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Completeness of the system of root functions of q-Sturm-Liouville operators

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Abstract. In this paper, we study q-Sturm-Liouville operators. We construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of q-Sturm-Liouville operators in terms of boundary conditions. Then we prove a theorem on completeness of the system of eigenfunctions and associated functions of dissipative operators by using the Lidskii's theorem.
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1. Introduction

Quantum calculus was initiated at the beginning of the 19th century and recently there has been great interest therein. The subject of q-difference equations has evolved into a multidisciplinary subject [6]. There are several physical models involving q-difference and their related problems (see [2, 5, 8, 9, 12, 23]). Many works have been devoted to some problems of a q-difference equation. In particular, Adıvar and Bohner [1] investigated eigenvalues and spectral singularities of non-self-adjoint q-difference equations of second order with spectral singularities. Huseynov and Bairamov [16] examined the properties of eigenvalues and eigenvectors of a quadratic pencil of q-difference equations. In [2], Annaby and Mansour studied a q-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. They also discussed properties of eigenvalues and eigenfunctions. One can see the reference [17] for some definitions and theorems on q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler-Maclaurin formula, etc.

Many problems in mechanics, engineering, and mathematical physics lead to the study of completeness and basic properties of all or part of eigenvectors and associated vectors corresponding to some operators. For instance, when we apply the method of separation of variables to solve an equation like the reduced wave equation or Helmholtz equation, we assume the solution expanded in a series of

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eigenfunctions of one of the variables. The coefficients depend upon the other variable. We substitute the expansion into the equation, thereby obtaining ordinary differential equations for the coefficients. The method relies upon completeness of eigenfunctions corresponding to one of the variables [19]. Dissipative operators are an important part of non-self-adjoint operators. In the spectral analysis of a dissipative operator, we should answer the question whether all eigenvectors and associated vectors of a dissipative operator span the whole space or not. Many authors investigated the problem of completeness of the system of eigenvectors and associated vectors of boundary value problems for differential and difference operators (see [3, 4, 7, 13, 14, 15, 24]).

The organization of this paper is as follows: In Section 2, some preliminary concepts related to q-difference equation and Lidskii's theorem essentials are presented for the convenience of the readers. In Section 3, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of q-Sturm-Liouville operators in terms of boundary conditions. Finally, in Section 4, we proved a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators under consideration.

2. Preliminaries

In this section, we introduce some of the required q-notations and Lidskii's theorem essentials.

Following the standard notations in [7] and [17], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$ and $a \in \mathbb{C}$. A q-difference equation is an equation that contains q-derivatives of a function defined on A. Let $y(x)$ be a complex-valued function on $x \in A$. The q-difference operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \quad \text{for all } x \in A, \tag{1}$$

where $\mu(x) = (q-1)x$. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x}, \quad x \in A, \tag{2}$$

if the limit exists and does not depend on x. A right inverse to D_q, the Jackson q-integration is given by

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in A, \tag{3}$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad a, b \in A. \tag{4}$$

Let $L_q^2(0,a)$ be the space of all complex-valued functions defined on $[0,a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|\, d_q x\right)^{1/2} < \infty. \tag{5}$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\,\overline{g(x)} d_q x, \quad f,g \in L_q^2(0,a). \tag{6}$$

Let A denote the linear non-self-adjoint operator in the Hilbert space with domain $D(A)$. A complex number λ_0 is called an eigenvalue of the operator A if there exists a non-zero element $y_0 \in D(A)$ such that $Ay_0 = \lambda_0 y_0$; in this case, y_0 is called the eigenvector of A for λ_0. The eigenvectors for λ_0 span a subspace of $D(A)$, called the eigenspace for λ_0.

The element $y \in D(A)$, $y \neq 0$ is called a root vector of A corresponding to the eigenvalue λ_0 if $(T - \lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. The root vectors for λ_0 span a linear subspace of $D(A)$, is called the root lineal for λ_0. The algebraic multiplicity of λ_0 is the dimension of its root lineal. A root vector is called an associated vector if it is not an eigenvector. The completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

An operator A is called dissipative if $Im(Ax,x) \geq 0$, $(\forall x \in D(A))$. A bounded operator is dissipative if and only if

$$ImA = \frac{1}{2i}(A - A^*) \geq 0. \tag{7}$$

Theorem 1 (see [25])**.** *Let A be an invertible operator. Then $-A$ is dissipative if and only if the inverse operator A^{-1} of A is dissipative.*

A linear bounded operator A defined on the separable Hilbert space H is said to be of trace class (nuclear) if the series

$$\sum_j (Ae_j, e_j) \tag{8}$$

converges and has the same value in any orthonormal basis $\{e_j\}$ of H. The sum

$$TrA = \sum_j (Ae_j, e_j) \tag{9}$$

is called the trace of A.

