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Growth of solutions of linear differential equations with meromorphic coefficients of $[p,q]$-order

Hui Hu[1] and Xiu-Min Zheng[1,*]

[1] *Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330 022, P. R. China*

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Abstract. In this paper, we investigate the growth of meromorphic solutions to a complex higher order linear differential equation whose coefficients are meromorphic functions of $[p,q]$-orders. We get the results about the lower $[p,q]$-order of solutions of the equation. Moreover, we investigate the $[p,q]$-convergence exponent and the lower $[p,q]$-convergence exponent of distinct zeros of $f(z)-\varphi(z)$.

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1. Introduction and notations

In this paper, a meromorphic function means being meromorphic in the whole complex plane. We also assume that readers are familiar with the standard notations and fundamental results of Nevanlinna's theory (see e.g. [5, 12, 16]). Let us define inductively for $r\in[0,+\infty), \exp_1 r=e^r$ and $\exp_{p+1} r=\exp(\exp_p r),\ p\in\mathbb{N}$. For all sufficiently large r, we define $\log_1 r=\log r$ and $\log_{p+1} r=\log(\log_p r),\ p\in\mathbb{N}$. We also denote $\exp_0 r=r=\log_0 r$ and $\exp_{-1} r=\log_1 r$. It is well known that there exist many functions which have infinite iterated orders. There are also many papers considering the iterated order of solutions of complex linear differential equations (see e.g. [3, 7, 8, 11, 15]). In order to discuss accurately the growth of these functions of fast growth, Juneja-Kapoor-Bajpai investigated some properties of entire functions of $[p,q]$-order in [9, 10]. In [14], in order to maintain accordance with general definitions of the entire function $f(z)$ of iterated p-order, Liu-Tu-Shi gave a minor modification of the original definition of the $[p,q]$-order given in [9, 10]. With this new concept of $[p,q]$-order, the $[p,q]$-order of solutions of complex linear differential equations are investigated (see e.g. [13, 14]). B. Belaïdi also considered the growth of solutions of higher order linear differential equations with analytic coefficients of $[p,q]$-order in the unit disc (see e.g. [1, 2]).

Now we introduce the definitions of the $[p,q]$-order as follows.

*Corresponding author. *Email addresses:* `zhengxiumin2008@sina.com` (X. M. Zheng), `h_h87_6@hotmail.com` (H. Hu)

Definition 1 (see [13, 14])**.** *Let p, q be integers such that $p \geq q \geq 1$ and let $f(z)$ be a meromorphic function. The $[p,q]$-order of $f(z)$ is defined by*

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

For an entire function $f(z)$, we also define

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Remark 1. *By Definition 1 we note that $\sigma_{[1,1]}(f) = \sigma(f)$, $\sigma_{[2,1]}(f) = \sigma_2(f)$, and $\sigma_{[p,1]}(f) = \sigma_p(f)$.*

Now, by Definition 1, we can get the definition of the lower $[p,q]$-order for entire and meromorphic functions.

Definition 2. *Let p, q be integers such that $p \geq q \geq 1$, and let $f(z)$ be a meromorphic function. The lower $[p,q]$-order of $f(z)$ is defined by*

$$\mu_{[p,q]}(f) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

For an entire function $f(z)$, we also define

$$\mu_{[p,q]}(f) = \underline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Definition 3 (see [13, 14])**.** *Let p, q be integers such that $p \geq q \geq 1$. The $[p,q]$-convergence exponent of the sequence of a-points of a meromorphic function $f(z)$ is defined by*

$$\lambda_{[p,q]}(f-a) = \lambda_{[p,q]}(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f-a})}{\log_q r},$$

and the $[p,q]$-convergence exponent of the sequence of distinct a-points of a meromorphic function $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f-a) = \overline{\lambda}_{[p,q]}(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f-a})}{\log_q r}.$$

Now, by Definition 3, we can get the definition of the lower $[p,q]$-convergence exponent for entire and meromorphic functions.

Definition 4. *Let p, q be integers such that $p \geq q \geq 1$. The lower $[p,q]$-convergence exponent of the sequence of a-points of a meromorphic function $f(z)$ is defined by*

$$\underline{\lambda}_{[p,q]}(f-a) = \underline{\lambda}_{[p,q]}(f,a) = \underline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f-a})}{\log_q r},$$

and the lower $[p,q]$*-convergence exponent of the sequence of distinct a-points of a meromorphic function* $f(z)$ *is defined by*

$$\underline{\overline{\lambda}}_{[p,q]}(f-a)=\underline{\overline{\lambda}}_{[p,q]}(f,a)=\varliminf_{r\to\infty}\frac{\log_p \overline{N}(r,\frac{1}{f-a})}{\log_q r}.$$

Furthermore, we can get the definitions of $\lambda_{[p,q]}(f-\varphi)$, $\overline{\lambda}_{[p,q]}(f-\varphi)$, $\underline{\lambda}_{[p,q]}(f-\varphi)$ and $\underline{\overline{\lambda}}_{[p,q]}(f-\varphi)$, when a is replaced by a meromorphic function $\varphi(z)$.

