by pairs of four points p_1, p_2, p_3 and p_4. Now

$$\begin{aligned}p_1p_2 &= (1 : -(t+u) : tu) \qquad & p_3p_4 &= (1 : -(v+w) : vw) \\ p_1p_3 &= (1 : -(t+v) : tv) \qquad & p_2p_4 &= (1 : -(t+w) : tw)\end{aligned}$$

so the general conic in the pencil through p_1, p_2, p_3 and p_4, has the form

$$\begin{aligned}0 =& p(x,y,z) = (x-(t+u)y+tuz)(x-(v+w)y+vwz) \\ &+\lambda(x-(t+v)y+tvz)(x-(u+w)y+uwz).\end{aligned}$$

Now since also $p(0,1,0) = 0$, we can solve for λ to get

$$\lambda = -\frac{(t+u)(v+w)}{(t+v)(u+w)}.$$

Substituting back and simplifying, we find that the equation of the required conic is (17). □

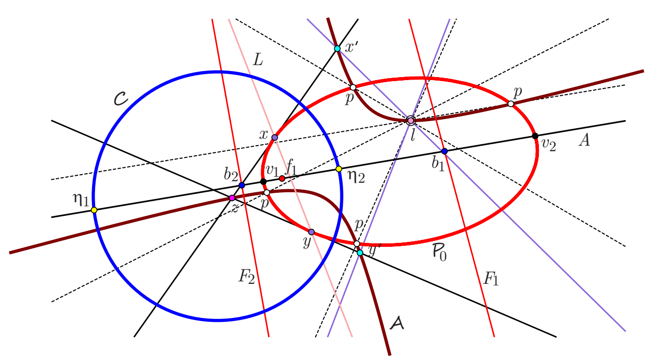

Figure 29: *Four points p with normals through l and associated conic $\mathcal{A}$*

There is a clear similarity between the form of this conic and the familiar identity

$$\begin{aligned}(x-t_1)(x-t_2)(x-t_3)(x-t_4) &= x^4-(t_1+t_2+t_3+t_4)x^3\\ &+(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4)x^2\\ &-(t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4)x+t_1t_2t_3t_4\end{aligned}$$

relating the coefficients of a degree four polynomial and the elementary symmetric functions of its zeros. This may be explained by noting that if $p=[x:y:z]=\left[t^2:t:1\right]$ is a point on the parabola, then the quantities x^2, xy, xz, yz and z^2 are respectively exactly t^4, t^3, t^2, t and 1, while the condition that the conic passes through a ensures that the coefficient of y^2 is necessarily 0.

5.3 The conic $\mathcal{A}_n$ and finding normals

Theorem 41 (Four normal conic) *Suppose that the normal lines at four points p_1, p_2, p_3, p_4 lying on $\mathcal{P}_0$ are concurrent at a point $l=[x_0, y_0, z_0]$ not lying on the axis A. Then the conic $\mathcal{A}_l$ with equation*

$$\begin{aligned}&\alpha^2\left(\alpha^2-1\right)y_0x^2+\alpha^2\left(x_0+2z_0-x_0\alpha^2\right)xy\\ &+\left(z_0-z_0\alpha^2-2x_0\alpha^2\right)yz+\left(\alpha^2-1\right)y_0z^2=0 \qquad (18)\end{aligned}$$

passes through the six points p_1, p_2, p_3, p_4, a and l, so in particular $\mathcal{A}_l=\mathcal{A}(p_1, p_2, p_3, p_4)$.

