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Conchoids on the Sphere

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ABSTRACT

The construction of planar conchoids can be carried over to the Euclidean unit sphere. We study the case of conchoids of (spherical) lines and circles. Some elementary constructions of tangents and osculating circles are stil valid on the sphere. Further, we aim at the illustration and a precise description of the algebraic properties of the principal views of spherical conchoids, *i.e.*, the conchoid's images under orthogonal projections onto their symmetry planes.

Key words: spherical curves, conchoids, algebraic curves, tangent, osculating circle, singularities, orthogonal projection

MSC2010: 51N20, 14H99, 70B99

Konhoide na sferi

SAŽETAK

Konstrukcija ravninskih konhoida može se prenijeti na euklidsku jediničnu sferu. Promatramo slučaj konhoida generiranih sfernim pravacima i kružnicama. Neke elementarne konstrukcije tangenata i kružnica zakrivljenosti vrijede i za sferne konhoide. Nadalje, naš je cilj ilustracija i precizan opis algebarskih svojstava glavnih pogleda sfernih konhoida, tj. slika konhoida pri ortogonalnom projiciranju na njihove ravnine simetrije.

Ključne riječi: krivulje na sferi, konhoide, algebarske krivulje, tangenta, kružnica zakrivljenosti, singulariteti, ortogonalna projekcija

1 Introduction

The construction of conchoids goes back to the early Greek mathematicians [5, 13]. Assume we are given a point F, called *focus* and a line l called *directrix* one can ask for the set c of all points in the Euclidean plane at fixed distance d from l measured on all lines through F, cf. Figure 1.

The set c turns out to be an algebraic curve of degree 4, namely the *conchoid* of the line l with respect to F at distance $d \in \mathbb{R}$. The conchoid c can be described by the equation

$$(x^2 - d^2)(f - x)^2 + x^2y^2 = 0$$

provided that a Cartesian coordinate system is chosen as depicted in Figure 1 with $F = (f, 0)$, $f \in \mathbb{R}$ and $l:\ x = 0$. The conchoid has two branches, one corresponding to the distance $+d$, while the other corresponds to the distance $-d$. The algebraic variety contains both branches.

The conchoid c has an ordinary double point at $F = (f, 0)$ if $|d| > |f|$ (or an isolated double point if $|d| < |f|$). In the case of $|d| = |f|$, F is a cusp of the first kind, *i.e.*, with the local expansion $(u^2 + o(u^3), u^3 + o(u^4))$, see [2, 3]. The cusped curve can also be seen in Figure 2.

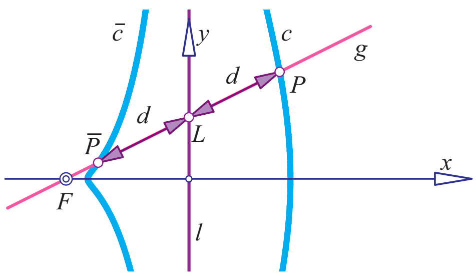

Figure 1: *The construction of the conchoid c of a line l in the plane.*

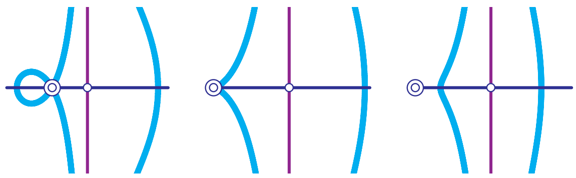

Figure 2: *The planar conchoid of a line has an ordinary double point if $|d| > |f|$ (left), a cusp if $|d| = |f|$ (in the middle), and an isolated double point if $|d| < |f|$ (right).*

Independent of the choice of d and f the curve c considered as a curve in the projective plane (cf. Figure 3) has a tacnode at the ideal point of the y-axis. There, two linear branches with the same tangent emanate. Therefore, the conchoid is of genus 0, and thus, it is a rational curve.

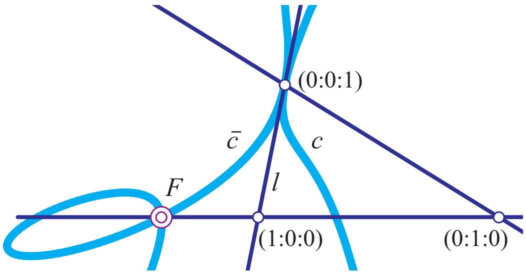

Figure 3: *The singularities of the conchoid considered as a curve in the projective plane.*

The name *conchoid* is due to the fact that its shape somehow reminds of a conch. The conchoid of a line (the directrix l is a line) is frequently called conchoid of Nikomedes, see [4, 5, 13]. The line l can be replaced by an arbitrary curve.

In former years, mathematicians developed elementary constructions of points, tangents, and osculating circles for some kinds of conchoids such as those of lines and circles. The kinematic point of view allows us to see the conchoids as traces of moving particles, and thus, further constructions of tangents and osculating circles can be deduced, see for example [6, 14].

In the last few years conchoids became popular in CAGD, see [1, 8, 9, 10, 11]. This is mainly due to the fact that under certain circumstances conchoids can be parametrized by means of rational functions which is mainly the content of [8, 9]. Thus, a huge class of possibly new surfaces is available for CAGD. The conchoids of spheres and ruled surfaces are not spheres or ruled surfaces anymore, except in some special cases. In order to overcome this flaw, an intrinsic construction of conchoids for some geometries is presented in [7].