Definition 5 (see [13, 14])**.** *Let* $p,\ q$ *be integers such that* $p\geq q\geq 1$*. The* $[p,q]$*-type of a meromorphic function* $f(z)$ *of* $[p,q]$*-order* $\sigma(0<\sigma<\infty)$ *is defined by*

$$\tau_{[p,q]}(f)=\varlimsup_{r\to\infty}\frac{\log_{p-1}T(r,f)}{(\log_{q-1}r)^{\sigma}}.$$

For an entire function $f(z)$*, we also define*

$$\tau_{[p,q]}(f)=\varlimsup_{r\to\infty}\frac{\log_{p}M(r,f)}{(\log_{q-1}r)^{\sigma}}.$$

Now, by Definition 5, we can get the definition of the lower $[p,q]$-type for entire and meromorphic functions.

Definition 6. *Let* $p,\ q$ *be integers such that* $p\geq q\geq 1$*. The lower* $[p,q]$*-type of a meromorphic function* $f(z)$ *of lower* $[p,q]$*-order* $\mu(0<\mu<\infty)$ *is defined by*

$$\underline{\tau}_{[p,q]}(f)=\varliminf_{r\to\infty}\frac{\log_{p-1}T(r,f)}{(\log_{q-1}r)^{\mu}}.$$

For an entire function $f(z)$*, we also define*

$$\underline{\tau}_{[p,q]}(f)=\varliminf_{r\to\infty}\frac{\log_{p}M(r,f)}{(\log_{q-1}r)^{\mu}}.$$

Remark 2. *Especially, when* $q=1$*, Definitions 5 and 6 are the definitions of the iterated p-type and the iterated p-lower type, and the condition "*$p\in\mathbb{N}\backslash\{1\}$*" in [7] conforms to the condition "*$p>q=1$*" in this paper.*

Moreover, we denote the linear measure of a set $E\subset[0,+\infty)$ by $mE=\int_E dt$ and the logarithmic measure $E\subset[1,+\infty)$ by $m_lE=\int_E dt/t$, respectively (see e.g. [6]).

2. Main results

We consider the following equation, for $n\geq 2$,

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1}$$

where $A_j(z),\ (j=1,\cdots,n-1),\ A_0(z)(\not\equiv 0)$ are meromorphic functions. For the case of entire coefficients, by definitions of $[p,q]$-order and $[p,q]$-type, Liu-Tu-Shi [14] got a result as follows.

Theorem 1. *Let $A_0(z), \ldots, A_{n-1}(z)$ be entire functions satisfying* $\max\{\sigma_{[p,q]}(A_j)|j = 1, \ldots, n-1\} \le \sigma_{[p,q]}(A_0) < \infty$ *and* $\max\{\tau_{[p,q]}(A_j)|\sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0, j \ne 0\} < \tau_{[p,q]}(A_0)$. *Then every nontrivial solution $f(z)$ of (1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.*

For the case of meromorphic coefficients, Li-Cao [13] obtained the same result by complementing some conditions on the poles of the coefficients and solutions of (1).

Theorem 2. *Let $A_0(z), \ldots, A_{n-1}(z)$ be meromorphic functions in the complex plane. Assume that $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0) < \infty$, $\max\{\sigma_{[p,q]}(A_j)|j = 1, \ldots, n-1\} = \sigma_{[p,q]}(A_0) < \infty$ and $\max\{\tau_{[p,q]}(A_j)|\sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0), j \ne 0\} < \tau_{[p,q]}(A_0)$. Then any nonzero meromorphic solution $f(z)$ of (1) whose poles are of uniformly bounded multiplicities, satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.*

In the above, Liu-Tu-Shi [14] and Li-Cao [13] used the $[p, q]$-type of $A_0(z)$ to dominate the $[p, q]$-types of other coefficients, and got the result about $\sigma_{[p+1,q]}(f)$. Thus, a natural question arises: If we use the lower $[p, q]$-type of $A_0(z)$ to dominate other coefficients, what can be said about $\mu_{[p+1,q]}(f)$? Another question is: Can we find some other conditions to dominate other coefficients? In the meantime, can we improve the condition on the poles of $f(z)$?

In this paper, we give our main results solving the above two questions. Moreover, we get the results about the $[p, q]$-convergence exponent and the lower $[p, q]$-convergence exponent of distinct zeros of $f(z) - \varphi(z)$.

