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PARAMETER ESTIMATION AND ACCURACY ANALYSIS OF THE FREE GEODETIC NETWORK ADJUSTMENT USING SINGULAR VALUE DECOMPOSITION

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Subject reviews
In order to determine the vector of the coordinates of the free geodetic networks by the least square adjustment, it is necessary to solve a singular system of linear equations and to find the inverse of a singular matrix. It is well known that there is a unique inverse of a regular square matrix. The main question is how to find the inverse of a singular matrix. This paper describes the traditional approach to the solving of the singular problem in opposition to the singular value decomposition (SVD). To exemplify, a free two-dimensional (2D) geodetic network was used to obtain and compare both the traditional and the SVD solutions. The analysis of the two approaches to the problem solving is of particular interest to this work. The advantages of the SVD approach are summarized and suggested to be used more frequently.

Keywords: *geodetic networks, least square method, linear systems, singular value decomposition*

Ocjena parametara i analiza točnosti izravnanja geodetske mreže pomoću dekompozicije vlastitih (karakterističnih) vrijednosti

Pregledni rad
U cilju određivanja vektora nepoznatih parametara izravnanjem slobodne geodetske mreže, neophodno je riješiti singularan sistem linearnih jednadžbi, odnosno pronaći inverznu matricu singularne matrice sistema. Dobro je poznata činjenica da za regularne kvadratne matrice postoji jedinstvena inverzija. Postavlja se pitanje kako doći do inverzne matrice jedne singularne matrice. U radu je opisan klasičan način rješavanja singularnih sistema linearnih jednadžbi i rješavanje singularnom dekompozicijom vlastitih vrijednosti (Singular Value Decomposition - SVD) matrice sistema linearnih jednadžbi. Na primjeru jedne slobodne dvodimenzijske (2D) geodetske mreže razmatraju se značajnosti razlika ova dva pristupa. Analiza dvaju pristupa rješenju problema ocjene nepoznatih parametara u linearnim geodetskim modelima je od posebnog značaja u ovom radu. Prednosti SVD pristupa se sumiraju i isti postupak se preporučuje.

Ključne riječi: *dekompozicija vlastitih vrijednosti, geodetska mreža, linearni sustavi, metoda najmanjih kvadrata*

1 Introduction

Matrix decomposition may reveal many interesting and useful properties of the original matrix. Namely, different techniques of matrix decomposition can be used, and **S**ingular **V**alue **D**ecomposition (SVD) is one of them. SVD is widely used in linear algebra, a mathematical discipline that deals with vectors and matrices, in general vector spaces and linear transformations [13, 9, 10]. In geodesy, the application of SVD is connected to the least square (LSQ) adjustment of the free geodetic networks, when the problem of singularity occurs [4]. In order to solve the singular systems of linear equations, it is necessary to determine the inverse of the singular matrices, called generalized inversion, which is not unique. A special form of generalized inversion is pseudo-inversion, and opposite to the generalized inversion, the pseudo-inversion is unique, has a minimum trace of a cofactor matrix, and is based on a solution with the minimum norm and the minimum sum of residuals. SVD could be applied with high reliability to determine the rank of the matrix, too. The rank of the matrix represents the maximum number of linearly independent species (columns) of the matrix, which is greatly important in geodetic applications. Due to the rounding errors of input data, the linearly independent matrix columns, whose rank is determined, are treated as linearly dependent, and vice versa, giving out the wrong information of the rank. Therefore, determining the rank of the matrix is not an easy task. Although there are faster methods to determine the rank of the matrix, it is safest to use the SVD. This mode of matrix decomposition is explained in [3, 9, 10, 12, 20].

In geodesy, SVD is increasingly used as a solution to the problems of adjustment. For example, Martic in [14] considers SVD in determining the Frobenius norm, rank and pseudo-inverse of the matrix. Vennebusch [19] describes the use of SVD in regression diagnostics, as a tool for geodetic application (cluster analysis), as a very long baseline interferometer technique. For further details and computational aspects of the SVD, readers are directed to Golub and Kahan [7], Press et al. [17] and Stewart [18].