It is somehow surprising that conchoids on the sphere have not attracted the researchers' interest. Many constructions that are valid in the Euclidean plane can easily be adapted for the Euclidean unit sphere. In this article, we shall demonstrate this at hand of the spherical analoga to conchoids of lines and circles. The spherical conchoids of lines are conchoids of greatcircles on the sphere. However, the spherical conchoids of circles are stil conchoids of circles but on the sphere.

We shall describe spherical conchoids of lines and circles and study their algebraic properties at hand of their equations. Then, we discuss the shape of the principal views of the spherical conchoids. The principal views are obtained as orthogonal projections to a triple of mutually orthogonal planes where at least one of these planes is a plane of symmetry of the spherical curve. The resulting image curves are at most of degree 8 as is the case for the space curves. For some image curves the degree reduces to 4. Further, we describe the singularities showing up on the principal views of the spherical conchoids.

2 Conchoids of a line

Assume Σ is the Euclidean unit sphere with the equation

$$\Sigma:\ x^2+y^2+z^2=1 \tag{1}$$

and let further l be a *line on* Σ, *i.e.*, a greatcircle of Σ. Without loss of generality, we can asssume that l is the equator of Σ in the plane $z=0$ (see Figure 4). Thus, a parametrization of l reads

$$L(\lambda)=(c_\lambda,s_\lambda,0)\quad \text{with } \lambda\in[0,2\pi[\tag{2}$$

where we have used the abbreviations $c_\lambda:=\cos\lambda$ and $s_\lambda:=\sin\lambda$.

The focus F of the conchoid shall be at spherical distance $\phi\in\,]0,\pi/2[$ from l. Therefore, its coordinates are

$$F=(c_\phi,0,s_\phi) \tag{3}$$

(with $c_\phi:=\cos\phi$ and $s_\phi:=\sin\phi$) since it means no restriction to assume that the greatcircle orthogonal to l through F lies in the plane $y=0$.

The points on the spherical conchoid c of l with respect to F at distance $\delta\in\,]0,\frac{\pi}{2}[$ are found via the analogous construction on the sphere: Choose a point L on the equator l, join it with F by a greatcircle, and determine the points P at spherical distance δ from L.

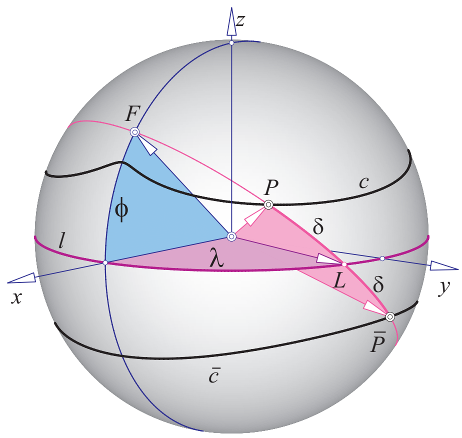

Figure 4: *Construction of a conchoid on the unit sphere and the choice of a coordinate system.*

We exclude the case $\phi=\frac{\pi}{2}$ which yields a pair of *distance curves* provided that $\delta\neq 0$. These distance curves are circles on Σ with spherical radius $\frac{\pi}{2}-\delta$ in planes parallel to the equator plane. The choice $\delta=0$ shows that the equator can be seen as a trivial conchoid $c=l$. The case $\phi=\frac{\pi}{2}$ also yields circles as spherical conchoids of l.

Now we are going to derive an analytical description of the spherical conchoid. Assume that (x,y,z) are the Cartesian coordinates of a point X on the conchoid of l at the spherical distance $\delta \in]0,\frac{\pi}{2}[$ with respect to the point F. These coordinates satisfy Eq. (1). Since $[L,F]$ is a greatcircle of Σ, the points F, L, and the point X on the conchoid are coplanar with the center $(0,0,0)$ of Σ. This is equivalent to

$$s_\lambda s_\phi\, x - c_\lambda s_\phi\, y - s_\lambda c_\phi\, z = 0. \tag{4}$$

Further, we have $\widehat{LX} = \delta$ which is measured along the greatcircle $[L,X]$. Thus, the canonical scalar product of the unit vectors $X = (x,y,z)$ and $L = (c_\lambda, s_\lambda, 0)$ yields the cosine of the angle subtained by $\widehat{LX}$, and therefore, we have

$$c_\lambda\, x + s_\lambda\, y = \cos\delta. \tag{5}$$

We can eliminate λ from Eqs. (4) and (5): These equations are linear in c_λ and s_λ, and thus, we can solve this system for c_λ and s_λ which gives

$$\begin{aligned} c_\lambda &= \frac{\cos\delta(s_\phi\, x - c_\phi\, z)}{s_\phi(x^2+y^2) - c_\phi\, xz}, \\ s_\lambda &= \frac{\cos\delta s_\phi\, y}{s_\phi(x^2+y^2) - c_\phi\, xz}. \end{aligned}$$

Since ${c_\lambda}^2 + {s_\lambda}^2 = 1$ holds for any $\lambda \in \mathbb{C}$, we arrive at an implicit equation of the spherical conchoids c of a (spherical) line l:

$$c: \left\{ \begin{aligned} \cos^2\delta\left((s_\phi\, x - c_\phi\, z)^2 + {s_\phi}^2 y^2\right) & \\ -(s_\phi(x^2+y^2) - c_\phi\, xz)^2 &= 0, \\ x^2+y^2+z^2 &= 1. \end{aligned} \right. \tag{6}$$

