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EXISTENCE OF SUPER CHAOTIC ATTRACTORS IN A GENERAL PIECEWISE SMOOTH MAP OF THE PLANE

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ABSTRACT

In this paper we give some rigorous conditions for the occurrence of super chaotic attractors in a general piecewise smooth map of the plane.

KEY WORDS

piecewise smooth map, super chaotic attractor, rigorous proof of chaos, Lyapunov exponents

CLASSIFICATION

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INTRODUCTION

There are many works that focused on the topic of chaotic behaviours of a discrete mapping. For example it has been studied from a control and anticontrol (chaotification) schemes or from the use of Lyapunov exponents [1-4], or by the use of several modified versions of the Marotto theorem [5], to prove the existence of chaos in n-dimensional dynamical discrete system, where the results in some way are to making an originally non-chaotic dynamical system chaotic, or enhancing the existing chaos of a chaotic system.

Robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighbourhood of the parameter space, since the existence of these windows in some chaotic regions imply that with small changes of the parameters would destroy the chaotic behaviour. This effect implies the fragility of this type of chaos. Contrary to this situation, there are many practical applications as in communication and spreading the spectrum of switch-mode power supplies to avoid electromagnetic interference [2-9], where it is necessary to obtain reliable operation in the chaotic mode where the robustness of chaos is required. A practical example can be found from electrical engineering to demonstrate robust chaos as shown in [6]. If all Lyapunov exponents are positive throughout the range, then the resulting attractors are called super-chaotic attractors. The importance of these attractors is that are more non-regular, and the iteration points are seemingly "almost" full of the considered space, which explains one of applications of chaos in fluid mixing, for example, refer to [7, 8]. A super-chaotification (or hyper-chaotification) scheme by making all Lyapunov exponents of a controlled dynamical system positive via the controller of some simple triangular function is given in [2].

In this paper, we shall determine rigorously a range in the parameters space for the existence of super chaotic attractors in a general 2-dimensional piecewise smooth discrete mapping, using an equivalence relation between the two matrices of the corresponding normal form of the considered map, and hence we compute analytically and assure the positivity of all the Lyapunov exponents. Since many practical applications require piecewise smooth map under discrete modeling [1, 6, 10], we consider a general two-dimensional piecewise smooth map $f(x, y; \rho)$, which depends on a single parameter ρ. Let Γ_ρ, given by $x = h(y, \rho)$ denotes a smooth curve that divides the phase plane into two regions R_1 and R_2. The map is given by:

$$f(x, y; \rho) = \begin{cases} f_1(x, y; \rho), & \text{if } x, y \in R_1, \\ f_2(x, y; \rho), & \text{if } x, y \in R_2. \end{cases} \tag{1}$$

For obtaining the general conditions of occurrence of super-chaotic attractors, it is assumed that the functions f_1 and f_2 are both continuous and have continuous derivatives. The map f is continuous but its derivative is discontinuous at the line Γ_ρ, called the "border". It is further assumed that the one-sided partial derivatives at the border are finite, and in each subregions R_1 and R_2 the map (1) has one fixed point P_1 and P_2, for certain value ρ_* of the parameter ρ. A normal form theory for border-collision bifurcations of two-dimensional piecewise smooth maps has been developed in [10]. Obviously, it has been shown that it is possible to choose an appropriate coordinate transformation so that the choice of axis is independent of the parameter. In so doing, the normal form of the map (1) is given by:

$$N(x, y) = \begin{cases} \begin{pmatrix} \tau_1 & 1 \\ -\delta_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu, & \text{if } x < 0, \\ \begin{pmatrix} \tau_2 & 1 \\ -\delta_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu, & \text{if } x > 0. \end{cases} \tag{2}$$

where μ is a parameter and τ_i, δ_i, i = 1, 2 are the traces and determinants of the corresponding matrices of the linearised map in the two subregions R_1 and R_2 evaluated at P_1 and P_2, respectively.

