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Stackelberg Equilibrium of the Client and the Producer of Embedded Software

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Abstract

Background: Our research assumes that the software quality affects the product validity. This assumption also refers to embedded software. **Objectives:** This paper analyses the Stackelberg equilibrium in which the consumer is the leader and the producer of embedded software is the follower. **Methods/Approach:** A comparative statics analysis of a producer's reaction is carried out and confirms our intuition that the product price is positively correlated to the number of employees and the software quality. **Results:** An increase in wage has an adverse effect on producer's reaction. Derived results are illustrated numerically and Stackelberg and cooperative equilibrium are compared. It is shown that the welfare loss is smaller with higher quality software for any number of employees. **Conclusions:** Although the equilibrium involves less employed workers, the optimal software quality is higher. The optimal product price is lower, which puts the consumer and the producer in a better position.

Keywords: embedded software, Stackelberg equilibrium, producer's reaction function, comparative statics analysis, welfare loss

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Introduction

Embedded systems are one of the technologies that have great market potential with the possible large areas of further improvement. Embedded systems consist on interconnected hardware and software, and are important emerging market (Schlett, 2000) with the great prospect for growth as well with number of possible implementations (Dey, 2010; BCC Research, 2012; Lakka et al., 2012).

Goal of the paper is to find equilibrium in the Stackelberg model in which the relationship between the client and the producer of embedded sofwer is analysed (Bierman, 1993; Jehle, 2001; Mas-Collel, Whinston, Green, 1995). In the Stackelberg leadership model the follower firm acts like in the Cournot model and for the given level of production of the leader firm determines its own production level which maximizes economic profit (Simaan, Cruz, 1973; Fudenberg, Tirole, 1992; Osborne, Rubinstein, 1994). The leader firm chooses quantity knowing that the follower firm will react to it according to its reaction function. The product quality affects the likelihood of product safety and the consumers' utility but depends on the invested effort. The goal is to find equilibrium in the Stackelberg model where the client is the leader, and the producer is the follower.

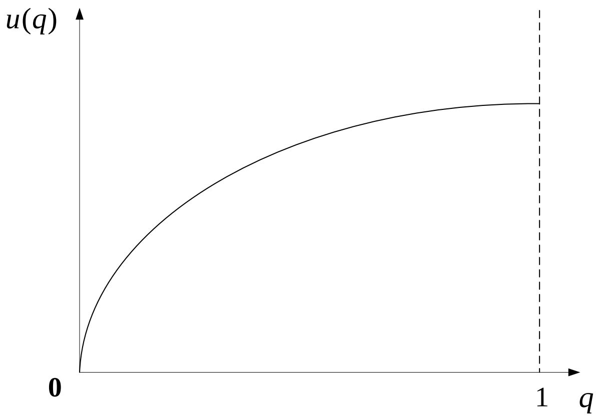
In the first step, the producer chooses the number of units of labour for the given price of a product in order to maximize the expected profit. In this way the producer indirectly affects the software quality. In the paper the effect of the product price change on producer's reaction is analysed. The comparative statics results confirm our intuition. The higher the product price, the greater the level of labour units employed by the producer and the better the product. Increase in labour costs has opposite effect on the producer's decision. In the second step the consumer determines the product price in order to maximize his or her expected utility. This decision is affected by the producer's reaction or the price effect on the product quality. The consumer's utility and the software quality are usually directly related in the existing literature. In this paper, this relationship is complemented by the probability of product safety.

Theoretical optimization models which are solved by the client and the producer are illustrated numerically and the total welfare in the Stackelberg and the cooperative equilibrium is compared. The effect of the change in relationship between the employed labour units and the software quality on the given results is analysed. According to our numerical example, the more favourable this relationship is, the higher is the client's and the producer's welfare in the Stackelberg equilibrium. Even though fewer workers are employed, produced software is better. The total welfare loss is smaller due to more dominant effect of more favourable relationship between employees and the software quality in the Stackelberg equilibrium than in the cooperative equilibrium. Given the sensitivity of the results, future research should be devoted to examining the functional relationship between given variables in specific phenomena. The cost of software maintenance could also be taken into account.