Theorem 3. *Let p, q be integers such that $p \ge q > 1$ or $p > q = 1$, and let $A_0(z), \ldots, A_{n-1}(z)$ be meromorphic functions. Assume that $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0) < \infty$, and that $\max\{\sigma_{[p,q]}(A_j)\ |j = 1, \ldots, n-1\} \le \mu_{[p,q]}(A_0)$ and $\max\{\tau_{[p,q]}(A_j)|\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \ne 0\} < \underline{\tau}_{[p,q]}(A_0) = \tau$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b \log_q r\}$ $(b \le \mu_{[p,q]}(A_0))$, then we have*

$$\underline{\overline{\lambda}}_{[p+1,q]}(f-\varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \le \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f-\varphi),$$

where $\varphi(z)(\not\equiv 0)$ is a meromorphic function with $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$.

Theorem 4. *Let p, q be integers such that $p \ge q > 1$ or $p > q = 1$, and let $A_0(z), \ldots, A_{n-1}(z)$ be meromorphic functions. Assume that $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0) < \infty$, and that $\max\{\sigma_{[p,q]}(A_j)|j = 1, \ldots, n-1\} \le \mu_{[p,q]}(A_0)$ and $\overline{\lim\limits_{r\to\infty}}\sum_{j=1}^{n-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b \log_q r\}$ $(b \le \mu_{[p,q]}(A_0))$, then we have*

$$\underline{\overline{\lambda}}_{[p+1,q]}(f-\varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \le \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f-\varphi),$$

where $\varphi(z)(\not\equiv 0)$ is a meromorphic function satisfying $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$.

Theorem 5. *Let p, q be integers such that $p \ge q \ge 1$, and let $A_0(z), \ldots, A_{n-1}(z)$ be meromorphic functions. Suppose that there exists one $A_s(z)$ $(0 \le s \le n-1)$*

with $\lambda_{[p,q]}(\frac{1}{A_s}) < \mu_{[p,q]}(A_s) < \infty$, *and that* $\max\{\sigma_{[p,q]}(A_j)|\ j \neq s\} \leq \mu_{[p,q]}(A_s)$ *and* $\max\{\tau_{[p,q]}(A_j)|\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), j \neq s\} < \underline{\tau}_{[p,q]}(A_s) = \tau$. *Then every transcendental meromorphic solution* $f(z)(\not\equiv 0)$ *of (1) satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b\log_q r\}$ $(b \leq \mu_{[p,q]}(A_s))$ *satisfies* $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s) \leq \mu_{[p,q]}(f)$ *and* $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s) \leq \sigma_{[p,q]}(f)$. *Moreover, every non-transcendental meromorphic solution* $f(z)$ *of (1) is a polynomial with degree* $\deg(f) \leq s-1$.

Remark 3. *All meromorphic solutions of (1) satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b\log_q r\}$ $(b \leq \mu_p(A_0))$ *in Theorems 3-5 are of regular growth* $\mu_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f)$, *when the coefficient* $A_0(z)$ *is of regular growth* $\mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$.

Remark 4. *The condition "*$p \geq q > 1$ *or* $p > q = 1$*" ensures to get rid of the case "*$p = q = 1$*", which is essential for the proof of Lemma 7. Since Lemma 7 is the part and parcel in the proofs of Theorems 3 and 4, the condition "*$p \geq q > 1$ *or* $p > q = 1$*" cannot be omitted in Theorems 3 and 4.*

Remark 5. *The condition that* $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0)$ *in Theorems 3-5 can be changed by* $N(r, A_0) = o(m(r, A_0))$ $(r \to \infty)$, $\delta(\infty, A_0) > 0$ *or*

$$\mu_{[p,q]}(A_0) = \underline{\lim}_{r\to\infty} \frac{\log_p m(r, A_0)}{\log_q r}.$$

3. Preliminary lemmas

Lemma 1 (see [4]). *Let* $f(z)$ *be a meromorphic solution of (1) assuming that not all coefficients* $A_j(z)$ *are constants. Given a real constant* $\gamma > 1$, *and denoting* $T(r) = \sum_{j=0}^{n-1} T(r, A_j)$, *we have*

$$\log m(r,f) < T(r)\big\{(\log r)\log T(r)\big\}^{\gamma}, \quad \text{if } p = 0$$

and

$$\log m(r,f) < r^{2p+\gamma-1}T(r)\left\{\log T(r)\right\}^{\gamma}, \quad \text{if } p > 0,$$

outside of an exceptional set E_p *with* $\int_{E_p} t^{p-1}dt < +\infty$.

Remark 6. *Especially, if* $p = 0$, *then the exceptional set* E_0 *has finite logarithmic measure* $\int_{E_0} \frac{dt}{t} = m_l E_0$.