At Belgrade University, the academic education of the future Geodetic Engineers traditionally involves the LSQ method of geodetic measurements adjustment [1, 2, 4, 5, 16]. Namely, the advantages of using the SVD in a network adjustment problem are specifically analyzed within the Adjustment Calculation courses.

This article aims to summarize the scope of work necessary to be done in order to find a solution for the problem of such great importance in the field of geodesy, where using the SVD would help to obtain better understanding of the geometry of the network adjustment routine, and the comprehensive diagnostic analysis of that process.

2 The meaning and properties of SVD

The quadratic form:

$$(\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{K}_{\hat{X}}^{-} (\boldsymbol{x}-\hat{\boldsymbol{x}}) = \chi^2_{1-\alpha,r}, \tag{1}$$

represents $(1-\alpha)$ hyper ellipsoid in r dimensional space, where $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are unknown parameter vectors, and its estimate. Matrix $\boldsymbol{K}_{\hat{X}}^{-}$ is the covariance matrix, and $\chi_{1-\alpha,r}^{2}$ is the percentile value of the Pearson distribution. When only one point is analysed (for example point k), then:

$$(\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{Q}_{kk}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}})=\sigma_0^2 \chi_{1-\alpha,2}^2, \tag{2}$$

figures as the hyper ellipse with the semi-axes a and b (singular values of matrix $\boldsymbol{A}$ are the lengths of the semi-axes of the hyper ellipse) calculated by:

$$\begin{aligned} a&=\sigma_0\sqrt{\lambda_1} \\ b&=\sigma_0\sqrt{\lambda_2} \end{aligned}, \tag{3}$$

where λ_i are the eigenvalues of $\boldsymbol{K}_{\hat{X}}^{-}$ and σ_0 is the *a priori* variance factor ($\boldsymbol{K}_X=\sigma_0 \boldsymbol{Q}_X$).

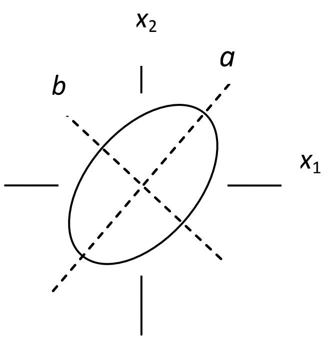

Figure 2 Error ellipse of point *k*

Hyper ellipse is not only a measure of quality related to their eigenvalues. There are more useful relations connected to eigenvalues λ_i, eigenvector $\mathbf{s}_i$ and covariance matrix $\boldsymbol{K}_{\hat{X}}$, such as [11]:

$$\begin{aligned} &\sum_{i=1}^{u}\lambda_i=\operatorname{tr}\boldsymbol{K}_{\hat{X}}, \\ &\prod_{i=1}^{u}\lambda_i=\det\boldsymbol{K}_{\hat{X}}, \\ &\frac{\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{K}_{\hat{X}}\boldsymbol{s}_i}{\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s}_i}=\lambda_i, \end{aligned} \tag{4}$$

which are particularly important for geodetic network analysis and to the process of geodetic network optimization.

When estimating the reliability of the geodetic measurements, maximum eigenvalues of $\boldsymbol{P}\boldsymbol{Q}_{\hat{v}}\boldsymbol{P}$ and $\boldsymbol{P}\boldsymbol{Q}_{\hat{l}}\boldsymbol{P}$ products of the matrices are used as the global measures of inner and outer reliability, respectively. Matrix $\boldsymbol{P}$ is a weight matrix, matrix $\boldsymbol{Q}_{\hat{v}}$ is a cofactor matrix of the residuals, and matrix $\boldsymbol{Q}_{\hat{l}}$ is a cofactor matrix of measurements. We have shown the eigenvalue decomposition above, which only exists for the square matrix, i.e. $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$. What we would like to know is whether a similar decomposition exists for the rectangular (non-square) matrices, i.e. $\boldsymbol{A}\in\boldsymbol{R}^{m\times n}$. For the kind of matrices we use, SVD is specifically found to be one of the best solutions.