Assume that:

$$\tau_1 > 1 + \delta_1, \text{ and } \tau_2 < -(1 + \delta_2). \tag{3}$$

Then there are no fixed points for the map (1) when $\mu < 0$, and there are altogether two fixed points, one in R_1 and the other in R_2, for $\mu > 0$, given by:

$$P_1 = \left(\frac{\mu}{1-\tau_1+\delta_1}, \frac{-\mu\delta_1}{1-\tau_1+\delta_1}\right), \left(\frac{\mu}{1-\tau_2+\delta_2}, \frac{-\mu\delta_2}{1-\tau_2+\delta_2}\right). \tag{4}$$

Since as the parameter μ is varied through zero, the local bifurcation of the map (1) depends only on the values of τ_i and δ_i, (i = 1, 2), then it suffices to study the bifurcations in the normal form (2), and it is shown in [6] that the map (1) has a robust chaotic attractor for some parameter space region, when in first the condition (3) is verified. On the other hand, it is also shown in [6] that there is periodic attractor in the piecewise smooth map of the form (2) when $\tau_1 > 1 + \delta_1$, and $-(1 + \delta_2) < \tau_2 < (1 + \delta_2)$, thus, we exclude this region from our analysis when looking for super chaotic behaviour.

MAIN RESULT

In this section, a rigorous proof for the occurrence of super chaotic attractors in the piecewise smooth map (1) is given, using an equivalence relation between the two matrices of the corresponding normal form (2), and hence we compute analytically all the Lyapunov exponents. The existence of fixed points P_1 and P_2 given in (4) imply that it is possible to write the map (1) under the normal form (2). Therefore, assume that

$$\delta_1 < \frac{\tau_1^2}{4} \text{ and } \delta_2 < \frac{\tau_2^2}{4}. \tag{5}$$

Then $\tau_1^2 - 4\delta_1 > 0$ and $\tau_2^2 - 4\delta_2 > 0$ and these inequalities imply that the eigenvalues in R_1 are $\lambda_{11} = (\tau_1 + \sqrt{\tau_1^2 - 4\delta_1})/2$, $\lambda_{12} = (\tau_1 - \sqrt{\tau_1^2 - 4\delta_1})/2$, while in R_2 the eigenvalues are $\lambda_{21} = (\tau_2 + \sqrt{\tau_2^2 - 4\delta_2})/2$, $\lambda_{12} = (\tau_2 - \sqrt{\tau_2^2 - 4\delta_2})/2$. On the other hand, there are many ways for realizing an equivalence between matrices A_1 and $A2$, with

$$A_1 = \begin{pmatrix} \tau_1 & 1 \\ -\delta_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \tau_2 & 1 \\ -\delta_2 & 0 \end{pmatrix}.$$

A simple way is to assume that their eigenvalues are equal. Suppose for example that $\lambda_{11} = \lambda_{21}$ and $\lambda_{12} = \lambda_{22}$, thus we obtain the following condition

$$2(\delta_1 + \delta_2) - \tau_1\tau_2 = -\sqrt{\tau_1^2 - 4\delta_1} \cdot \sqrt{\tau_2^2 - 4\delta_2}. \tag{6}$$

Hence equation (6) has a solution if $2(\delta_1+\delta_2) - \tau_1\tau_2 < 0$, or

$$\delta_2 < \frac{\tau_1\tau_2 - 2\delta_1}{2}, \tag{7}$$

so that equation (6) becomes

$$\delta_2^2 + (-\tau_1\tau_2 - 2\delta_1 + \tau_1^2)\delta_2 + \tau_2^2\delta_1 + \delta_1^2 - \tau_1\tau_2\delta_1 = 0, \tag{8}$$

with the discriminant $\Delta = (\tau_2 - \tau_1)^2(\tau_1^2 - 4\delta_1) > 0$. Since $(\tau_1^2 - 4\delta_1) > 0$, solutions of (8) with respect to δ_2 are

$$\begin{aligned} \delta_2^{(1)} &= \frac{\tau_1\tau_2 + 2\delta_1 - \tau_1^2 + \sqrt{(\tau_2-\tau_1)^2(\tau_1^2-4\delta_1)}}{2}, \\ \delta_2^{(2)} &= \frac{\tau_1\tau_2 + 2\delta_1 - \tau_1^2 - \sqrt{(\tau_2-\tau_1)^2(\tau_1^2-4\delta_1)}}{2}. \end{aligned} \tag{9}$$

For brevity, we consider only the case $\delta_2 = \delta_2^{(1)}$, the case $\delta_2 = \delta_2^{(2)}$ being quite similar.