The kernel $G(x,t)$, $x,t \in \mathbb{R}$, of the integral operator K on $L^2(\mathbb{R})$

$$Kf = \int_{\mathbb{R}} G(x,t) f(x)\, dx, \quad f \in L^2(\mathbb{R}), \tag{10}$$

is a Hilbert-Schmidt kernel if $|G(x,t)|^2$ is integrable on $\mathbb{R}^2$, i.e.,

$$\int_{\mathbb{R}^2} |G(x,t)|^2 dxdt < \infty. \tag{11}$$

If $G(x,x)$ is measurable and summable, then it is called a trace-class kernel (see [21],[22]). Recall that the integral operator with a trace class-kernel is nuclear.

Theorem 2 (see [10, 18])**.** *If the dissipative operator A is the nuclear operator, then the system of root functions is complete in H.*

3. Dissipative extensions of q-difference operators

In this section, we describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of q-Sturm-Liouville operators. We will consider a q-Sturm-Liouville operator

$$l\left(y\right) := -\frac{1}{q} D_{q^{-1}} D_q y\left(x\right) + w\left(x\right) y\left(x\right), \quad 0 \leq x \leq a < +\infty, \tag{12}$$

where $w\left(x\right)$ is defined on $[0,a]$ and continuous at zero. The q-Wronskian of $y_1\left(x\right),\ y_2\left(x\right)$ is defined to be

$$W_q\left(y_1, y_2\right)\left(x\right) := y_1\left(x\right) D_q y_2\left(x\right) - y_2\left(x\right) D_q y_1\left(x\right), \quad x \in [0,a].$$

Let L_0 denote the closure of the minimal operator generated by (12) and by D_0 its domain. Besides, by D we denote the set of all functions $y\left(x\right)$ from $L_q^2\left(0,a\right)$ such that $y\left(x\right)$ and $D_q y\left(x\right)$ are continuous in $[0,a)$ and $l\left(y\right) \in L_q^2\left(0,a\right)$; D is the domain of the maximal operator L. Furthermore, $L = L_0^*$ [20]. Suppose that the operator L_0 has a defect index $(2,2)$, so the case of Weyl's limit-circle occurs for l.

For every $y, z \in D$ we have q-Lagrange's identity ([2])

$$(Ly, z) - (y, Lz) = [y,z]_a - [y,z]_0$$

where $[y,z]_x := y\left(x\right) \overline{D_{q^{-1}} z\left(x\right)} - D_{q^{-1}} y\left(x\right) \overline{z\left(x\right)}$.

By $u\left(x,\lambda\right),\ v\left(x,\lambda\right)$ denote the solutions of the equation $l\left(y\right) = \lambda y$ satisfying the initial conditions

$$u\left(0,\lambda\right) = \cos\alpha, \quad D_{q^{-1}} u\left(0,\lambda\right) = \sin\alpha, \tag{13}$$

$$v\left(0,\lambda\right) = -\sin\alpha, \quad D_{q^{-1}} v\left(0,\lambda\right) = \cos\alpha, \tag{14}$$

where $\alpha \in \mathbb{R}$. The solutions $u\left(x,\lambda\right)$ and $v\left(x,\lambda\right)$ form a fundamental system of solutions of $l\left(y\right) = \lambda y$ and they are entire functions of λ (see [2]). Let $u\left(x\right) = u\left(x,0\right)$ and $v\left(x\right) = v\left(x,0\right)$ be the solutions of the equation $l\left(y\right) = 0$ satisfying the initial conditions

$$u\left(0\right) = \cos\alpha, \quad D_{q^{-1}} u\left(0\right) = \sin\alpha, \tag{15}$$

$$v\left(0\right) = -\sin\alpha, \quad D_{q^{-1}} v\left(0\right) = \cos\alpha. \tag{16}$$

Lemma 1. *For arbitrary $y, z \in D$, one has the equality*

$$[y,z]_x[u,v]_x = [y,u]_x[z,v]_x - [y,v]_x[z,u]_x, \quad 0 \le x \le +\infty \tag{17}$$

Proof. Direct calculations verify equality (17). □

Let us consider the functions $y \in D$ satisfying the conditions

$$y(0)\cos\alpha + D_{q^{-1}}y(0)\sin\alpha = 0, \tag{18}$$

$$[y,u]_a - h[y,v]_a = 0, \tag{19}$$

where $Imh > 0, \ \alpha \in \mathbb{R}$.