Obviously, c is a space curve of degree 8, since it is the intersection of a quartic surface Φ (an example of which is displayed in Figure 5) with the unit sphere. Thus, we can say:

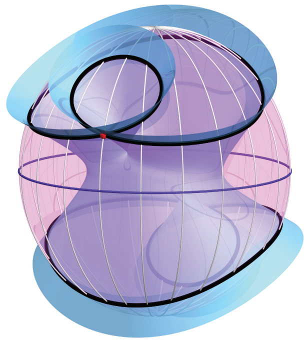

Figure 5: *A spherical conchoid is the intersection of the unit sphere with a quartic surface.*

Theorem 1. *The spherical conchoid c of a (spherical) line l with respect to the focus F at (spherical) distance $\delta \in]0,\frac{\pi}{2}[$ is an algebraic space curve of degree* 8 *and can be given by the two equations* (6).

It is clear that these curves are spherical so that it is not worth to be mentioned that Eq. (1) is fulfilled by the coordinates (x,y,z) of a generic point on the conchoid. Therefore, only the first equation of (6) matters. Thus, such curves are often called of *spherical degree four*.

The three different shapes of conchoids of a line that can be observed in a plane also appear on the sphere as can be seen in Figure 6. There are conchoids with loops, *i.e.*, they have a spherical double point (actually a pair of opposite double points) with real tangents at the double point F if $\delta > \phi$. The conchoids with spherical cusps (a pair of opposite cusps) appear if, and only if, $\delta = \phi$. In the case of $\delta < \phi$, we observe that F is an isolated (spherical double) point on the conchoid.

As can be seen from Figures 4 and 6 the spherical conchoids always consist of two branches. This is caused by the fact that points in spherical geometry are actually a pair of antipodal points on the sphere. Therefore, any singular point on a conchoid also shows up twice. Even the spherical singularity is a pair of antipodal points.

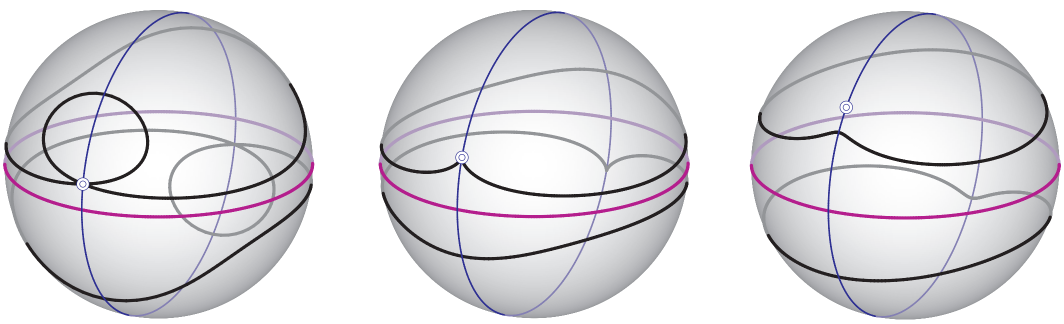

Figure 6: *Three different appearances of spherical conchoids of a the equator: $\delta > \phi$ (left), $\delta = \phi$ (middle), $\delta < \phi$ (right).*

2.1 Principal views of spherical conchoids

The orthogonal projections of c onto the three planes $z=0$, $x=0$, and $y=0$ shall be called top view, front view, and (right) side view. We can state:

Theorem 2. *The front and top view of a spherical conchoid given by Eq.* (6) *with* $\delta \in]0, \frac{\pi}{2}[$ *are of algebraic degree* 8 *and of genus* 1, i.e., *they are elliptic. The right side view is a rational quartic.*

Proof. The equations of c's principal views can be obtained from (6) by simply eliminating z, x, or y. Since c is of degree 8, the principal views of c are at most of degree 8. Reductions of the degree occur only in cases where the image plane is a plane of symmetry of each branch, *i.e.*, each point of the image curve is the image of two points on c. Because of the special choice of the coordinate system, we see that c is symmetric with respect to the plane $y=0$, and therefore, the side view is covered twice. Hence, it is of degree 4. When computing the resultants of both equations in Eq. (6) with respect to y, we find the square of

$$q: \ (c_\lambda x + s_\lambda z)^2 z^2 - 2 s_\lambda c_\lambda \sin^2\delta xz$$
$$-(c_{2\lambda}\cos^2\delta + 2{s_\lambda}^2)z^2 + {s_\lambda}^2\sin^2\delta = 0$$

as the equation of the right side view of the spherical conchoid.

The computations can be carried out by Maple. The *algcurves* package allows us to compute the singularities and the genus of an algebraic curve. We summarize the results in tables: Besides the degree we give the singularities in terms of homogeneous coordinates (with the homogenizing factor always in the first position), the invariants $[m,d,b]$, where m is the multiplicity, d is the δ-invariant, and b is the branching number.