Proof. The condition (16) on t for $p=\left[t^2:t:1\right]$ on $\mathcal{P}_0$ to have a normal line passing through $l\equiv[x_0, y_0, z_0]$ may be rewritten, since $y_0\neq 0$, as

$$t^4+\frac{\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)}{\alpha^2\left(\alpha^2-1\right)y_0}t^3+\frac{\left(z_0\left(1-\alpha^2\right)-2x_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}t+\frac{1}{\alpha^2}=0.$$

If we have four distinct solutions t, u, v, w of this equation, then

$$\begin{aligned}t+u+v+w &= -\frac{\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)}{\alpha^2y_0\left(\alpha^2-1\right)}\\ tu+tv+tw+uv+uw+vw &= 0\\ tuv+tuw+tvw+uvw &= -\frac{z_0\left(1-\alpha^2\right)-2x_0\alpha^2}{\alpha^2y_0\left(\alpha^2-1\right)}\\ tuvw &= \frac{1}{\alpha^2}.\end{aligned}$$

From the previous theorem, the conic passing through the five points $p_1=p(t), p_2=p(u), p_3=p(v), p_4=p(w)$ and a then has the form

$$x^2+\frac{\alpha^2\left(x_0+2z_0-x_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}xy+\frac{\left(z_0-2x_0\alpha^2-z_0\alpha^2\right)}{\alpha^2\left(\alpha^2-1\right)y_0}yz+\frac{1}{\alpha^2}z^2=0$$

which we can rewrite as the conic $\mathcal{A}_l$ (18). But now we can check that also l lies on this conic, since identically

$$\begin{aligned}&\alpha^2\left(\alpha^2-1\right)y_0x_0^2+\alpha^2\left(x_0\left(1-\alpha^2\right)+2z_0\right)x_0y_0\\ &+\left(z_0\left(1-\alpha^2\right)-2x_0\alpha^2\right)y_0z_0+\left(\alpha^2-1\right)y_0z_0^2=0. \qquad \square\end{aligned}$$

Theorem 42 (Conic construction of common normals) *Let l be a point of the hyperbolic plane with the property that the dual line L of l meets $\mathcal{P}_0$ at two points x and y. Then the meet z of the tangent lines to $\mathcal{P}_0$ at x and y, the meet x' of the tangent line at x and the dual line of x, and the meet y' of the tangent line at y and the dual line of y, all line on the conic $\mathcal{A}_l$.*

Proof. Suppose that the dual line L of l meets $\mathcal{P}_0$ at two points $x=\left[t^2:t:1\right]$ and $y=\left[u^2:u:1\right]$. Then the meets of the tangent lines is $z=[2tu:t+u:2]$ from the Tangent meets theorem. Also $L=\langle 1:-(t+u):tu\rangle$ and

$$l=\left[\alpha^2-1:\alpha^2(t+u):-\alpha^2tu\left(\alpha^2-1\right)\right].$$

In this case the equation (18) for the conic $\mathcal{A}_l$ simplifies, after some cancellation, to

$$\begin{aligned}&\alpha^2(t+u)x^2+\left(1-2tu\alpha^2-\alpha^2\right)xy\\ &+\left(tu\alpha^2-tu-2\right)yz+(t+u)z^2=0. \qquad (19)\end{aligned}$$

The dual line of x meets the tangent line at x at

$$x'=\left[t\left(\alpha^2t^2-t^2+2\right):\alpha^2t^4+1:t\left(2\alpha^2t^2-\alpha^2+1\right)\right]$$

and the dual line of y meets the tangent line at y at

$$y'=\left[u\left(\alpha^2u^2-u^2+2\right):\alpha^2u^4+1:u\left(2\alpha^2u^2-\alpha^2+1\right)\right].$$

We check that both of these points identically satisfy the equation (19). $\square$