There are a lot of reasons for taking the approach that offers SVD as a very powerful tool in geodetic network adjustment process and quality analysis. Let us try to give a short introduction into the procedure for the calculation of the eigen and the singular values of the matrix.

Let the $m\times n$ matrix $\boldsymbol{A}$ with $m\geq n$ belong to the $\boldsymbol{R}^{m\times n}$ space. Then, the decomposition of the matrix $\boldsymbol{A}$ will be:

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}, \tag{5}$$

where:
$\boldsymbol{U}$ – is the orthogonal $m\times n$ matrix with columns called the left singular vectors of matrix $\boldsymbol{A}$;
$\boldsymbol{V}$ – is the orthogonal $n\times n$ matrix with columns called the right singular vectors of matrix $\boldsymbol{A}$;
$\boldsymbol{\Sigma}=\operatorname{diag}(\delta_1,\delta_2,\ldots,\delta_{\min(m,n)})$ is the $n\times n$ diagonal matrix with diagonal nonnegative elements δ_i, representing the singular values of $\boldsymbol{A}$, arranged in the descending order:

$$\delta_1\geq\delta_2\ldots\geq\delta_{\min(m,n)}>0. \tag{6}$$

The decomposition of the form Eq. (5) is called the SVD of matrix $\boldsymbol{A}$. If matrix $\boldsymbol{A}$ is singular, then

$$\operatorname{rang}(\boldsymbol{A})=r<n, \tag{7}$$

and $(n-r)$ singular values of the matrix $\boldsymbol{A}$ will be zero.

Thus, the number of the singular values of the matrix **A** different from zero define the rank of the matrix $\boldsymbol{A}$. This information is of vital importance to the surveyors dealing with control surveys, especially in the deformation analysis.

In [7] Embree includes matrix 2-norm (called by some the *spectral norm,* the 2-norm is invoked in MatLab by the statement norm ($\boldsymbol{A}$,2)), which is simply the first (*largest*) singular value $\delta_{\max}$, as follows:

$$\|\boldsymbol{A}\|_2=\max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}, \tag{8}$$

where

$$\|\boldsymbol{x}\|_2=\sqrt{x_1^2+x_2^2+\ldots+x_n^2},$$

or

$$\|\boldsymbol{A}\|_2=\delta_{\max}=\delta_1. \tag{9}$$

Since $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices, they do not change the Euclidean norm of the matrix $\boldsymbol{\Sigma}$ which coincides with the norm of the matrix $\boldsymbol{A}$. Norm of the nonsingular inverse matrix $\boldsymbol{A}$ is equal to its largest singular values, i.e. the reciprocal of the least singular value of matrix $\boldsymbol{A}$ [7]:

$$\left\|\boldsymbol{A}^{-1}\right\|_2 = \left\|\boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^{\mathrm{T}}\right\|_2 = \frac{1}{\delta_{\max}} = \frac{1}{\delta_n}. \quad (8)$$

Product $\|\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|$ represents the condition of the matrix $\boldsymbol{A}$ where the $\|\cdot\|$ above could be any of the norms defined for matrices. Using the 2-norm, the conditional number of $\boldsymbol{A}$ is given by ratio:

$$\|\boldsymbol{A}\|_2 \cdot \left\|\boldsymbol{A}^{-1}\right\|_2 = \frac{\delta_1}{\delta_n}. \quad (9)$$

If the condition value is close to 1, then the matrix $\boldsymbol{A}$ is well conditioned. A poorly conditioned matrix is nearly singular and a conditionally singular matrix is infinite [3]. The condition of the unit and the orthogonal matrix is equal to one. The condition of the matrix $\boldsymbol{A}$ in MatLab can be obtained by function 'cond($\boldsymbol{A}$)'.

3 The procedure of SVD matrix determination

Here we will show the process of determining the SVD of an mxn arbitrary matrix $\boldsymbol{A}$, where $m \geq n$, consists of five steps:

1) Determination of the eigenvalues $(\lambda_1, \lambda_2, ..., \lambda_n)$ of the matrix product $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$, via the solution of the characteristic equation:

$$\det(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} - \lambda\boldsymbol{I}) = 0, \quad (10)$$

where $\boldsymbol{I}$ is the identity matrix of n order, and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is the $n\times n$ symmetric matrix. Matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a positive semi-definite matrix, and its eigenvalues λ_i are nonnegative numbers ($\lambda_i \geq 0$).