Note that for an ordinary m-fold point the equation $m=b$ holds. In any other case we have $m>d$. The genus g of a planar algebraic curve c of degree n is the integer

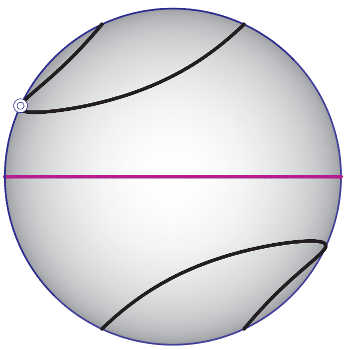

Figure 7: *Right side view of the spherical conchoid shows no singularity in the affine part. Note that the image of the focus is not singular.*

$$g = \frac{1}{2}(n-1)(n-2) - \sum_{\mathcal{S}} d_{\mathcal{S}},$$

where $\mathcal{S}$ is the set of singular points on c and $d_{\mathcal{S}}$ are the δ-invariants of all singularities on c. According to the Milnor-Jung formula, the δ-invariant d can be computed from the Milnor number μ and the branching number b of a singularity as $d=\frac{1}{2}(\mu+b-1)$. Thus, an ordinary k-fold point has invariants $[k, \frac{1}{2}k(k-1), k]$, see [2, 3].

We have to distinguish between two cases whether $\phi \neq \delta$ or $\phi = \delta$.

(1) Let us first assume that $\phi \neq \delta$:

The singularities of the right side view are given in Table 1. Since the genus equals zero, the curve showing up in the right side view is rational. Note that both singularities are ideal points of the $[x,z]$-plane. The point $(0:1:0)$ is an isolated tacnode, *i.e.*, a point where a pair of complex conjugate linear branches touches a real tangent at the real point $(0:1:0)$. The remaining singularity is an ordinary double point. The right side view of the spherical conchoid is displayed in Figure 7.

right side view		
	$\deg(c)=4$	
S_1	$(0:1:0)$	[2,2,2]
S_2	$(0:1:-\cot\phi)$	[2,1,2]
	$\text{genus}(c)=0$	

Table 1: *Singularities on the right side view.*

In Figure 8 we can observe another phenomenon which may not only appear in connection with spherical conchoids. The algebraic image curve carries points that are outside the silhouette of the unit sphere. Thus, these points cannot be the images of points on the spherical curve. The points on these parts of the curve are called *parasitic*.

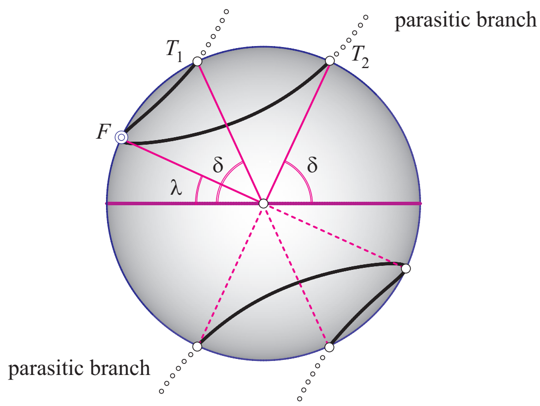

Figure 8: *Singularities on the principal views of spherical conchoids of lines.*

The front view shows a curve of degree eight (shown in Figure 9). It has a pair of complex conjugate ordinary double points $(0:\pm i:c_\phi)$ at the ideal line of the $[y,z]$-plane. Further, there is an ideal 4-fold point with δ-invariant $d=12$. Among the four singularities in the affine part of the curve (the part we can see in Figure 9) there are two tacnodes $(1:0:\pm\sin\delta)$ which are the images of the top most points T_1 and T_2 of the conchoid on the front and back side of the sphere (cf. Figure 8). The fact that the two linear branches are in contact at the common image of the top most point is caused by the fact that the spherical conchoid has horizontal tangents at both points, T_1 and T_2. The image of the spherical focus F (antipodal pair) completes the list of singular points, cf. Table 2.

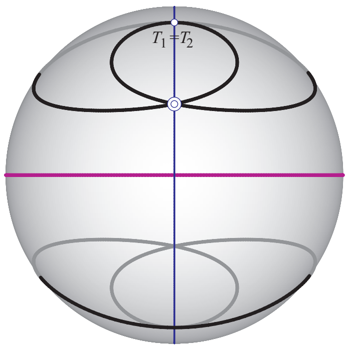

Figure 9: *The front view of the spherical conchoid shows up to four singularties.*

	front view	
	$\deg(c)=8$	
$S_{1,2}$	$(1:0:\pm s_\phi)$	[2,1,2]
$T_{1,2}$	$(1:0:\pm\sin\delta)$	[2,2,2]
S_5	$(0:1:0)$	[4,12,4]
$S_{6,7}$	$(0:\pm i:c_\phi)$	[2,1,2]
	$\mathrm{genus}(c)=1$	

Table 2: *Singularities on the front view.*

The top view has six real ordinary double points (see Figure 10). These are the image points $(\pm c_\phi,0)$ of F and its antipode. Further, there are four ordinary double points at $(0,w)$ where w is a solution of the quartic equation

$$t^4 {s_\phi}^2 + t^2\cos^2\delta({c_\phi}^2 - {s_\phi}^2) - {c_\phi}^2\cos^2\delta = 0.$$

Two of these double points are real, two are complex conjugate. The ideal points $(0:1:\pm i)$ of the $[x,y]$-plane are double points on the top view of the spherical conchoid. However, they are not ordinary double points, for their δ-invariant equals four. At these points the curve hyperosculates itself. Further, we find tacnodes at $(1:\pm\cos\delta:0)$ being the images of the front and back most points of the conchoid on the upper and lower hemisphere, see Figures 8 and 10. The singularities of the spherical conchoid's top view are listed in Table 3.