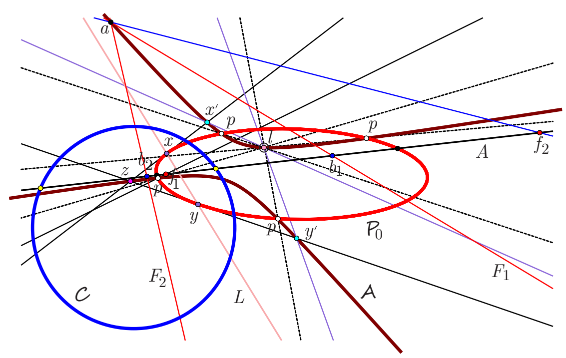

Figure 30: *Construction of points p on $\mathcal{P}_0$ with normals through n*

This also provides us with an elegant method to find all normals through a given point l. Firstly, find the dual line L of the point l and then find the meets x, y of this line L with the parabola $\mathcal{P}_0$. Construct the tangents P_x, P_y to $\mathcal{P}_0$ at x and y and find their meet z. Construct the dual lines X and Y of x and y, then find the meet of the tangent at x and the dual line of x, that is $x'=P_xX$ and the meet of the tangent at y and the dual line of y, that is $y' = P_y Y$. According to the above theorem, the five points l, x', y', z, a lie on a conic A_l which may meet the parabola $\mathcal{P}_0$ in at most four points which have the property that their normals meet at l. We see that the number of normals passing through l is determined by the meet of the conic A_l with the parabola $\mathcal{P}_0$. So if we can find the meets of these two conics, we have the normals which pass through l.

This construction shows that some aspects of hyperbolic geometry are surprisingly more simple than in Euclidean geometry. In the latter, finding normals to points on a parabola from a particular point is quite cumbersome, as shown in [8].

Furthermore, the four normals drawn from a particular points are also the normals to four points on the twin parabola $\mathcal{P}^0$. These points are the dual points of the tangents to four points on the original parabola $\mathcal{P}_0$. This observation is the result of duality between lines and points.

5.4 Normal conjugate points

If p_0 is a point on $\mathcal{P}_0$ with tangent line P^0 and normal line P, then the other meet of P with the parabola gives a point p_0', which we call the **normal conjugate point** of p_0. Then the tangent line $P^{0\prime}$ to p_0' meets with P^0 at the point

$$\begin{aligned}k_0 &= P^0P^{0\prime}\\ &=\left\langle t^2\alpha^4\left(t^2\alpha^2-t^2+2\right)^2 : 2t\alpha^2\left(t^2\alpha^2-t^2+2\right)\left(2t^2\alpha^2-\alpha^2+1\right)\right.\\ &\quad\left. : \left(2t^2\alpha^2-\alpha^2+1\right)^2\right\rangle \times \left\langle 1:-2t:t^2\right\rangle\\ &=\left[-2t\left(2t^2\alpha^2-\alpha^2+1\right):\left(\alpha^2-1\right)\left(t^4\alpha^2+1\right):2t\alpha^2\left(t^2\alpha^2-t^2+2\right)\right].\end{aligned}$$

Figure 31 shows the **normal conjugate curve** $\mathcal{K}_0$: the locus of k_0 as p_0 moves. This a higher degree curve which passes through a as well as d_0 and $\overline{d_0}$, and is tangent to $\mathcal{P}_0$ at those latter two points. It seems an interesting future direction to investigate more fully such associated algebraic curves connected with $\mathcal{P}_0$.

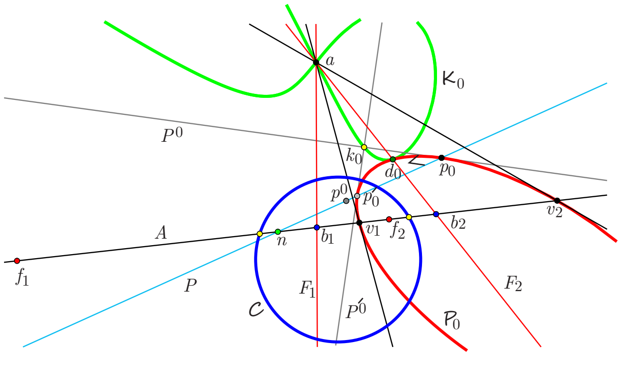

Figure 31: *The normal conjugate conic $\mathcal{K}_0$*

5.5 The evolute and centers of curvature

Recall that the **evolute** of a curve is the envelope of the normals to that curve, or equivalently the locus of the centers of curvature. Following the technique described in [4], here is a pleasant construction of the center of curvature c_0 to the hyperbolic parabola $\mathcal{P}_0$ at the point p_0.