Model formulation

Let $q \in [0,1]$ denote the embedded software quality and let $\pi(q)$ denote the probability of product safety. $\pi(q)$ is an increasing function, and the following equalities hold $\pi(0) = 0$ and $\pi(1) = 1$. It is assumed that the preferences of a consumer who uses valid product are described by the quasilinear utility function $v(q, m) = u(q) + m$, where m is the composite commodity good. Expression $u(q)$ is then a concave function describing how software quality affects consumer's utility. It is assumed that there is no benefit from completely unprofessional software, $u(0) = 0$ which is illustrated in Figure 1.

Figure 1
Concave utility function of quality



Source: Authors

Quasilinear preferences enable monetary measurement of consumer's utility. The price which is payed by the consumer for valid product is denoted by p and the consumer's utility is equal to $u(q)-p.$ This is actually the change in utility in relation to consumer's wealth. Since for quasilinear preferences consumer's wealth doesn't affect his or her decisions, it is omitted from the analysis. For the product which does not work properly the consumer does not pay.

The consumer's expected utility is equal to

$$\pi(q)[u(q)-p]. \quad (1)$$

Let w denote the potential mass of wages which describes the amount that the producer would spend on wages when employing a working potential. Let $e \in [0,1]$ denote part of a working potential that the producer employs. Then the mass of wages is described by the expression we. The producer earns income only if the product works. This income is equal to the price of the product. The expected profit of the producer is described by the following expression

$$p\pi(q)-we. \quad (2)$$

It is assumed that the software quality is an increasing function of the employed labour units, $q=q(e)$. Furthermore we assume that the consumer or a client first determines the price of a product which the producer then takes into accounts and for given value of price maximizes the expected profit,

$$\max_{e\in[0,1]} p\pi(q)-we. \quad (3)$$

We seek the Stackelberg equilibrium where the consumer is the leader, and the producer is the follower. The producer's reaction function $e^*(p,w)$ describes the optimal number of labour units which maximizes producer's profit. By choosing the product price, the consumer indirectly affects the quality

$$q^*(p,w)=q\left[e^*(p,w)\right]. \quad (4)$$

The consumer takes into account the producer's reaction and maximizes his or her expected utility

$$\max_{p} \pi\left[q(e^{*})\right]\left\{u\left[q(e^{*})\right]-p\right\} \tag{5}$$

In this Stackelberg model the sensitivity of the problem's solution to changes in its parameters is examined.

Comparative static analysis

If the problem has interior solution, the producer derives the reaction function from the first order necessary conditions

$$p\frac{d\pi}{dq}[q(e)]\cdot\frac{dq}{de}(e)-w=0. \tag{6}$$

Second order necessary conditions are as follows

$$p\left[\frac{d^{2}\pi}{dq^{2}}\left(\frac{dq}{de}\right)^{2}+\frac{d\pi}{dq}\frac{d^{2}q}{de^{2}}\right]\leq 0. \tag{7}$$

First order necessary conditions can be understood as an identity

$$p\frac{d\pi}{dq}\left[q(e^{*})\right]\cdot\frac{dq}{de}(e^{*})-w=0. \tag{8}$$

By deriving it with respect to product price we get

$$\frac{d\pi}{dq}\frac{dq}{de}+p\left[\frac{d^{2}\pi}{dq^{2}}\left(\frac{dq}{de}\right)^{2}\frac{de^{*}}{dp}+\frac{d\pi}{dq}\frac{d^{2}q}{de^{2}}\frac{de^{*}}{dp}\right]=0. \tag{9}$$

From the first order necessary conditions it follows that the expression $\frac{d\pi}{dq}\frac{dq}{de}$ is equal to the real potential mass of wages

$$\frac{d\pi}{dq}\frac{dq}{de}=\frac{w}{p}. \tag{10}$$

It follows

$$\frac{w}{p}+p\frac{de^{*}}{dp}\left[\frac{d^{2}\pi}{dq^{2}}\left(\frac{dq}{de}\right)^{2}+\frac{d\pi}{dq}\frac{d^{2}q}{de^{2}}\right]=0. \tag{11}$$

Taking into account the second order necessary conditions leads to the following conclusion

$$\frac{de^*}{dp} > 0. \tag{12}$$

This comparative statics result confirms our intuition that the higher the product price, the more labor units the producer will hire which will produce high-quality software. Notice that $e^*(p,w)$ is a homogeneous function in p,w with the degree of homogeneity equal to zero. By applying Euler's theorem we get

$$p\frac{de^*}{dp} + w\frac{de^*}{dw} = 0. \tag{13}$$

It follows $\frac{de^*}{dw} < 0$. An increase in wages adversely affects the share of employees and indirectly the software quality.