2) Knowing the eigenvalues of the matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$, singular values δ_i of matrix $\boldsymbol{A}$ could be obtained as its nonnegative square roots:

$$\delta_i = \sqrt{\lambda_i};\ i = 1,2,...,n. \quad (11)$$

3) The creation of a diagonal matrix $\boldsymbol{\Sigma}$, with singular values diagonally arranged in the descending order. Off-diagonal elements are zero.

4) The creation of an orthogonal matrix $\boldsymbol{V}$, where the columns of this matrix are the eigenvectors of the matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

The columns of matrix $\boldsymbol{V}$ are ortho-normalized vectors:

$$\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_1, ..., \boldsymbol{v}_n]. \quad (12)$$

5) The creation of an orthogonal matrix $\boldsymbol{U}$, with columns consisting of the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$. The columns of $\boldsymbol{U}$ are orthogonal and unit vectors (ortho-normalized):

$$\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_1, ..., \boldsymbol{u}_m]. \quad (13)$$

For the above matrices, the following identity could be established:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}, \quad (14)$$

defining the SVD of $\boldsymbol{A}$. The SVD of $\boldsymbol{A}$ could be obtained directly, using the function 'SVD' in MatLab package, following the next syntax:

$$\left[\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}\right] = \mathrm{svd}\,(\boldsymbol{A},0). \quad (15)$$

4 Singular system solution using the datum matrix

In order to estimate the vector of an unknown parameter $\boldsymbol{x}$, it is necessary to resolve the inconsistent and singular system of linear equations [11]:

$$\boldsymbol{A}\,\boldsymbol{x} \cong 1' = 1 + \boldsymbol{v}. \quad (16)$$

In a free 2D geodetic network, the vector of the unknown parameter $\boldsymbol{x}$ is the vector of the y and x coordinates.

In order to solve the system, we would normally use the LSQ method and force the vector $\boldsymbol{Ax}-1$ towards the zero vector, ie. minimize the vector norm. Similarly, the system (18) could be described as a singular system of normal equations:

$$\boldsymbol{N}\boldsymbol{x} = \boldsymbol{n}. \quad (17)$$

To solve the system in Eq. (19) and estimate the unknown vector of parameter $\boldsymbol{x}$, it is necessary to determine the inverse of a singular matrix: $\boldsymbol{N} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$. To find the inverse of the singular matrix $\boldsymbol{N}$, in geodesy we would usually form the so-called matrix of datum/data conditions $\boldsymbol{B}$, or the datum/data matrix [4, 11, 16]:

$$\boldsymbol{N}^{+} = (\boldsymbol{N} + \boldsymbol{B}\,\boldsymbol{B}^{\mathrm{T}})^{-1} - \boldsymbol{B}\,\boldsymbol{B}^{\mathrm{T}}, \quad (18)$$

where $\boldsymbol{N}^{+}$ is the pseudo-inverse matrix of $\boldsymbol{N}$. $\mathbf{N}^{+}$ matrix is called the cofactor matrix of unknown parameters $\boldsymbol{Q}_x$, which contains the information on reliability and the quality of geodetic networks. $\boldsymbol{Q}_x$ is the key matrix in geodesy. This method of determining the inverse matrix requires the knowledge of the network geometry and the plan of measurements.