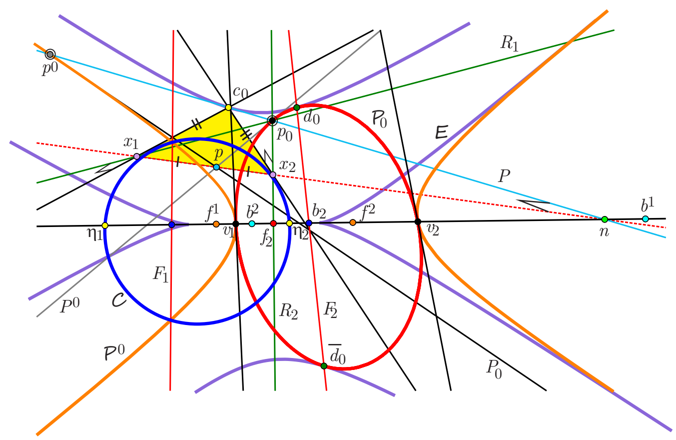

Figure 32: *Evolute of a parabola*

Theorem 43 (Center of curvature construction) *Let P be the normal at p_0 to the parabola $\mathcal{P}_0$, and construct the altitude line Q to P through $n = AP$. Suppose that the meets of Q with the focal lines R_1 and R_2 are respectively x_1 and x_2. Then the meet of the perpendicular line to R_1 through x_1 and the perpendicular line to R_2 through x_2 is the required center of curvature c_0 to $\mathcal{P}_0$ at the point p_0.*

Proof. Let $p_0 = \left[t^2 : t : 1\right]$ and $n = \left[2t^2\alpha^2 - \alpha^2 + 1 : 0 : \alpha^2\left(t^2\alpha^2 - t^2 + 2\right)\right]$, then the perpendicular to P through $l = n$ is

$$\begin{aligned}Q \equiv pn = \Big[&\alpha^2\left(t^4\alpha^2+1\right)\left(t^2\alpha^2-t^2+2\right)\\ &: t\left(2\alpha-t^2\alpha-2t^2\alpha^2+t^2\alpha^3+\alpha^2-1\right)\\ &\quad\cdot\left(-2\alpha+t^2\alpha-2t^2\alpha^2-t^2\alpha^3+\alpha^2-1\right)\\ &: \left(t^4\alpha^2+1\right)\left(-2t^2\alpha^2+\alpha^2-1\right)\Big]\end{aligned}$$

This line will meet the line R_1 at

$$\begin{aligned}x_1 = \Big[&-2\alpha^4t^6+\left(\alpha^5+3\alpha^4-3\alpha^2-\alpha\right)t^4\\ &+\left(2\alpha^3-\alpha^4+4\alpha^2+2\alpha-1\right)t^2+\left(1-\alpha^2\right)\\ &: t\alpha\left(\alpha^2+1\right)\left(t^4\alpha^2+1\right)\\ &: \alpha\left(-\alpha^3\left(\alpha^2-1\right)t^6+\alpha\left(2\alpha-4\alpha^2+2\alpha^3+\alpha^4+1\right)t^4\right.\\ &\quad\left.-\left(\alpha^2-1\right)\left(-3\alpha+\alpha^2+1\right)t^2+2\alpha\right)\Big]\end{aligned}$$