Numerical example

If we normalize the potential mass of wages, $w = 1$, the product price, the expected consumer's utility and the expected producer's profit are then expressed in terms of the potential mass of wages. Let's assume that the function which expresses the influence of software quality on consumer's utility is represented by the following expression $u(q) = \sqrt{2q}$, the probability of product validity with embedded software is represented by the function $\pi(q) = q$ and the quality of software in relation to the part of hired working potential is represented by $q(e) = e^2$. In the first step the producer maximizes the expected profit

$$\max_{e\in[0,1]} pe^{\frac{1}{2}} - e\,. \tag{14}$$

The goal function is a concave function in the share of a hired working potential. If the problem has an interior solution, from the first order conditions represented by the following expression

$$\frac{1}{2}pe^{-\frac{1}{2}} - 1 = 0 \tag{15}$$

it follows $e^* = \frac{p^2}{4}$. Thus, the producer's reaction function is

$$e^* = \begin{cases} \frac{p^2}{4}, & p < 2 \\ 1, & p \geq 2\,. \end{cases} \tag{16}$$

It can be seen that in the relevant interval increase in price increases the share of hired labour units. The relationship between quality and the software price is described by the following

$$q^* = \begin{cases} \frac{p}{2}, & p < 2 \\ 1, & p \geq 2. \end{cases} \tag{17}$$

Thus, as long as the product price is not greater than twice the mass of potential wages, increase in prices increases software quality. In the second step, based on producer's reaction, consumer determines the price that maximizes his or her expected utility,

$$\max_{p\leq 2}\frac{p}{2}(p^{\frac{1}{2}}-p)=\frac{1}{2}p^{\frac{3}{2}}-\frac{1}{2}p^{2}. \tag{18}$$

From the first order necessary conditions represented by the following equality

$$\frac{3}{4}p^{\frac{1}{2}}-p=p^{\frac{1}{2}}(\frac{3}{4}-p^{\frac{1}{2}})=0, \tag{19}$$

it follows that the optimal product price which is determined by the consumer is $p^{*}=\frac{9}{16}=0{,}5625$. The producer's reaction is described by the optimal hired share of the working potential $e^{*}=\frac{81}{1024}\approx 0{,}079101562$ and by the optimal software quality $q^{*}=\frac{9}{32}=0{,}28125$. The expected consumer's utility is equal to $\frac{27}{512}\approx 0{,}052734375$ and the expected producer's profit is $\frac{81}{1024}\approx 0{,}079101562$.

Comparison of Stackelberg and cooperative equilibrium

In the cooperative equilibrium the total expected consumer's and producer's utility is maximized

$$\max_{e\in[0,1]}\pi(q)u(q)-we. \tag{20}$$

In the example from the previous section, this problem is

$$\max_{e\in[0,1]}q\sqrt{2q}-e=2^{\frac{1}{2}}q^{\frac{3}{2}}-e=2^{\frac{1}{2}}e^{\frac{3}{4}}-e. \tag{21}$$

The goal function in this optimization problem is concave and reaches its extreme when the producer hires all workers $e^{*}=1$. The software quality is then the highest $q^{*}=1$. The sum of expected utility of a consumer and expected profit of a producer is $\sqrt{2}-1\approx 0{,}414213562$. In Stackelberg equilibrium this sum was

$$\frac{27}{512}+\frac{81}{1024}=\frac{135}{1024}\approx 0{,}131835937.$$

Welfare loss in relation to the potential mass of wages is described by the difference $\sqrt{2}-1-\frac{135}{1024}\approx 0{,}282377625$. Consumer's utility in the cooperative equilibrium is represented by $u(1)-p=\sqrt{2}-p$. In order for a consumer to accept the cooperative equilibrium, this utility shouldn't be less than the expected utility in Stackelberg equilibrium,

