# WORLD AS THE EMPTY SET

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What is a world? How many worlds are there? What is in common to all possible worlds? Is there a point to which everything that *is* can be reduced? And finally, is it possible for a world to perish, in what sense can a world come to its end? We will discuss these questions from the standpoint of elementary set theory supported by first–order logic.

## 1. *Set*

Set is a fundamental mathematical concept which means that we do not define it. It is considered that the meanings of such concepts are simply self–evident. Although we do not define the set itself we know how to distinguish one set from the other — we differentiate sets by their elements (members). So every set is determined by its elements, and (if sets are properly given) for every object it should be clear whether it is a member of some particular set (or not). Sets are usually denoted by capital letters and their members are placed inside of braces. Every object is an element of some set. If an object x is a member of certain set S, we symbolize that as $x \in S$.

**Definition 1** A set $S_1$ is a *subset* of a set $S_2$ ($S_1 \subseteq S_2$) iff[1] every member of $S_1$ is also a member of $S_2$.

**Definition 2** A set $S_1$ is *equal* to a set $S_2$ ($S_1 = S_2$) iff $S_1$ is a subset of $S_2$ and $S_2$ is a subset of $S_1$.

Therefore, sets are equal iff they have identical members.

**Axiom 1** *There is* a set $S$ which contains no members ($\exists S \forall x\ (x \notin S)$).[2]

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1 IFF = if and only if; *A if and only if B* means that A is true only if B is true and B is true only if A is true. So there is no A without B and no B without A.

2 Axioms are propositions that we accept as true, although we do not prove them. Perhaps we could choose to demonstrate the existence of such set, e.g. as intersection of two sets

We call this set the *empty set* and we use the symbol $\varnothing$ to designate it.
**Corollary 1** Empty set is a subset of every set (*def. 1*)[3]
**Theorem 1** There is only one empty set.
**Proof**: From the *definition 1* it follows that there is no set (other than the empty set itself) which is a subset of $\varnothing$. Consequently, there can be no set equal to the empty set (*def. 2*).[4]

**Example 1** A, B, C, D, E, F and G are some sets:

$A = \{3, 2, 1, 4\}$
$B = \{x: x \in \mathbb{N}\ \&\ x < 5\}$
$C = \{2, 4, 6, 8, 10...\}$
$D = \{x: x \in \mathbb{N}\ \&\ x < 0\}$
E= {x: x is a capital of some European state}
F= {Ljubljana, Rome, Wien, Zagreb}
G= {1, k, Tuesday, ship}

As we can see, sets can be finite and infinite (C). Example of two equal sets are A and B. D is the set of all natural numbers less than zero. Since there is no such natural number, we can conclude that D is the empty set. Further, sets E and F are in a subset relation ($F \subseteq E$). G is also one very interesting set; it is obvious that members of different sets can be various, but we can also place some unrelated objects into the same set. In these cases it is just important to list all elements or the set will not be properly given. So, there can be no such set as {1, k, Tuesday, ship…} because it is not evident which additional objects could belong to that set.[5]

Next we will show how this elementary set theory can be related to a theory of worlds.[6]

with no common elements, but it is very important for us to stress that the empty set does not presuppose the existence of any other set; moreover, we will show that every other set is built on the assumption of its existence. Hence, if there is any set at all, then there must be also the empty set.

3 We can ask ourselves how every member of $\varnothing$ is a member of any other set when we know that the empty set has no members. But if we converse the question, and ask whether there is any member of $\varnothing$ which is not in every other set, it is clear that there is no such object and therefore the empty set is truly a subset of every set. Logicians would say that the proposition $\forall x \forall S\ (x \in \varnothing \rightarrow x \in S)$ is true because the antecedent of this implication is necessarily false.

4 Perhaps it remains unclear, what is really so special about the empty set? For example, is it not so that the set of natural numbers is also a unique one? Of course it is, but the empty set is the only set with no members, and there are infinitely many one, two, three (etc.) member sets. It is also evident that there are infinitely many sets which share the cardinality of $\mathbb{N}$; for instance — set of all even numbers, set of all primes, set of all integers ...

5 We could object to this condition of “set clarity” for if we, for example, want to establish the set of all big trees, it is not obvious which tree would belong to that set and which would not. Does that mean that it is impossible to have such set? No, but it does mean that we would need a proper definition of a big tree. Obviously, here the problem does not arise so much from the set concept itself, rather from the lack of other terms’ definition.