and the line R_2 at

$$\begin{aligned}x_2 = \Big[& \left(2\alpha^4\right) t^6 + \left(\alpha^5 - 3\alpha^4 + 3\alpha^2 - \alpha\right) t^4 \\ & + \left(\alpha^4 + 2\alpha^3 - 4\alpha^2 + 2\alpha + 1\right) t^2 + \left(\alpha^2 - 1\right) \\ & : t\alpha\left(\alpha^2+1\right)\left(t^4\alpha^2+1\right) \\ & : \alpha\left(\alpha^3\left(\alpha^2-1\right)t^6 + \left(2\alpha^4 - \alpha^5 + 4\alpha^3 + 2\alpha^2 - \alpha\right)t^4 \right. \\ & \left. + \left(3\alpha - 3\alpha^3 - \alpha^4 + 1\right)t^2 - 2\alpha\right)\Big].\end{aligned}$$

The perpendicular line to R_1 through x_1 is $X_1 = x_1 r_1$ and the perpendicular line to R_2 through x_2 is $X_2 = x_2 r_2$ which meet at

$$\begin{aligned}& c_0 = X_1 X_2 = \\ & \Big[\left(\alpha^2-1\right)\left(2\alpha^4 t^6 + 3\alpha^2\left(1-\alpha^2\right)t^4 - 6\alpha^2 t^2 + \left(\alpha^2-1\right)\right) \\ & \quad : -2t^3\alpha^2\left(\alpha^2+1\right)^2 \\ & \quad : \alpha^2\left(\alpha^2-1\right)\left(\alpha^2\left(\alpha^2-1\right)t^6 + 6\alpha^2 t^4 + 3\left(1-\alpha^2\right)t^2 - 2\right)\Big].\end{aligned}$$

To evaluate the center of curvature, we note that adjacent normals, say at $p(t)$ and $p(r)$, meet at

$$\begin{aligned}& f(t,r) = \\ & \Big[\left(\alpha^2-1\right)\left(-2r^3t^3\alpha^4 + r^3t\alpha^4 - r^3t\alpha^2 + r^2t^2\alpha^4 - r^2t^2\alpha^2 \right. \\ & \qquad \left. + 2r^2\alpha^2 + rt^3\alpha^4 - rt^3\alpha^2 + 2rt\alpha^2 + 2t^2\alpha^2 - \alpha^2 + 1\right) \\ & \quad : rt\alpha^2\left(r+t\right)\left(\alpha^2+1\right)^2 \\ & \quad : -\alpha^2\left(\alpha^2-1\right)\left(r^3t^3\alpha^4 - r^3t^3\alpha^2 + 2r^3t\alpha^2 + 2r^2t^2\alpha^2 \right. \\ & \qquad \left. - r^2\alpha^2 + r^2 + 2rt^3\alpha^2 - rt\alpha^2 + rt - t^2\alpha^2 + t^2 - 2\right)\Big]\end{aligned}$$

where we have removed a common factor of $r-t$. Now let $r=t$ to find that $f(t,t) = c_0$. □

5.6 Formula for the evolute

Can we get a formula for the evolute? Working with affine coordinates (setting $z=1$), we need eliminate t from the equations

$$x = \frac{\left(2t^6\alpha^4 - 3t^4\alpha^4 + 3t^4\alpha^2 - 6t^2\alpha^2 + \alpha^2 - 1\right)}{\alpha^2\left(t^6\alpha^4 - t^6\alpha^2 + 6t^4\alpha^2 - 3t^2\alpha^2 + 3t^2 - 2\right)}$$

$$y = \frac{-2t^3\left(\alpha^2+1\right)^2}{\left(\alpha^2-1\right)\left(t^6\alpha^4 - t^6\alpha^2 + 6t^4\alpha^2 - 3t^2\alpha^2 + 3t^2 - 2\right)}.$$