The unknown vector of parameter $\boldsymbol{x}$ is estimated using the next equation:

$$\boldsymbol{x} = \boldsymbol{N}^{+}\boldsymbol{n}. \quad (19)$$

Having in mind that the pseudo-inverse matrix $\boldsymbol{N}^{+}$ is used, the solution computed is the unique minimum norm solution

$$\|Ax-1\|_2 = \min, \tag{20}$$

and vector x also minimizes

$$\|x\|_2 = \min. \tag{21}$$

5 SVD in LSQ problem solving

The solution for the system given in Eq. (18) is requested so that the 2-norm of vector $Ax-1$ be minimal, Eq. (22). Let the SVD of A have the form (5). Then:

$$\begin{aligned} &Ax-1= \\ &=U\Sigma V^{\mathrm{T}}x-1=U(\Sigma V^{\mathrm{T}}x)-U(U^{\mathrm{T}}l)=U(\Sigma y-c), \end{aligned} \tag{24}$$

where:

$$y=V^{\mathrm{T}}x, \tag{25}$$

and:

$$c=U^{\mathrm{T}}L. \tag{26}$$

U as an orthogonal matrix keeps the Euclidean norm of vectors in Eq. (24), ie. we get:

$$\|U(\Sigma y-c)\|=\|\Sigma y-c\|, \tag{27}$$

and find y that minimizes $\|\Sigma y-c\|$. In order to solve this criterion, let the non-zero diagonal elements of the matrix Σ:δ_i for $1\le i\le r$, and y_i be the components of y for $1\le i\le n$, then:

$$\Sigma y=\begin{bmatrix}\delta_1 y_1\\ \delta_2 y_2\\ \vdots\\ \delta_r y_r\\ 0\\ \vdots\\ 0\end{bmatrix}, \tag{28}$$

or:

$$\Sigma y-c=\begin{bmatrix}\delta_1 y_1-c_1\\ \delta_2 y_2-c_2\\ \vdots\\ \delta_r y_r-c_r\\ -c_{r+1}\\ \vdots\\ -c_n\end{bmatrix}, \tag{29}$$

It is obvious when $\delta_i y_i=c_i$ for $1\le i\le r$, vector $\Sigma y-c$ has the minimum Euclidean norm, represented by:

$$\|\Sigma y-c\|=\sqrt{\sum_{i=r+1}^{n}c_i^2}. \tag{30}$$

If $r=n$ (A is a square regular matrix with full rank), then the sum in Eq. (30) is zero, and the vector of unknown parameters in Eq. (18) is estimated as:

$$\begin{aligned} &x=A^{-1}l=(U\Sigma V^{\mathrm{T}})^{-1}l=(V^{\mathrm{T}})^{-1}\Sigma^{-1}U^{-1}l= \\ &=(V^{\mathrm{T}})^{-1}\Sigma^{-1}U^{\mathrm{T}}l=V\Sigma^{-1}U^{\mathrm{T}}l, \end{aligned} \tag{31}$$

where:

$$A^{-1}=V\Sigma^{-1}U^{\mathrm{T}}, \tag{32}$$

is the regular inverse of A, obtained using the transposed matrices V^{T}, U and the regular inverse of Σ.

If $r<n$ (matrix A is singular and with rank deficiencies), then there are no preconditions for $y_{r+1},\dots,y_n$ components, i.e. their values do not influence the Euclidean norm of vector $\Sigma y-c$, because δ_i for $r+1\le i\le n$ are zero.

As $c=U^{\mathrm{T}}L$, that is $y=\Sigma^{+}c$. Matrix Σ^{+} is obtained from the transposed matrix Σ, and non–null singular values of Σ are inversed.

Finally, the vector of an unknown parameter x is estimated as:

$$x=Vy=V\Sigma^{+}U^{\mathrm{T}}l=A^{+}l, \tag{33}$$

where:

$$A^{+}=V\Sigma^{+}U^{\mathrm{T}}, \tag{34}$$

is the pseudo inverse of A.

The cofactor matrix Q_x is determined as follows:

$$\begin{aligned} &Q_x=(A^{\mathrm{T}}A)^{+}=\left((U\Sigma V^{\mathrm{T}})^{\mathrm{T}}U\Sigma V^{\mathrm{T}}\right)^{+}= \\ &=\left(V\Sigma^{\mathrm{T}}U^{\mathrm{T}}U\Sigma V^{\mathrm{T}}\right)^{+}=\left(V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}\right)^{+}= \\ &=\left(V\Sigma^{2}V^{\mathrm{T}}\right)^{+}=V(\Sigma^{2})^{+}V^{\mathrm{T}}. \end{aligned} \tag{35}$$

If A is a regular square matrix, the SVD of A is obtained from Eq. (33). The same result is obtained using the function 'inv' in MatLab software. But, if A is a singular matrix, the SVD, Eq. (34) are used to get the inverse matrix which is, at the same time, the pseudo inverse of A. Using the MatLab function 'pinv', we get the same solution.