Lemma 2 (see [6, 12]). *Let* $g : [0, +\infty) \to \mathbb{R}$ *and* $h : [0, +\infty) \to \mathbb{R}$ *be monotone increasing functions. If (i)* $g(r) \leq h(r)$ *outside of an exceptional set of finite linear measure, or (ii)* $g(r) \leq h(r)$, $r \notin E_1 \cup (0, 1]$, *where* $E_1 \subset [1, \infty)$ *is a set of finite logarithmic measure, then for any constant* $\alpha > 1$, *there exists* $r_0 = r_0(\alpha) > 0$ *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

Lemma 3 (see [13]). *Let* p, q *be integers such that* $p \geq q \geq 1$ *and let* $A_0(z), A_1(z), \ldots, A_{n-1}(z), F(z)(\not\equiv 0)$ *be meromorphic functions. If* $f(z)$ *is a meromorphic solution of*

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z) \qquad (2)$$

satisfying

$$\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j)|j = 0, \dots, n-1\} < \sigma_{[p,q]}(f) = \sigma < \infty,$$

then we have $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

Lemma 4. *Let* p, q *be integers such that* $p \geq q \geq 1$ *and let* $A_0(z), A_1(z), \dots, A_{n-1}(z)$, $F(z)$ $(\not\equiv 0)$ *be meromorphic functions. If* $f(z)$ *is a meromorphic solution of (2) satisfying*

$$\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j)|j = 0, \dots, n-1\} < \mu_{[p,q]}(f),$$

then we have $\overline{\underline{\lambda}}_{[p,q]}(f) = \underline{\lambda}_{[p,q]}(f) = \mu_{[p,q]}(f)$.

Proof. By (2), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(n)}}{f} + A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_0(z)\right). \tag{3}$$

If $f(z)$ has a zero at z_0 of order $\gamma(> n)$ and $A_0(z), A_1(z), \cdots, A_{n-1}(z)$ are all analytic at z_0, then $F(z)$ has a zero at z_0 of order at least $\gamma - n$. Hence, we have

$$N(r, \frac{1}{f}) \leq n\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}) + \sum_{j=0}^{n-1} N(r, A_j). \tag{4}$$

By (3), we get that

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{F}) + \sum_{j=0}^{n-1} m(r, A_j) + \sum_{j=1}^{n} m(r, \frac{f^{(j)}}{f}) + O(1). \tag{5}$$

Therefore, by (4), (5) and the first fundamental theorem,

$$\begin{aligned} T(r,f) &= T(r, \frac{1}{f}) + O(1) \\ &\leq n\overline{N}(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{n-1} T(r, A_j) + O(\log(rT(r,f))) \end{aligned} \tag{6}$$

holds for all $r \notin E$, where E is a set of r of finite linear measure. By $\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j)|j = 0, \dots, n-1\} < \mu_{[p,q]}(f)$, for sufficiently large r, we have

$$T(r, F) = o(T(r,f)) \quad \text{and} \quad T(r, A_j) = o(T(r,f)),\ j = 0, 1, \dots, n-1. \tag{7}$$

Moreover, for sufficiently large r, we have $O(\log(rT(r,f))) = o(T(r,f))$. By combining this, (6) and (7), we have

$$(1 - o(1))T(r,f) \leq n\overline{N}(r, \frac{1}{f}),\ r \notin E,\ r \to \infty. \tag{8}$$

Hence, by Lemma 2 and (8), we have

$$\overline{\underline{\lambda}}_{[p,q]}(f) \geq \mu_{[p,q]}(f).$$

Since $\overline{\underline{\lambda}}_{[p,q]}(f) \leq \underline{\lambda}_{[p,q]}(f) \leq \mu_{[p,q]}(f)$, the result holds. □

Lemma 5. *Let p, q be integers such that $p \geq q \geq 1$ and let $f(z)$ be a meromorphic function with $0 < \mu_{[p,q]}(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset (1, \infty)$ of infinite logarithmic measure such that*

$$\mu = \mu_{[p,q]}(f) = \lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log_p T(r,f)}{\log_q r},$$

and for any given $\varepsilon > 0$ and sufficiently large $r \in E_2$,

$$T(r,f) < \exp_p\{(\mu + \varepsilon)\log_q r\}.$$

Proof. We use a similar proof as [14, Lemma 8]. By the definition of lower $[p,q]$-order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $r_n < (1 - \frac{1}{n+1})r_{n+1}$, and

$$\lim_{r_n\to\infty} \frac{\log_p T(r_n,f)}{\log_q r_n} = \mu_{[p,q]}(f).$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$, for any $r \in [(1-\frac{1}{n})r_n, r_n]$, we have

$$\frac{\log_p T(r,f)}{\log_q r} \leq \frac{\log_p T(r_n,f)}{\log_q r_n} \frac{\log_q r_n}{\log_q r}.$$