6 More on the key concepts of the set theory (intersection, union, cardinality ...) can be found in:

## 2. *World*

**Definition 3** A *world* is:

a) a set of first–order objects — domain (D), and

b) a set of all true propositions regarding elements of D (T)

Logic differentiates many classes of objects and when we speak of first–order objects we think of concrete entities — of this Socrates, of this chair, of this particular tree.[7] A question remains, is it possible that any set of objects constitutes a world? If we accept such a standpoint, we also accept the idea that there are many subworlds of one world, which in some way helps us to better understand reality, but on the other hand leaves us at risk of constituting a whole world from every single object. Now, concerning condition *b*, it is most certain that the set of all true propositions must be wider than it is suggested. For instance, statements like *Plato wrote dialogues on Socrates' teaching* or *Courage is a virtue* are definitely true for this world although neither Plato, nor Socrates are any longer its elements and something as courage cannot even be a member of such determined domain. So we will consider this as one working definition. Nevertheless, we can accept that each world is a set of certain objects which are bearers of some distinct characteristics, but it is evidently much more than just that. We have listed necessary conditions for something to be called a world, however they are not sufficient.

**Example 2**

a) D = {Socrates, Plato, Aristotle}

T = {Socrates is a publisher, Plato supports democracy, Aristotle follows Plato's teaching, Socrates is immortal but Plato and Aristotle are not ...}

b) D = {x: x is human}

T = {Everyone is happy, Everyone is healthy, Everyone helps someone, Someone helps everyone, Everyone is a friend of everyone ...}

These are examples of two very distant, but possible worlds. The first world is a simple one, it contains only three elements and members of the second world are all objects with an attribute of being human. We have left

— Lipschutz, Seymour: Set Theory and Related Topics (Schaum's Outline Series), McGraw–Hill Companies 1998 (2nd ed.)

— Bezhanishvili, Guram; Landreth, Eachan: An Introduction to Elementary Set Theory, http://www.cs.nmsu.edu/historical–projects/Projects/20920110331SetTheoryRevised.pdf (30.01.2013)

7 Second–order objects are considered to be properties and a third– or higher–order object is a property of a property. However, to make our point we will have to stay on this basic interpretation.

all these sets of true propositions unfinished, but it is also possible to reduce properties of one world to some finite number and then specify which members of D are the holders of some particular characteristic and which are not. For instance, we could assign only one possible property to the second domain — e.g. *being happy* and then the entire T could be reduced to the sentence *Everyone is happy.* But this would also mean that, in this world, the only accidental characteristic of people is happiness.

**Axiom 2** There are no two identical elements of D.

Therefore, if some objects $x$ and $y$ are elements of one world's domain then it necessarily follows that $x \neq$ y. Likewise, if some particular Socrates is an element of the world $P$, there can be no replication of him in that same world, but there is no obstacle for this Socrates to be a member of many other worlds.

This has brought us to the question of world plurality. By the definition itself, we are inclined to think that there can be more than just one world. For example, there could be two worlds with exactly the same set of objects but differing in the part of their true properties (e.g. in this world this table is brown, but in some other world it can be green, but still the same object). We will take a stand that there is only one existing world $E$ (which does not necessarily have to be the case) and numerously (even infinitely) many possible worlds — $P_1 \ldots P_n$ (of course, this existing world is also a possible one). However, it is obvious that $E$ is in persistent change concerning conditions $a$ and $b$ (e.g. people are getting born and die; sometimes we are reasonable, sometimes we are not). So it seems natural to conclude that the existing world is not some constant, rather it always becomes one of $P_1$ to $P_n$ and it is in every moment some other $P$ (although repetition is not excluded).[8]

Further, all objects included in these possible worlds are in fact members of some (super)set $S$ of all possibly existing things, and every $P$ (considering $a$) is a subset of that set. Now, we are particularly interested in one such subset (and we do know that it is a subset of $S$ for it is a subset of every

8 When we speak about contemporary views of possible worlds, names of three philosophers should definitely be mentioned — David Lewis (1941—2001), Alvin Plantinga and Saul Kripke. Simplified, the position of Lewis is that every possible world is also an existing one (*modal realism*); on the other side, Plantinga and Kripke consider that there is only one existing world (this actual one) and possible worlds are just our mind constructions of how things could have been or could be (*actualism*). Suitable introduction to the theories of possible worlds can be found in:
— Loux, Michael J.: Metaphysics *A contemporary introduction*; London, New York: Routledge 2002 (2nd ed.)
— Yagisawa, Takashi: Possible Objects, The Stanford Encyclopedia of Philosophy (Winter 2009 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/win2009/entries/possible–objects (30.01.2013)

conceivable set) — ∅, the empty set. Can the empty set become the set of objects for the existing world?