$$\sqrt{2} - p \geq \frac{27}{512}. \quad (22)$$

Similarly, the producer accepts the cooperative equilibrium if his or her profit isn't less than the expected profit in Stackelberg equilibrium,

$$p - 1 \geq \frac{81}{1024}. \quad (23)$$

Thus, the cooperative equilibrium is described by the highest quality software and the price from an interval $p \in \left[1 + \frac{81}{1024}, \sqrt{2} - \frac{27}{512}\right] \approx [1{,}079101563{,}1{,}361479187]$. This interval describes the space for client-producer negotiation in the cooperative equilibrium. Its size is equal to the total welfare loss in Stackelberg equilibrium in relation to the cooperative equilibrium.

Now consider how the changes in employees-software quality relationship affect the obtained results. Let's assume that $q(e) = e^{\frac{1}{3}}$. Then, for each share of employees between 0 and 1, quality of embedded software is higher than in the previous case. For a given price of the product the producer maximizes the expected profit.
The producer's reaction function is

$$e^* = \begin{cases} \left(\frac{p}{3}\right)^{\frac{3}{2}}, & p \leq 3 \\ 1, & p > 3 \end{cases}. \quad (24)$$

The software's quality-price relationship is described by the following equality

$$q^* = \begin{cases} \left(\frac{p}{3}\right)^{\frac{1}{2}}, & p \leq 3 \\ 1, & p > 3 \end{cases}. \quad (25)$$

Based on producer's reaction consumer determines the price that maximizes his or her expected utility

$$p^* = 2^{-\frac{2}{3}} 3^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{12}} \approx 0{,}436790232 \quad (26)$$

The software price determined by the producer is lower. The optimal proportion of employed workers is $e^* \approx 0{,}055555555$ and the optimal software quality is $q^* \approx 0{,}381571414$. Although the proportion of employed workers is lower, software quality is higher. Expected consumer's utility is higher $q^*(\sqrt{2q^*} - p^*) \approx 0{,}166666666$ and higher is also expected producer's profit $p^* q^* - e^* \approx 0{,}111111111$. Let's notice that changes in obtained results in Stackelberg equilibrium are consequence of more favourable relationship between the software quality and proportion of employed workers.

In the cooperative equilibrium total consumer's and producer's expected utility is maximized

$$\max_{e\in[0,1]} 2^{\frac{1}{2}} e^{\frac{1}{2}} - e \,. \tag{27}$$

Optimal proportion of employees is $e^* = \frac{1}{2}$ whereas the optimal quality software is $q^* = \left(\frac{1}{2}\right)^{\frac{1}{3}} \approx 0{,}793700526$. In this case not all workers are employed and total expected welfare of consumers and producers is equal half the mass of potential wages. In Stackelberg equilibrium total consumer's and producer's welfare is approximately $0{,}166666666 + 0{,}111111111 = 0{,}277777777$. It can be seen that welfare loss is equal to $0{,}222222223$ of mass of potential wages. This welfare loss is lower due to more dominant effect of more favourable proportion of employees-software quality relationship in Stackelberg equilibrium than in cooperative equilibrium.

Conclusion

In this paper the relationship between the producer of embedded software and the consumer is analysed using Stackelberg leadership model in which the consumer is the leader and the producer is the follower. It is assumed that the software quality is affected by the number of employees and that the producer maximizes the expected profit. It is also assumed that the software quality affects the product validity. The comparative statics analysis confirms our intuition that the higher the product price, more workers are employed and better software is produced. Adverse effect on producer's reaction has an increase in wage. Based on producer's reaction, the consumer determines the product price which maximizes his or her expected utility.

Derived results are illustrated by a numerical example in which Stackelberg and cooperative equilibrium are compared and the welfare loss is determined. This loss is smaller with higher quality software for any number of employees. Although less employed workers are involved in equilibrium, optimal software quality is higher. Optimal product price is lower what brings the consumer and the producer in better position. Future research will be devoted to derivation of general relationship between the software quality, probability of software validity, number of employees and the product price where the cost of software maintenance should also be taken into account.

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