Condition (7) with $\delta_2 = \delta_2^{(1)}$ gives the following inequality

$$\delta_1 < \frac{2\tau_1\tau_2 - 4\tau_2^2}{2}, \tag{10}$$

while condition (5) with $\delta_2 = \delta_2^{(1)}$ leads to

$$\sqrt{(\tau_2-\tau_1)^2(\tau_1^2-4\delta_1)} < -\left(2\delta_1 + \tau_1\tau_2 - \tau_1^2 - \frac{\tau_2^2}{2}\right), \tag{11}$$

with the additional condition

$$\delta_1 < \frac{-\tau_1\tau_2 + \tau_1^2 + \frac{\tau_2^2}{2}}{2}, \tag{12}$$

so that solution of the inequality (11) is possible. It is easy to verify that the inequality (11) still holds for all values of τ_1, τ_2, and δ_1, thus we consider only the condition (12) for this case.

Condition (3) with $\delta_2 = \delta_2^{(1)}$ gives

$$\delta_1 < \frac{\tau_1^2 - \tau_1\tau_2 - 2\tau_2 - 2}{2}. \tag{13}$$

Thus inequalities (3), (10), (12) and (13) imply that

$$\delta_1 < \min\left(\frac{2\tau_1\tau_2 - \tau_2^2}{4}, \frac{-\tau_1\tau_2 + \tau_1^2 + \frac{\tau_2^2}{2}}{2}, \frac{\tau_1^2 - \tau_1\tau_2 - 2\tau_2 - 2}{2}, \tau_1 - 1\right). \tag{14}$$

We remark that

$$\frac{2\tau_1\tau_2 - \tau_2^2}{4} - \frac{-\tau_1\tau_2 + \tau_1^2 + \frac{\tau_2^2}{2}}{2} = -\frac{(\tau_2-\tau_1)^2}{2} < 0.$$

because of what the condition (14) transforms into

$$\delta_1 < \min\left(\frac{2\tau_1\tau_2 - \tau_2^2}{4}, \frac{\tau_1^2 - \tau_1\tau_2 - 2\tau_2 - 2}{2}, \tau_1 - 1\right). \tag{15}$$

Note that $\lambda_{11} = \lambda_{21} > 1$, if (3) holds. i.e., $\delta_1 < \tau_1 - 1$, and $\lambda_{12} = \lambda_{22} < -1$, if and only if:

$$\delta_1 < -(\tau_1 + 1). \tag{16}$$

Lyapunov exponents of the map (1) in the region R_1 are $\varpi_1(X_0) = \ln|\lambda_{11}|$, and $\varpi_2(X_0) = \ln|\lambda_{12}|$, and in the region R_2 the Lyapunov exponents are $\eta_1(X_0) = \ln|\lambda_{21}|$ and $\eta_2(X_0) = \ln|\lambda_{22}|$, for all $X_0 \in R_2$, thus, according to (3) and (16) we obtain:

$$\varpi_1(X_0) = \eta_1(X_0) = \ln|\lambda_{11}| > 0 \text{ and } \varpi_2(X_0) = \eta_2(X_0) = \ln|\lambda_{12}| > 0 \tag{17}$$

Finally, the Lyapunov exponents are identical in both regions R_1 and R_2, then the map (1) has a super chaotic attractor when conditions (9), (15) and (16) are verified.

The main result of this article is now given.

Theorem 1. Consider the piecewise smooth map (1) written in the normal form (2), and assume the following:

(1) $\delta_1 < \min\left(\dfrac{2\tau_1\tau_2 - \tau_2^2}{4},\ \dfrac{\tau_1^2 - \tau_1\tau_2 - 2\tau_2 - 2}{2},\ \tau_1 - 1,\ -\tau_1 - 1\right).$

(2) $\delta_2 = \dfrac{\tau_1\tau_2 + 2\delta_1 - \tau_1^2 + \sqrt{(\tau_2 - \tau_1)^2(\tau_1^2 - 4\delta_1)}}{2}.$

Then the map (1) converges to a super chaotic attractor.