Before we try to answer this question we will take a look is there something what unites all these possible worlds no matter how different they actually are.

## 3. *Natural numbers as a "natural" domain*

We are going to introduce a common language for propositions of all worlds — the language of first–order logic (we will use abbreviation FOL).[9] This language is constructed of the following symbols:

| *Symbol(s)* | *Meaning* | *Example* |
|---|---|---|
| a, b, c…v, $a_1$, $b_1$, $c_1$…$v_1$, $a_2$… (individual constants) | first–order objects | **a** can stand for (this) Aristotle, **s** for (this) Socrates and further – **p**: Plato **$s_1$**: Seneca |
| w, x, y, z, $w_1$ , $x_1$, $y_1$, $z_1$, $w_2$… (individual variables) | arbitrary first–order objects (they are bounded by quantifiers) | |
| A, B, C...$A_1$, $B_1$, $C_1$...$A_2$… (predicate letters) | properties (properties can be defined concerning one or more objects) | Hx: x is healthy $H_1xy$: x is helping y Hp (Plato is healthy.) $H_1ap$ (Aristotle is helping Plato.) |
| ∃ followed by variable (existential quantifier) | some property holds for at least one member of the domain | ∃xHx (Someone is healthy.) $\exists y H_1yp$ (Someone is helping Plato.) |
| ∀ followed by variable (universal quantifier) | some property holds for every object of the domain | ∀xHx (Everyone is healthy.) $\exists y \forall z H_1yz$ (Someone is helping everyone.) |
| ¬ (negation) | absence of some property; negation of sentence | ¬Hp (Plato is not healthy.) |

9 For further insights on the language of first–order logic:
— Bergmann, Merrie; Moor, James; Nelson, Jack: The Logic Book, New York: McGraw–Hill Companies 2004 (4th ed.)
— Brachman, Ronald J.; Levesque, Hector J.: Knowledge Representation and Reasoning, San Francisco: Morgan Kaufmann Publishers 2004; slides from the book are available online at:
http://www.cs.toronto.edu/~hector/PublicKRSlides.pdf (30.01.2013.),
http://www.informatik.uni–leipzig.de/~brewka/lehre1/FOL.pdf (30.01.2013.)

| | | |
|---|---|---|
| & (conjunction) | and, but, although… | $\neg$Hp & $H_1$ap (Plato is not healthy but Aristotle is helping him.) |
| $\vee$ (disjunction) | or | $H_1$ap $\vee$ $H_1$pa (Either Aristotle is helping Plato or Plato is helping Aristotle.) |
| $\rightarrow$ (material conditional) | if…then | $\forall y(\neg Hy \rightarrow \exists x H_1 xy)$ (If there is anyone who is not healthy, then there is someone helping him.) |
| $\leftrightarrow$ (material biconditional) | if and only if (if *p* then *q* and if *q* then *p*) — IFF | $H_1$ap $\leftrightarrow$ $H_1$pa (Aristotle is helping Plato if and only if Plato is also helping Aristotle) |

**Definition 4** In the language of first–order logic a set of propositions is *satisfiable* iff there is an interpretation on which all members of the set are true.[10]

As we have pointed out, FOL uses small and capital alphabetical letters as symbols. When we give to those letters certain meaning, we give one possible interpretation of some sentence stated in that formal language.[11]

**Corollary 2** Every world is a satisfiable set of propositions.

Since every world is one interpretation on which all propositions of the set are true (*def. 3, b*), it is evident that every world is also a satisfiable set.

**Theorem 2** "If *F* is satisfiable, it is satisfiable in the domain of natural numbers."[12]

This is one formulation of so–called Löwenheim–Skolem theorem.[13] Although, in original text, *F* stands for a formula of the FOL, to us it will desig-

10 This means that a set is satisfiable if it is not possible to deduce a contradiction from it. For instance, the set {Everyone is happy, Socrates is not happy} is not a satisfiable one.

11 One sentence of the FOL can be interpreted in many ways, e.g. Cs can stand for *Socrates is courages*, but it can also stand for *Aristotle is reasonable*. So it is very important to specify the interpretation of the used letters. Vice–versa, one proposition of the natural language can be translated into FOL in different ways but it is important not to use the same letter to designate different properties (or objects) of the same world (although different letters can symbolize one property or object).

12 Van Heijenoort, Jean: From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931; Cambridge, London: Harvard University Press 1967, 253.