When $q \geq 1$, we have $\frac{\log_q r_n}{\log_q r} \to 1 (n \to \infty)$. Let $E_2 = \bigcup_{n=n_1}^{\infty}[(1-\frac{1}{n})r_n, r_n]$, then we have

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log_p T(r,f)}{\log_q r} \leq \lim_{r_n\to\infty} \frac{\log_p T(r_n,f)}{\log_q r_n} = \mu_{[p,q]}(f)$$

and $m_l E_2 = \sum_{n=n_1}^{\infty} \int_{(1-\frac{1}{n})r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1 + \frac{1}{n-1}) = \infty$. Therefore, by the evident fact that

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log_p T(r,f)}{\log_q r} \geq \underline{\lim}_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r} = \mu_{[p,q]}(f),$$

we have

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log_p T(r,f)}{\log_q r} = \mu_{[p,q]}(f)$$

and for any given $\varepsilon > 0$ and sufficiently large $r \in E_2$,

$$T(r,f) < \exp_p\left\{(\mu+\varepsilon)\log_q r\right\}.$$

□

Lemma 6. *Let p, q be integers such that $p \geq q \geq 1$ and let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be meromorphic functions such that $\max\{\sigma_{[p,q]}(A_j) | j \neq s\} \leq \mu_{[p,q]}(A_s) < \infty$. If $f(z) (\not\equiv 0)$ is a meromorphic solution of (1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b \log_q r\}$ $(b \leq \mu_{[p,q]}(A_s))$, then we have $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s)$.*

Proof. By (1), we know that the poles of $f(z)$ can only occur at the poles of $A_0(z)$, $A_1(z), \ldots, A_{n-1}(z)$. By $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\{b \log_q r\}$ $(b \leq \mu_{[p,q]}(A_s))$, we have

$$N(r,f) < \exp_{p+1}\{b \log_q r\}\overline{N}(r,f) \leq \exp_{p+1}\{b \log_q r\} \sum_{j=0}^{n-1} \overline{N}(r, A_j)$$
$$\leq \exp_{p+1}\{b \log_q r\} \sum_{j=0}^{n-1} T(r, A_j). \tag{9}$$

Then by (9), we have

$$T(r,f) \leq m(r,f) + \exp_{p+1}\{b \log_q r\} \sum_{j=0}^{n-1} T(r, A_j). \tag{10}$$

By Lemma 5, there exists a set E_2 of infinite logarithmic measure such that for any given $\varepsilon > 0$ and sufficiently large $r \in E_2$, we have

$$T(r, A_s) \leq \exp_p\{(\mu_{[p,q]}(A_s) + \varepsilon) \log_q r\}. \tag{11}$$

Since max $\{\sigma_{[p,q]}(A_j)|\ j \neq s\} \leq \mu_{[p,q]}(A_s)$, for the above $\varepsilon > 0$ and sufficiently large r, we have

$$T(r, A_j) \leq \exp_p\{(\mu_{[p,q]}(A_s) + \varepsilon) \log_q r\},\ j \neq s. \tag{12}$$

By (11), (12) and Lemma 1, there exists a set E_0 of r of finite logarithmic measure such that for sufficiently large $r \in E_2 \backslash E_0$

$$m(r,f) \leq \exp\left\{ \sum_{j=0}^{n-1} T(r, A_j) \left[(\log r) \log \left(\sum_{j=0}^{n-1} T(r, A_j) \right) \right]^{\gamma} \right\}$$
$$\leq \exp_{p+1}\{(\mu_{[p,q]}(A_s) + 2\varepsilon) \log_q r\}. \tag{13}$$

By (10) and (13), we have

$$\varliminf_{r \to \infty} \frac{\log_{p+1} T(r,f)}{\log_q r} \leq \varliminf_{\substack{r \to \infty \\ r \in E_2 \backslash E_0}} \frac{\log_{p+1} T(r,f)}{\log_q r} \leq \mu_{[p,q]}(A_s) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s)$. □

Lemma 7. *Let p, q be integers such that $p \geq q > 1$ or $p > q = 1$, and let $A_0(z), \ldots, A_{n-1}(z)$ be meromorphic functions. Assume that $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0)$ and that*

$$\max\{\sigma_{[p,q]}(A_j) | j = 1, \ldots, n-1\} \leq \mu_{[p,q]}(A_0) = \mu\ (0 < \mu < \infty),$$
$$\max\{\tau_{[p,q]}(A_j)|\ \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\} < \underline{\tau}_{[p,q]}(A_0) = \tau\ (0 < \tau < \infty).$$

If $f(z) (\not\equiv 0)$ is a meromorphic solution of (1), then we have $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$.