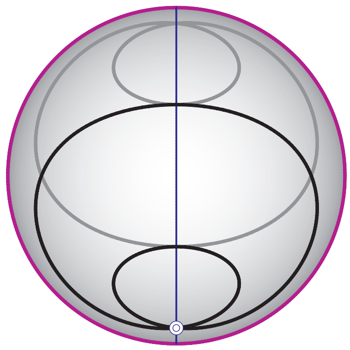

Figure 10: *The top view of the spherical conchoid shows up to six singular points.*

	top view	
	$\deg(c)=8$	
$S_{1,2}$	$(1:\pm\cos\delta:0)$	[2,2,2]
$S_{3,4}$	$(1:\pm c_\phi:0)$	[2,1,2]
$S_{5,6,7,8}$	$(1:0:w)$	[2,1,2]
$S_{9,10}$	$(0:1:\pm i)$	[2,4,2]
$S_{11,12}$	$(0:\pm s_\phi:1)$	[2,1,2]
	$\mathrm{genus}(c)=1$	

Table 3: *Singularities on the top view.*

(2) Finally, we deal with the case $\phi=\delta$, *i.e.*, the curves with cusps.

We do not have to go through all the details. There are some minor changes in the types of some singularitiers showing up on the different views. Figure 11 shows the right side view, the front view, and the top view.

	right side view	
	$\deg(c)=4$	
S_1	$(0:1:0)$	[2,2,2]
	$\mathrm{genus}(c)=1$	

Table 4: *Singularities of the right side view of the curve with cusp.*

The right side view of the spherical conchoid with cusp shows no singularity in the affine part. There is only one ideal point which is a tacnode, cf. Table 4. In this case the curve is of degree four, but nevertheless, it has genus 1 and is, therefore, elliptic since the only singularity has δ-invariant $d=2$.

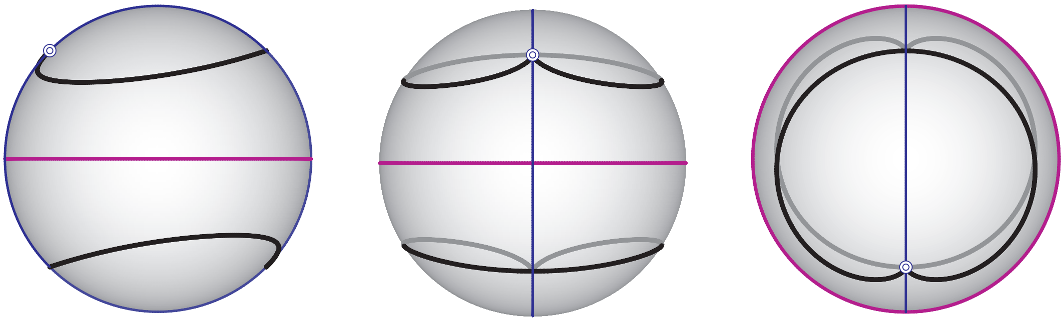

Figure 11: *From left to right: the right side view, the front view, and the top view of the spherical conchoid with cusp. The front and top view show triple points that are composed of cusps and linear branches.*

front view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : \pm\sin\delta : 0)$	[3,3,2]
S_3	$(0 : 1 : 0)$	[4,12,4]
$S_{4,5}$	$(0 : \pm i : \cos\delta)$	[2,1,2]
	$\mathrm{genus}(c) = 1$	

Table 5: *Singularities of the front view of the curve with cusp.*

The front view shows a pair of triple points. Here, the images of the top most points and the image of the focus F coincide. These triple points have δ-invariant $d = 3$ and branching number $b = 2$, cf. Table 5. Thus, these triple points are composed singularities, consisting of an ordinary cusp sitting on a linear branch. Further, there are two complex conjugate ideal singular points on the front view.

top view		
	$\deg(c) = 8$	
$S_{1,2}$	$(1 : \pm\cos\delta : 0)$	[3,3,2]
$S_{3,4}$	$(0 : 1 : \pm i)$	[2,1,2]
$S_{5,6}$	$(0 : \pm i\sin\delta : 1)$	[2,1,2]
$S_{7,8,9,10}$	$(1 : 0 : w)$	[2,1,2]
	$\mathrm{genus}(c) = 1$	

Table 6: *Singularities of the top view of the curve with cusp.*

Again, the top view shows more singularities then any other view. The two triple points (see Table 6) showing up are composed singularities of the same type as those in the front view. Furthermore, there are four ordinary double points (two real ones and a pair of complex conjugate) at $(1 : 0 : w)$ where w is a solution of the quartic equation

$$t^4 {s_\phi}^2 - t^2 \cos^2\delta(2 - \cos^2\delta) - \cos^4\delta = 0.$$

According to the genus formula the front and top view are of genus 1, and thus, elliptic. □

There is a special type of spherical conchoid if we choose $\delta = \frac{\pi}{2}$. In this case the conchoid construction assigns to each point $L \in l$ the absolute polar point, *i.e.*, the *orthogonal point*. Hence, the two branches to $\delta = -\frac{\pi}{2}$ and to $\delta = \frac{\pi}{2}$ are identic since opposite points represent the same point. All the three principal views of *orthogonal conchoids* are curves of degree four. Figure 12 shows an axonometric view of some orthogonal conchoids together with the three principal views of them.