13 Leopold Löwenheim (1878—1957), German mathematician, originally stated this theorem: *"If a domain is at least denumerably infinite, it is no longer the case that a first order fleeing equation is satisfied for arbitrary values of the relative coefficients."* (Ibid. 235.) Slightly converted that would mean: "*If a first order proposition is satisfied in any domain at all, it is already satisfied in a denumerably infinite domain*." (Ibid., 293.) Thoralf Skolem (1887—1963), Norwegian mathematician, improved Löwenheim's research and proved the theorem using axiomatized set theory. One version of his reformulation of the theorem is: *"A (denumerable or nondenumerable) sum of infinitely many first–order propositions ei-*

nate a set. It is proven that the theorem holds for the sets of first–order propositions without identity.[14] Since there are no two identical objects in any of our worlds (*axiom 2*), there are no obstacles for us to apply it. This means that every single domain of these worlds is corresponding to the set of natural numbers and for each of these worlds we can construct an interpretation, using natural numbers as our domain, on which they remain satisfiable.

**Example 3**

We will show how this principle works on the worlds introduced in *example 2*:

a)

| *Proposition* | *(Possible) FOL translation* | *(Possible) interpretation for* $D=\mathbb{N}$ |
|---|---|---|
| Socrates is a publisher. | Ps | s: seven<br>Px: $x \in \{y: y = 2z-1 \ \& \ z \in \mathbb{N}\}$<br>(x is odd)<br>Seven is odd. |
| Plato supports democracy. | Sp | p: four<br>Sx: $x \in \{y: y=2z \ \& \ z \in \mathbb{N}\}$<br>(x is even)<br>Four is even. |
| Aristotle follows Plato's teaching. | Fap | a: ten<br>Fxy: $x > y$<br>(x is greater than y)<br>Ten is greater than four. |
| Socrates is immortal, but Plato and Aristotle are not. | Is & (¬Ip & ¬Ia) | Ix: $x \in \{y \in \mathbb{N}: (\forall z)\ [(z \in \mathbb{N} \ \& \ z \neq 1 \ \& \ y \neq z) \rightarrow y/z \notin \mathbb{N}]\}$<br>(x is a prime)<br>Seven is a prime, but four and ten are not. |

b)

| *Proposition* | *(Possible) FOL translation* | *(Possible) interpretation for* $D=\mathbb{N}$ |
|---|---|---|
| Everyone is happy. | $\forall x Hx$ | Hx: $x \rightarrow x+1$<br>Each natural number has its immediate successor. |

*ther is not satisfiable at all or is already satisfied in denumerably infinite domain for certain values of the relative symbols that occur in the propositions.*" (Ibid. 260)

14 Due to the complexity of this proof we will not demonstrate it here. Proofs of slightly different (more general) theorems which are, however, applicable to this one are presented in the cited book (Van Heijenoort, From Frege to Gödel). Appropriate online recourse for the proof:
Shapiro, Stewart: Classical Logic, *The Stanford Encyclopedia of Philosophy* (Winter 2009 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/win2009/entries/logic–classical/ (30.01.2013.)

| | | |
|---|---|---|
| Everyone is healthy. | $\forall x H_1 x$ | $H_1x$: $x \to x+1$<br>Each natural number has its immediate successor.[15] |
| Everyone helps someone. | $\forall x \exists y H_2 xy$ | $H_2xy$: $x \times y = z$, $z \in \{2w: w \in \mathbb{N}\}$<br>For each $x \in \mathbb{N}$ there is $y \in \mathbb{N}$ such that x multiplied by y is equal to some even number.[16] |
| Someone helps everyone. | $\exists x \forall y H_2 xy$ | There is some $x \in \mathbb{N}$ which multiplied by any natural number is equal to some even number. |
| Everyone is a friend of everyone. | $\forall x \forall y Fxy$ | Fxy: $x+y = y+x$<br>For each two natural numbers x,y— x plus y is equal to y plus x. |

Now we have seen how each possible world can be understood as some reflection of natural numbers, yet it is left to show how the empty set makes this whole story complete.

## 4. *Natural numbers as a reflection of the empty set*

First of all it is important to differentiate $\varnothing$ from $\{\varnothing\}$. Former is the empty set (which contains no elements) and the latter is the set containing the empty set as its member. So $\{\varnothing\}$ is one member set and more than just that, $\{\varnothing\}$ is *the one* itself.