Proof. Suppose that $f(z)(\not\equiv 0)$ is a meromorphic solution of (1). By (1), we get

$$-A_0(z) = \frac{f^{(n)}(z)}{f(z)} + A_{n-1}(z)\frac{f^{(n-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}. \tag{14}$$

By $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0)$, we have $N(r, A_0) = o(T(r, A_0))(r \to \infty)$. Then by (14), we have

$$T(r, A_0) = m(r, A_0) + N(r, A_0) \leq \sum_{j=1}^{n-1} m(r, A_j) + \sum_{j=1}^{n} m(r, \frac{f^{(j)}}{f}) + o(T(r, A_0)). \tag{15}$$

Hence, we have by (15) that

$$T(r, A_0) \leq O\left(\sum_{j=1}^{n-1} m(r, A_j) + \log(rT(r, f))\right), \tag{16}$$

for sufficiently large $r \to \infty$, $r \notin E$, where E is a set of r of finite linear measure.

Set $b = \max\{\sigma_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) = \mu, j = 1, \ldots, n-1\}$. If $\sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) = \mu$, then for any $\varepsilon(0 < 2\varepsilon < \mu - b)$ and all $r \to \infty$, we have

$$\begin{aligned} m(r, A_j) &\leq T(r, A_j) \leq \exp_p\{(b + \varepsilon)\log_q r\} \\ &< \exp_p\{(\mu - \varepsilon)\log_q r\} = \exp_{p-1}\{(\log_{q-1} r)^{\mu-\varepsilon}\}. \end{aligned} \tag{17}$$

Set $\tau_1 = \max\{\tau_{[p,q]}(A_j) |\ \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\}$, then $\tau_1 < \tau$. If $\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0)$, $\tau_{[p,q]}(A_j) \leq \tau_1 < \tau$, then for $r \to \infty$ and any ε $(0 < 2\varepsilon < \tau - \tau_1)$, we have

$$m(r, A_j) \leq T(r, A_j) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)(\log_{q-1} r)^{\mu}\right\}. \tag{18}$$

By the definition of the lower $[p, q]$-type, for $r \to \infty$, we have

$$T(r, A_0) > \exp_{p-1}\left\{(\tau - \varepsilon)(\log_{q-1} r)^{\mu}\right\}. \tag{19}$$

When $p \geq q > 1$ or $p > q = 1$, we have

$$\exp_{p-1}\left\{(\tau_1 + \varepsilon)(\log_{q-1} r)^{\mu}\right\} = o(\exp_{p-1}\left\{(\tau - \varepsilon)(\log_{q-1} r)^{\mu}\right\}),\ r \to \infty.$$

By substituting (17)-(19) into (16), we have

$$\exp_{p-1}\left\{(\tau - 2\varepsilon)(\log_{q-1} r)^{\mu}\right\} \leq O(\log(rT(r, f))),\ r \notin E,\ r \to \infty. \tag{20}$$

Then, by Lemma 2, we have $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$. □

Lemma 8. *Let p, q be integers such that $p \geq q \geq 1$ and let $f(z)$ be a meromorphic function with $0 < \sigma_{[p,q]}(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, \infty)$ of infinite logarithmic measure such that*

$$\tau = \tau_{[p,q]}(f) = \lim_{\substack{r \to \infty \\ r \in E_3}} \frac{\log_{p-1} T(r, f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}}.$$

Proof. By the definition of the $[p,q]$-type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$, and

$$\tau_{[p,q]}(f) = \lim_{r_n\to\infty} \frac{\log_{p-1} T(r_n,f)}{(\log_{q-1} r_n)^{\sigma_{[p,q]}(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\log_{p-1} T(r_n,f)}{(\log_{q-1} r_n)^{\sigma_{[p,q]}(f)}} \left(\frac{\log_{q-1} r_n}{\log_{q-1}(1+\frac{1}{n})r_n}\right)^{\sigma_{[p,q]}(f)} \leq \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}}.$$

When $q \geq 1$, we have $\frac{\log_{q-1} r_n}{\log_{q-1}(1+\frac{1}{n})r_n} \to 1,\ r_n \to \infty$. Let $E_3 = \bigcup_{n=n_1}^{\infty} [r_n, (1+\frac{1}{n})r_n]$, then we have

$$\lim_{\substack{r\to\infty \\ r\in E_3}} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}} \geq \lim_{r_n\to\infty} \frac{\log_{p-1} T(r_n,f)}{(\log_{q-1} r_n)^{\sigma_{[p,q]}(f)}} = \tau_{[p,q]}(f),$$

and $\int_{E_3} \frac{dr}{r} = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty$. Therefore, by the evident fact that

$$\lim_{\substack{r\to\infty \\ r\in E_3}} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}} \leq \overline{\lim_{r\to\infty}} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}} = \tau_{[p,q]}(f),$$

we have

$$\lim_{\substack{r\to\infty \\ r\in E_3}} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{[p,q]}(f)}} = \tau_{[p,q]}(f).$$