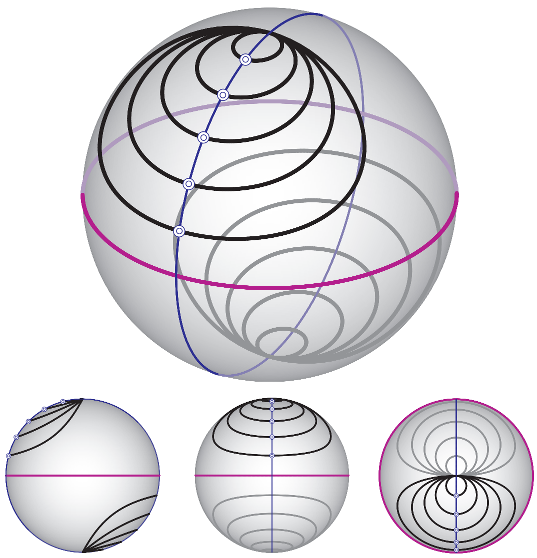

Figure 12: *Above: Some orthogonal conchoids of the equator. Below: Right side view, front view, and top view of some orthogonal conchoids.*

The curves in the right side view are two-fold hyperbolae in a pencil of the second kind with the images of the north and south pole as well as the ideal point of the x-axis for the base points.

2.2 Constructive approach

2.2.1 Planar and spherical tangents

The kinematic generation of conchoids allows us to construct tangents to conchoids in the plane, see for example [14]. The same holds true in the spherical case, cf. [6, 12].

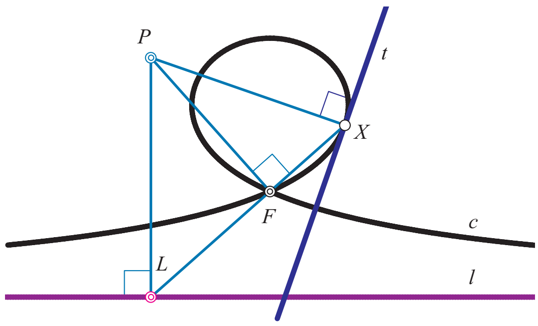

Figure 13: *The instantaneous pole P of the motion of the line $[L,X]$ with respect to the fixed system is found as the intersection of two normals.*

In Figure 13, the construction of the tangent to the planar conchoid c at some point X is shown. The kinematic generation of the curve shows the way: In order to find the instantaneous pole P of the motion of the line $[L,F]$ we observe that L is gliding on the line l, and thus, the pole of the motion of $[L,F]$ with respect to the fixed system l is the ideal point of the lines orthogonal to l. Since $[L,F]$ is gliding through F and rotating about F at the same time the instantaneous pole P is also contained in the line orthogonal to $[L,F]$ through F, see [14]. The construction also works at the double point since this is a singularity of the algebraic curve but not for the trace of X. The tangent t of c at X is orthogonal to $[P,X]$.

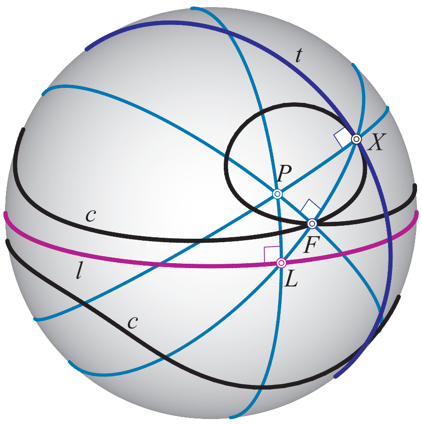

Figure 14: *The construction of the instantaneous pole P and the tangent t on the sphere.*

Figure 14 illustrates the construction of the tangent t to the spherical conchoid at some point X. Actually, the planar construction has to be translated into the spherical setting: We intersect the greatcircle orthogonal to the equator l through the point L with that greatcircle through F that is orthogonal to the greatcircle joining L and F and obtain the instantaneous spherical pole P (actually a pair of antipodal points). The spherical normal of the conchoid at X is the great circle joining X and P. Finally, the spherical tangent t is the greatcircle orthogonal to the spherical normal through the point X.

2.2.2 Planar and spherical osculating circles

Figure 15 shows the construction of the osculating circle o at a generic point X on a planar conchoid c. We use Bobillier's construction (see [14]). For that purpose we have to find two pairs of assigned points of the quadratic transformation that maps a point U to its center of curvature $U^{\star}$. The point L is moving on a straight line l, and thus, the center of its path is the ideal point $L^{\star}$ of all lines orthogonal to l. Further, we observe that the line $[L,F]$ is rotating about F while gliding through F. Thus, F is the envelope of $[L,F]$ and $F = A^{\star}$ is the center of curvature for the trace of the ideal point $A = [L,F]^{\perp}$ of all lines orthogonal to $[L,F]$. The two pairs $(L,L^{\star})$ and $(A,A^{\star})$ uniquely define the *quadratic curvature mapping*.