It is interesting to investigate whether there is a difference between the cofactor matrix determined by the datum matrix B, Eq. (20), and the cofactor matrices obtained using the SVD, Eq. (35), or if there are any differences between the components of the vector of an unknown parameter x obtained from the datum matrix and using the SVD. The next chapter contains the example of a free 2D geodetic network for which the cofactor matrix and the parameter vector $\boldsymbol{x}$ are determined via the two methods mentioned above.

6 SVD application in the free network adjustment

To illustrate the use of SVD in geodetic networks adjustment, the results of the measurements in the free 2D survey network of 11 directions, two angles and two distances are simulated. The standard deviation of the directions measurements is supposed to be 5" and 3 mm + 2 mm/km for distance measurements. The measurement planes are shown in Tabs. 1, 2 and 3.

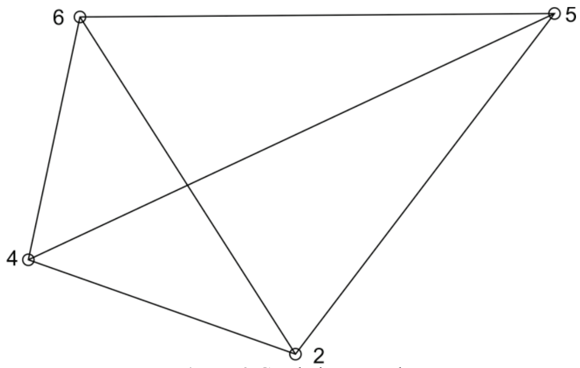

Figure 2 Geodetic network

Table 1 Direction measurements plan

From point	To point
6	2
	4
	5
2	4
	6
4	5
	6
	2
5	2
	6
	4

Table 2 Distance measurements plan

From point	To point
6	2
6	4

Table 3 Angle measurements plan

From	At	To
5	6	4
6	4	2

Table 4 The differences between SVD and classical parameter vector, and $\boldsymbol{Q}_x$ diagonal elements estimates

	Estimation of the coordinate vector increment / m	Diagonal elements of the cofactor matrix
1	5,77E−15	−8,53E−14
2	−8,66E−15	4,44E−14
3	4,00E−15	−8,88E−14
4	−8,10E−15	4,80E−14
5	−3,55E−15	1,28E−13
6	0,00E+00	−3,20E−14
7	3,77E−15	−6,75E−14
8	−7,88E−15	3,55E−14
9	−1,11E−15	0
10	1,05E−15	0
11	6,12E−15	0
12	0,00E+00	0

Applying the procedure mentioned above and using Eq. (20), (21), (33) and (35), the appropriate unknown parameter vectors and the cofactor matrices are determined.

Comparing the results of the two solutions (Tab. 4), it can be stated that there is no significant difference between them.

Singular values of matrix $\boldsymbol{A}$ ($\boldsymbol{\Sigma}$= diag (0,59 0,53 0,49 0,40 0,35 0,31 0,26 0,16 0,08)) and condition (cond($\boldsymbol{A}$) = 7,375) are of considerable importance in optimizing geodetic networks.

7 Conclusion

The direct application of the SVD matrix leads to the key matrix in geodesy – the cofactor matrix of unknown parameters. At the same time, this is the main source of information on the quality of the geodetic network, independent of the geodetic network geometry and the type of measurements. The vector of an unknown parameter **x,** and all other estimates regarding the design and the network diagnostics are easily accessible by the means of the SVD procedure. It is shown through the example that there are no significant differences between the values of the matrix $\boldsymbol{Q}_x$ and the vector $\boldsymbol{x}$ obtained using the SVD and in a conventional manner. The method based on the application of the SVD is simpler, faster and more convenient for the making of software routines. The MatLab program has the characteristics suitable for this problem, and can be used successfully in the most complex geodetic calculations.

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