□

4. Proofs of Theorems 3 - 5

Proof of Theorem 3. By Lemma 1 and (10), we can get

$$T(r,f) \leq \exp_{p+1}\{(\sigma_{[p,q]}(A_0) + 3\varepsilon)\log_q r\},$$

for any $\varepsilon > 0$ and $r \notin E_0,\ r \to \infty$, where E_0 is a set of r of finite logarithmic measure. And by Lemma 2, we can get $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0)$. Set $d = \max\{\sigma_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) < \sigma_{[p,q]}(A_0)\}$. If $\sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0)$ or $\sigma_{[p,q]}(A_j) \leq \mu_{[p,q]}(A_0) < \sigma_{[p,q]}(A_0)$, then for any given $\varepsilon (0 < 2\varepsilon < \sigma_{[p,q]}(A_0) - d)$ and sufficiently large r, we have

$$T(r,A_j) \leq \exp_p\{(d+\varepsilon)\log_q r\} = \exp_{p-1}\{(\log_{q-1} r)^{d+\varepsilon}\}. \tag{21}$$

Set $\tau_1 = \max\{\tau_{[p,q]}(A_j) |\ \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\}$. If $\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$, then we have $\tau_1 < \tau \leq \tau_{[p,q]}(A_0)$. Therefore,

$$T(r,A_j) \leq \exp_{p-1}\left\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_{[p,q]}(A_0)}\right\} \tag{22}$$

holds for $r \to \infty$ and any given ε $(0 < 2\varepsilon < \tau_{[p,q]}(A_0) - \tau_1)$. By the definition of the $[p,q]$-type and Lemma 8, for sufficiently large r, $r \in E_3$, where E_3 is a set of r of infinite logarithmic measure, we have

$$T(r, A_0) > \exp_{p-1}\left\{(\tau_{[p,q]}(A_0) - \varepsilon)(\log_{q-1} r)^{\sigma_{[p,q]}(A_0)}\right\}. \tag{23}$$

Then by (16) and (21)-(23), for all sufficiently large r, $r \in E_3 \backslash E$ and the above ε, we have

$$\exp_{p-1}\left\{(\tau_{[p,q]}(A_0) - 2\varepsilon)(\log_{q-1} r)^{\sigma_{[p,q]}(A_0)}\right\} \leq O(\log T(r,f))), \tag{24}$$

where E is a set of r of finite linear measure. Then, we have $\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. Thus, we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

By Lemmas 6 and 7, we have $\mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$.

Now we need to prove $\underline{\lambda}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f)$. Setting $g = f - \varphi$, since $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$, we have $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$, $\mu_{[p+1,q]}(g) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$, $\overline{\lambda}_{[p+1,q]}(g) = \overline{\lambda}_{[p+1,q]}(f - \varphi)$ and $\underline{\lambda}_{[p+1,q]}(g) = \underline{\lambda}_{[p+1,q]}(f - \varphi)$. By substituting $f = g + \varphi, f' = g' + \varphi', \cdots, f^{(n)} = g^{(n)} + \varphi^{(n)}$ into (1), we get

$$g^{(n)} + A_{n-1}(z) g^{(n-1)} + \cdots + A_0(z) g = -[\varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \cdots + A_0(z)\varphi]. \tag{25}$$

If $F(z) = \varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \cdots + A_0(z)\varphi \equiv 0$, then by Lemma 7, we have $\mu_{[p+1,q]}(\varphi) \geq \mu_{[p,q]}(A_0)$, which is a contradiction. Since $F(z) \not\equiv 0$ and $\sigma_{[p+1,q]}(F) \leq \sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0) = \mu_{[p+1,q]}(f) = \mu_{[p+1,q]}(g) \leq \sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f)$, by Lemma 3 and (25), we have $\overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g) = \sigma_{[p,q]}(A_0)$, i.e. $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. By Lemma 4 and (25), we have $\underline{\lambda}_{[p+1,q]}(g) = \mu_{[p+1,q]}(g)$, i.e. $\underline{\lambda}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$. Therefore, $\underline{\lambda}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi)$.

Then the proof is complete. □

Proof of Theorem 4. By the first part of the proof of Theorem 3, we can get $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0)$. By

$$\overline{\lim_{r\to\infty}} \sum_{j=1}^{n-1} \frac{m(r, A_j)}{m(r, A_0)} < 1, \tag{26}$$

we have for $r \to \infty$

$$\sum_{j=1}^{n-1} m(r, A_j) < \delta m(r, A_0), \tag{27}$$

where $\delta \in (0, 1)$. By $\lambda_{[p,q]}(\frac{1}{A_0}) < \mu_{[p,q]}(A_0)$, we have $N(r, A_0) = o(T(r, A_0))(r \to \infty)$. By (15) and (27), for $r \to \infty$, $r \notin E$, we have

$$T(r, A_0) = m(r, A_0) + N(r, A_0) \leq \delta T(r, A_0) + O(\log(rT(r,f))) + o(T(r, A_0)), \tag{28}$$

where E is a set of r of finite linear measure. By Lemma 2 and (28), we have $\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. Then we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

By (28) and Lemma 2, we have $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$. By Lemma 6, we have $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_0)$, then we get $\mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$.