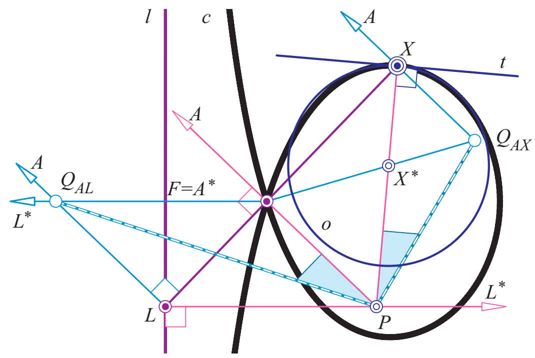

Figure 15: *Bobilier's construction simplifies in the case of the conchoid.*

Now, we can apply Bobbilier's construction to any of the pairs $(L,L^{\star})$ or $(A,A^{\star})$ in order to complete $(X,X^{\star})$ with the yet unknown point $X^{\star}$. Note that $[L,A] \cap [L^{\star},A^{\star}] =: Q_{AL}$ defines an auxiliary line $q_{AL} := [Q_{AL},P]$ with the property $\sphericalangle(q_{AL},p) = \sphericalangle(q_{AX},p)$ (after proper orientation), see [14], where p is the pole tangent, *i.e.*, the common tangent to the two polhodes at P.

In the case of the conchoid it is not necessary to construct the pole tangent p since we only have to add an angle as shown in Figure 15. On the auxiliary line q_{AX} we find the point $Q_{AX} := [A,X] \cap q_{AX}$, and finally, $X^{\star} = [X,P] \cap [A^{\star},Q_{AX}]$.

In order to find the spherical osculating circle o (as shown in Figure 16) we translate all the constructions done in the planar case to the sphere. We are allowed to do this since the quadratic curvature mapping can be lifted to the sphere. We consider the Euclidean unit sphere to be placed such that it touches the Euclidean plane (carrying the planar figure) at the instantaneous pole P. Then, we perform a gnomonic projection from the plane to the sphere. The center of the projection is the center of the sphere, and thus, the projectively extended Euclidean plane is mapped to the sphere model of projective geometry. The gnomonic projection is locally (around P) conformal, and therefore, the quadratic curvature mapping is lifted to that on the sphere.

Figure 16 shows the construction of the spherical center of curvature. At this point we shall remark that the spherical osculating circle o is not a greatcircle on Σ, except in those cases where X is a spherical point of inflection. The spherical radius of curvature equals the spherical distance of X and ist center of curvature $X^\star$.

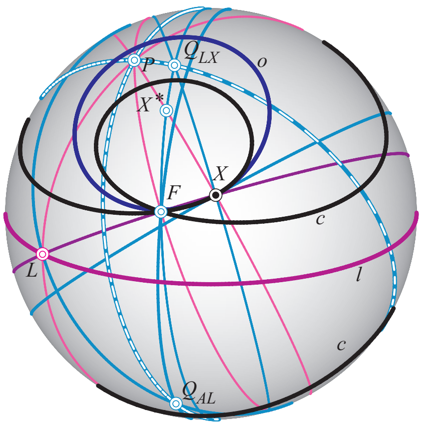

Figure 16: *The spherical version of Bobillier's construction yields the spherical center of curvature $X^\star$ for an arbitrary point X on the spherical conchoid.*

3 Conchoids of a circle

The construction of a conchoid is independent of the choice of the directrix curve. If we replace the line l by a circle, we obtain the conchoids of circles. The analytic as well as the constructive treatment of conchoids of circles does not differ that much from the affore mentioned types of conchoids. Since circles can also be found on a sphere, we can also find conchoids of circles on the sphere. We will not discuss the conchoids of a circle in the plane and on the sphere in all details. We shall just show that the equations of these special spherical curves can be derived in a similar way.

Conchoids of a circle in the Euclidean plane are of algebraic degree 6. Surprsingly, their spherical counter parts are of algebraic degree 8 (or, equivalently, of spherical degree 4), although we would expect them to be of degree 12. Some spherical conchoids of a circle are displayed in Figure 17.

The computation of an equation of spherical conchoids slightly differs from that of spherical conchoids of (spherical) lines.

Again, we assume that the focus F lies in $y=0$ at latitude $\phi \in [0, \frac{\pi}{2}[$. It means no restriction to assume that F is a point on the upper hemisphere. There is a change in the directrix l which shall henceforth be the circle of latitiude $\beta \neq 0, \frac{\pi}{2}$. Thus, the directrix is given by

$$L(\lambda) = (c_\beta c_\lambda, c_\beta s_\lambda, s_\beta) \text{ with } \lambda \in [0, 2\pi[\tag{7}$$

(with $c_\beta := \cos\beta$ and $s_\beta := \sin\beta$). Here, we should remark that this restricts the class of spherical conchoids of a circle. In this case, there exists a greatcircle through F in a plane parallel to the plane of l which, in general, needs not be true. However, we deal with the simpler type.