As we previously mentioned, every natural number has its immediate successor (for one that is two, for two — three; generally, successor of *n* is equal to *n+1*.). Even more, we define natural numbers as a collection of these successors (and one).[17] In words of the set theory we can formulate the successor function as $s(n) = \{n\}$.[18] Therefore, every natural number is actually a set containing its predecessor as the element.[19]

15 See note 11.

16 Obviously, y can be any even number (and if x is even, y can be any natural number).

17 Italian mathematician and logician Giuseppe Peano (1858—1932) defined natural numbers through several axioms. One version of these axioms is:
1. $1 \in \mathbb{N}$
2. $\forall x\ (x \in \mathbb{N} \to x+1 \in \mathbb{N})$ If x is an element of $\mathbb{N}$, its successor is also an element of $\mathbb{N}$.
3. $\forall x\ (x \in \mathbb{N} \to x+1 \neq 1)$ One is not successor of any natural number. (One is the first element of $\mathbb{N}$.)
4. $\forall x \forall y\ ((x,y \in \mathbb{N}\ \&\ x+1 = y+1) \to x=y)$ Each element of $\mathbb{N}$ has its own successor.
5. $\forall S\ ((S \subseteq \mathbb{N}\ \&\ 1 \in S\ \&\ \forall x\ (x \in S \to x+1 \in S)) \to S = \mathbb{N})$ The last axiom (the axiom of mathematical induction) insures that every subset of $\mathbb{N}$ which shares that same characteristics is actually the set of natural numbers.

18 Guljaš, Boris: Matematička analiza 1&2, http://web.math.pmf.unizg.hr/~guljas/skripte/MATANALuR.pdf (30.01.2013), 2.

19 Set notation of natural numbers hails from Hungarian mathematician John (János) Von Neumann (1903—1957). Ordinal numbers were his basis. Thus, in his symbolization

**Example 4**

$s\,(1) = s\,(\{\emptyset\}) = \{\{\emptyset\}\} = 2$

$s\,(2) = s\,(\{\{\emptyset\}\}) = \{\{\{\emptyset\}\}\} = 3$

$s\,(6) = s\,(\{\{\{\{\{\{\emptyset\}\}\}\}\}\}) = \{\{\{\{\{\{\{\emptyset\}\}\}\}\}\}\} = 7$

$\mathbb{N} = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\{\{\emptyset\}\}\}\}\ldots\}$

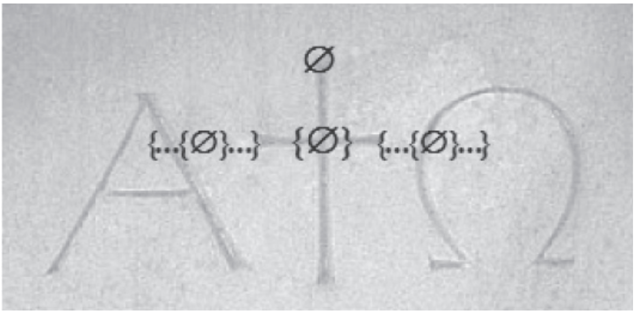
We have identified every natural number, except *one*, as a successor of some other natural number. So there is no 2 without 1, there is no 3 without 2. By simple syllogism it follows that there is no 3 without 1, and there is no any other natural number without 1.

And finally, what is *one*? *One* is the first natural number, and the only number without a predecessor. $\{\emptyset\}$ carries the code of all existence — the empty set.

## 5. *For the end (and a beginning)*

What does this all mean when we say that the world is coming to its end, when we imply that everything will just disappear, that reality is going to be reduced to the empty set? If that is about to happen, and our interpretation of the world is worth something, it simply means that everything is coming back to its origin, to its source and cause of existence.

If we are about to say that this origin is God himself, it definitely doesn't entail the reduction of God to nothing, but indicates that there is no part in Him, it indicates how He actually could create something out of nothing, and of course, if *theorem 1* is worth something, it undoubtedly indicates that there is only one God. [20]




$1=\{\emptyset\}$, $2 = \{\emptyset,\{\emptyset\}\}$, $3=\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ …
Generally, $s(n) = n \cup \{n\}$.

20 Background of the illustration has been downloaded from: http://www.derkleinegarten.de/800_lexikon/825_symbole/jesus_christus/auferstehung_lamm_gottes_hirte_fisch_sieger_schlange.htm (30.01.2013)

*Summary*

## WORLD AS THE EMPTY SET

The article discusses how the empty set reflects reality. Every possible world is observed as a set of objects which are holders of various properties. By applying the Löwenheim-Skolem theorem, every world is reduced to the set of natural numbers. Furthermore, it is shown that each natural number can be presented as a reflection of the empty set. Finally, the empty set concept is used to support some basic ideas of neoplatonic and Christian philosophy.

KEYWORDS: empty set, world, Löwenheim-Skolem theorem, natural numbers, creatio ex nihilo, God