By using the similar proof of Theorem 3, we can get

$$\begin{aligned}\overline{\lambda}_{[p+1,q]}(f-\varphi) &= \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)\\ &\leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi).\end{aligned}$$

Then the proof is complete. □

Proof of Theorem 5. Suppose that $f(z)$ is a rational solution of (1). If $f(z)$ is either a rational function with a pole of multiplicity $n \geq 1$ at z_0 or a polynomial with degree $\deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. If $\max\{\sigma_{[p,q]}(A_j) | \, j \neq s\} < \mu_{[p,q]}(A_s) = \mu$, then we have $\mu_{[p,q]}(0) = \mu_{[p,q]}(f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_0(z)f) = \mu_{[p,q]}(A_s) = \mu > 0$, which is a contradiction. Set $\tau_1 = \max\{\tau_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), j \neq s\}$, then we may choose constants $\delta_1, \ \delta_2$ such that $\tau_1 < \delta_1 < \delta_2 < \tau$. If $\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), \ \tau_{[p,q]}(A_j) \leq \tau_1 < \tau$, then for sufficiently large r, we have

$$m(r, A_j) \leq T(r, A_j) \leq \exp_{p-1}\{\delta_1 (\log_{q-1} r)^{\mu}\}. \tag{29}$$

If $\sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_s)$, then for sufficiently large r and any given $\varepsilon (0 < 2\varepsilon < \mu_{[p,q]}(A_s) - \sigma_{[p,q]}(A_j))$, we have

$$m(r, A_j) \leq T(r, A_j) \leq \exp_p\{(\sigma_{[p,q]}(A_j) + \varepsilon) \log_q r\}. \tag{30}$$

Under the assumption that $\lambda_{[p,q]}(\frac{1}{A_s}) < \mu_{[p,q]}(A_s)$, for sufficiently large r, we have

$$N(r, A_s) = o(T(r, A_s)). \tag{31}$$

By the definition of the lower $[p,q]$-type, for sufficiently large r, we have

$$T(r, A_s) \geq \exp_{p-1}\{\delta_2 (\log_{q-1} r)^{\mu}\}. \tag{32}$$

By (1), we have

$$T(r, A_s) \leq N(r, A_s) + \sum_{j \neq s} m(r, A_j) + O(\log r), \tag{33}$$

for sufficiently large r. Hence, by substituting (29) and (30) into (33), we have the contradiction. Therefore, if $f(z)$ is a non-transcendental meromorphic solution, then it must be a polynomial with degree $\deg(f) \leq s-1$.

Now we assume that $f(z)$ is a transcendental meromorphic solution of (1). By (1), we have

$$-A_s(z) = \frac{f}{f^{(s)}} \left[\frac{f^{(n)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f} + A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_0(z)\right]. \tag{34}$$

Noting that

$$m(r, \frac{f}{f^{(s)}}) \le T(r, f) + T(r, \frac{1}{f^{(s)}}) = T(r, f) + T(r, f^{(s)}) + O(1),$$

by the logarithmic derivative lemma and (34), we obtain that

$$T(r, A_s) \le N(r, A_s) + \sum_{j \ne s} m(r, A_j) + (s+3)T(r, f), \tag{35}$$

for sufficiently large $r \notin E$, where E is a set of r of finite linear measure. By (29)-(32),(35) and Lemma 2, we can get $\mu_{[p,q]}(f) \ge \mu_{[p,q]}(A_s)$ and $\sigma_{[p,q]}(f) \ge \sigma_{[p,q]}(A_s)$. By Lemma 1 and (10), we have

$$T(r, f) \le \exp_{p+1}\{(\sigma_{[p,q]}(A_s) + 3\varepsilon) \log_q r\}, \tag{36}$$

for any $\varepsilon > 0$, and $r \notin E_0$, $r \to \infty$, where E_0 is a set of r of finite logarithmic measure. Then by (36) and Lemma 2, we have $\sigma_{[p+1,q]}(f) \le \sigma_{[p,q]}(A_s)$. By Lemma 6, we obtain $\mu_{[p+1,q]}(f) \le \mu_{[p,q]}(A_s)$. Then we get $\sigma_{[p+1,q]}(f) \le \sigma_{[p,q]}(A_s) \le \sigma_{[p,q]}(f)$ and $\mu_{[p+1,q]}(f) \le \mu_{[p,q]}(A_s) \le \mu_{[p,q]}(f)$.

Then the proof is complete. □

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