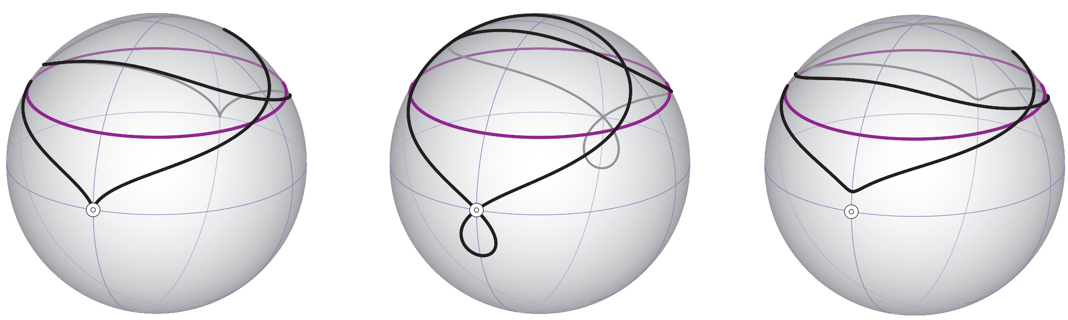

Figure 17: *Spherical conchoids of a circle show cusps, and two types of double points.*

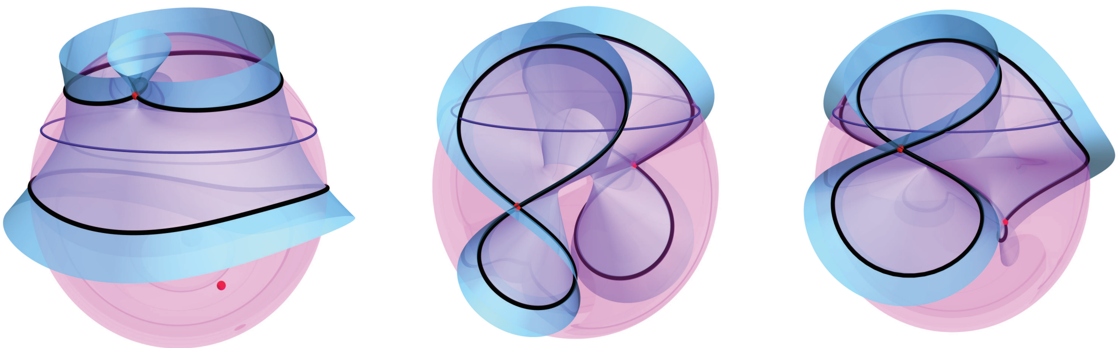

Figure 18: *Spherical conchoids as intersections of a quartic and the unit sphere.*

Let $X = (x,y,z)$ be the point on the conchoid of l with respect to F at spherical distance $\delta \in [0, \frac{\pi}{2}[$. Note that X is also a point on the unit sphere, and therefore, $x^2+y^2+z^2 = 1$ holds. The collinearity condition of F, X, and L from Eq. (4) now changes to

$$s_\phi s_\lambda\, x + (c_\phi t_\beta - c_\lambda s_\phi)y - c_\phi s_\lambda\, z = 0 \tag{8}$$

with $t_\beta := \tan\beta$. Between the point $l(t)$ on the directrix and the point X on the conchoid we measure the spherical distance δ which is a value with sign. Consequently, Eq. (5) modifies to

$$c_\lambda c_\beta\, x + s_\lambda c_\beta\, y + s_\beta\, z = \cos\delta. \tag{9}$$

Like in the case of the spherical conchoids of lines, we solve the system of linear equations (8), (9) with respect to c_λ and s_λ. Since ${c_\lambda}^2 + {s_\lambda}^2 = 1$ for all $\lambda \in \mathbb{C}$, we have the following two equations that have to be satisfied by the coordinates of a point on the spherical conchoid c of a circle l:

$$c: \begin{cases} \begin{aligned} &(2{c_\phi}^2-1)x^2 - {s_\phi}^2 y^4 \\ &+({c_\phi}^2-2{s_\phi}^2)x^2y^2 \\ &+2c_\phi s_\phi(x^2z+y^2)x \\ &-4c_\phi s_\phi s_\beta \cos\delta(y+x)y^2 \\ &+2s_\beta\cos\delta(2{c_\phi}^2-1)x^2z \\ &-2{s_\phi}^2 s_\beta\cos\delta y^2 z \\ &+((\cos^2\delta+{s_\beta}^2)(1-2{c_\phi}^2)-{c_\phi}^2)x^2 \\ &+(\cos^2\delta(1+2{c_\phi}^2)+{s_\phi}^2{s_\beta}^2)y^2 \\ &-2c_\phi s_\phi(\cos\delta^2+{s_\beta}^2)xz \\ &+2c_\phi s_\beta\cos\delta(2s_\phi x - c_\phi z) \\ &{c_\phi}^2(\cos^2\delta+{s_\beta}^2) = 0, \\ &x^2+y^2+z^2 = 1. \end{aligned} \end{cases} \tag{10}$$

From that we can infer in analogy to Theorem 1:

Theorem 3. *The spherical conchoids of a circle at latitude β with respect to a point F is an algebraic curve of degree* 8 *or of spherical degree* 4. *The coordinates of all points on the spherical conchoid fulfill Equation* (10).

The spherical conchoid of a circle is the intersection of a quartic surface with the sphere Σ. Some examples of the quartic surface are displayed in Figure 18. Like in the case of spherical and planar conchoids of lines, the spherical conchoids of circles can have cusps, isolated, and ordinary double points, see Figure 17.

Equations of the principal views (right side view, front view, top view) can be easily derived by eliminating coordinates (y, x, z) from the two equations given in Eq. (10). It is not necessary to go into all the details of the computations and discussions. They are similar to those in the previous section. Now, we can state (cf. Theorem 2):

Theorem 4. *The front and top view of spherical conchoids of circle are algebraic curves of degree* 8 *and genus* 1, i.e., *they are elliptic. The right side view is an elliptic quartic.*

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