ELEMENTARY EXAMPLE

Let us consider the following piecewise linear map of the plane:

$$f(x,y) = \begin{cases} \begin{pmatrix} 1 & a \\ -b & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0,2 \end{pmatrix}, & if\ x < 0, \\ \begin{pmatrix} 1 & -a \\ b & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0,2 \end{pmatrix}, & if\ x \geq 0. \end{cases} \tag{18}$$

It is easy to verify that the map (18) is a special case of the map (1) with all its assumptions given in this paper. Indeed, using the main theorem of this article we find that if $a = 1{,}2$ then a portion of the range for the occurrence of super chaotic attractor is $b < -1{,}6667$, and in this case the Lyapunov exponents are

$$\varpi_1(X_0) = \ln\left|\frac{1}{2}\sqrt{1-4{,}8b} + \frac{1}{2}\right| > 0, \text{ and } \varpi_2(X_0) = \ln\left|\frac{1}{2} - \frac{1}{2}\sqrt{1-4{,}8b}\right| > 0.$$

On the one hand, the super chaotic attractor and the bifurcation diagram of the map (18) are shown in Fig. 1a) and Fig. 1b), respectively. On the other hand, we remark that the given attractor is more non-regular, and the iteration points are seemingly "almost" full of the considered space, which explains an application of chaos in fluid mixing, for example, see [7, 8].

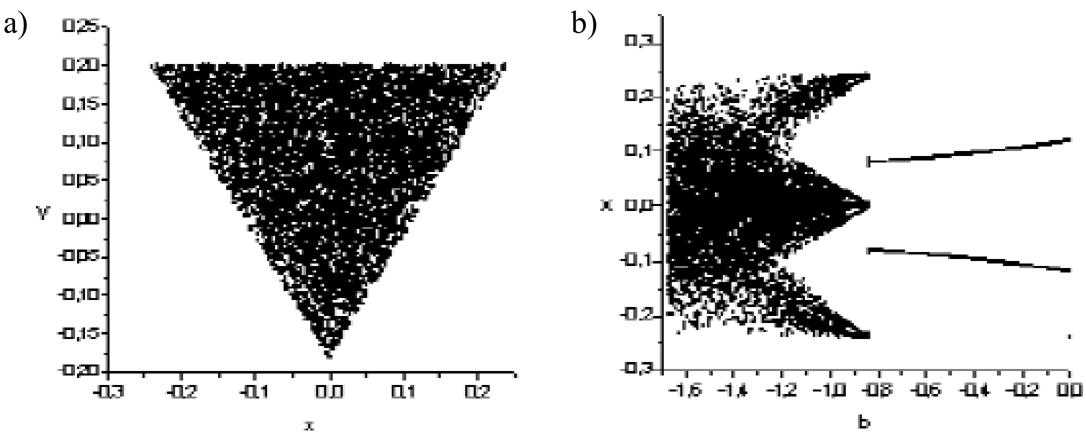

Figure 1. a) The super chaotic attractor obtained for $a = 1{,}2$, $b = -1{,}6668$, and the initial condition $x = y = 0{,}001$. b) The border collision bifurcation for the map (18) for $-1{,}66685 < b < 0$ and $a = 1{,}2$.

CONCLUSION

We have reported some analytical results on the existence of super chaotic attractors in a general piecewise smooth map of the plane. An elementary example is also given and discussed.

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EGZISTENCIJA SUPERKAOTIČNIH ATRAKTORA U PO DIJELOVIMA GLATKOJ RAVNINI

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SAŽETAK

U radu su dani rigorozni uvjeti za pojavu superkaotičnih atraktora u po dijelovima glatkoj mapi ravnine.

KLJUČNE RIJEČI

po dijelovima glatka mapa, superkaotični atraktor, rigorozni dokaz kaosa, eksponent Lyapunova