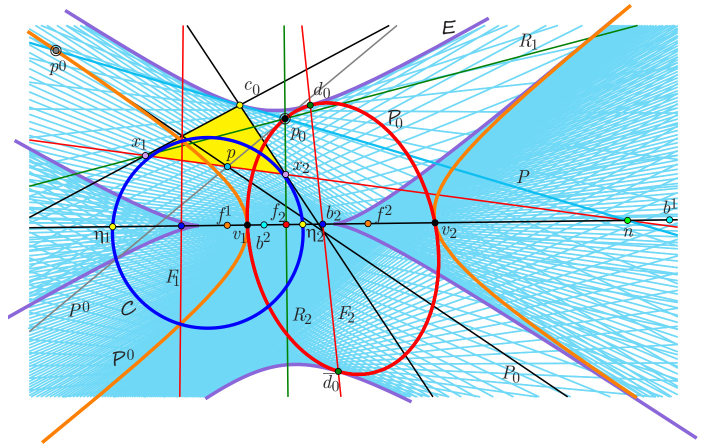

Figure 33: *Normals to a parabola*

We could use a Gröbner basis to calculate this, but the polynomials are small enough to do it by hand with classical elimination. We get, after some calculation, that x and y satisfy the affine equation

$$\begin{aligned}0 = h(x,y) = {} & 32\alpha^8\left(\alpha^2-1\right)^3 x^6 - 256\alpha^2\left(\alpha^2-1\right)^6 y^6 \\ & + 3\alpha^4\left(8\alpha + 6\alpha^2 - 8\alpha^3 + 3\alpha^4 + 3\right)\left(-8\alpha + 6\alpha^2 + 8\alpha^3 + 3\alpha^4 + 3\right) \\ & \quad \cdot\left(\alpha-1\right)^2\left(\alpha+1\right)^2 x^4y^2 \\ & + 384\alpha^4\left(\alpha^2-1\right)^5 x^2y^4 + 48\alpha^6\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^2 x^5 \\ & - 192\alpha^4\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^3 x^3y^2 \\ & + 192\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^4 xy^4 \\ & + 24\alpha^4\left(\alpha^2-1\right)\left(-2\alpha-6\alpha^2+2\alpha^3+\alpha^4+1\right)\left(2\alpha-6\alpha^2-2\alpha^3+\alpha^4+1\right)x^4 \\ & - 384\alpha^2\left(\alpha^2-1\right)^5 y^4 \\ & + 6\alpha^2\left(196\alpha^2 - 378\alpha^4 + 196\alpha^6 + \alpha^8 + 1\right)\left(\alpha^2-1\right)^2 x^2y^2 \\ & + 4\alpha^2\left(2\alpha+\alpha^2-1\right)\left(-2\alpha+\alpha^2-1\right)\left(-36\alpha^2+86\alpha^4-36\alpha^6+\alpha^8+1\right)x^3 \\ & + 192\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^3 xy^2 \\ & - 24\alpha^2\left(\alpha^2-1\right)\left(2\alpha-6\alpha^2-2\alpha^3+\alpha^4+1\right)\left(-2\alpha-6\alpha^2+2\alpha^3+\alpha^4+1\right)x^2 \\ & + 3\left(-8\alpha+6\alpha^2+8\alpha^3+3\alpha^4+3\right)\left(8\alpha+6\alpha^2-8\alpha^3+3\alpha^4+3\right)\left(\alpha^2-1\right)^2 y^2 \\ & + 48\alpha^2\left(-2\alpha+\alpha^2-1\right)\left(2\alpha+\alpha^2-1\right)\left(\alpha^2-1\right)^2 x - 32\alpha^2\left(\alpha^2-1\right)^3.\end{aligned}$$

So the evolute is a six degree curve, with coefficients that depend in a pleasant way on α. Note that all the coefficients are divisible by α^2-1, with the exception of the coefficient of x^3.

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Ali Alkhaldi
e-mail: aalkaldy@hotmail.com

Norman John Wildberger
e-mail: n.wildberger@unsw.edu.au

School of Mathematics and Statistics UNSW
Sydney 2052 Australia