We recall that a triple $(\mathbb{H}, \Gamma_1, \Gamma_2)$ is called a space of boundary values of a closed symmetric operator A on a Hilbert space H if Γ_1, Γ_2 are linear maps from $D(A^*)$ to H with equal deficiency numbers and

i) for every $f, g \in D(A^*)$

$$(A^*f, g)_H - (f, A^*g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathbb{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathbb{H}},$$

ii) for any $F_1, F_2 \in H$ there is a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1, \ \Gamma_2 f = F_2$ ([11]).

Let us define by Γ_1, Γ_2 linear maps from D to $\mathbb{C}^2$ by the formula

$$\Gamma_1 y = \begin{pmatrix} -y(0) \\ [y,u]_a \end{pmatrix}, \ \Gamma_2 y = \begin{pmatrix} D_{q^{-1}}y(0) \\ [y,v]_a \end{pmatrix}, \quad y \in D. \tag{20}$$

For any $y, z \in D$, using Lemma 1, we have

$$\begin{aligned}(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^2} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^2} &= -y(0)\overline{D_{q^{-1}}z(0)} + z(0)\overline{D_{q^{-1}}y(0)} \\ &\quad +[y,u]_a[z,v]_a - [z,u]_a[y,v]_a \\ &= [y,z](a) - [y,z](0) \\ &= (Ly,z) - (y,Lz).\end{aligned} \tag{21}$$

Theorem 3. *The triple $\left(\mathbb{C}^2, \Gamma_1, \Gamma_2\right)$ defined by (20) is a boundary space of the operator L_0.*

Proof. The proof is obtained from (21) and the definition of the boundary value space. □

The following result is obtained from Theorem 1.6 in Chapter 3 of [11].

Theorem 4. *For any contraction K in $\mathbb{C}^2$ the restriction of the operator L to the set of functions $y \in D$ satisfying either*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0 \tag{22}$$

or

$$(K - I)\Gamma_1 y - i(K + I)\Gamma_2 y = 0 \tag{23}$$

is a maximal dissipative and accretive extension of the operator L_0, respectively. Conversely, every maximally dissipative (accretive) extension of L_0 is the restriction of L to the set of functions $y \in D$ satisfying (22)((23)), and the contraction K is uniquely determined by the extension. Conditions (22)((23))) define self-adjoint extensions if K is unitary.

In particular, boundary conditions

$$\cos \alpha y\left(0\right) + \sin \alpha D_{q^{-1}} y\left(0\right) = 0 \tag{24}$$

$$[y, u]_a - h[y, v]_a = 0 \tag{25}$$

with $Imh > 0$, describe the maximal dissipative (self-adjoint) extensions of L_0 with separated boundary conditions.

We know that all eigenvalues of a dissipative operator lie in the closed upper half-plane. By virtue of Theorem 4, all the eigenvalues of L lie in the closed upper half-plane $Im\lambda \geq 0$.

4. Completeness theorem for the q-difference operators

Theorem 5. *The operator L has no any real eigenvalue.*

Proof. Suppose that the operator L has a real eigenvalue λ_0. Let $\eta_0\left(x\right) = \eta\left(x, \lambda_0\right)$ be the corresponding eigenfunction. Since

$$Im\left(L\eta_0, \eta_0\right) = Im\left(\lambda_0 \left\|\eta_0\right\|^2\right) = Imh\left([\eta_0, v]_a\right)^2,$$

we get $[\eta_0, v]_a = 0$. By boundary condition (25), we have $[\eta_0, u]_a = 0$. Thus

$$[\eta_0\left(t, \lambda_0\right), u]_a = [\eta_0\left(t, \lambda_0\right), v]_a = 0. \tag{26}$$

Let $\xi_0\left(t\right) = v\left(t, \lambda_0\right)$. Then

$$1 = [\eta_0, \xi_0]_a = [\eta_0, u]_a[\xi_0, v]_a - [\eta_0, v]_a[\xi_0, u]_a.$$

By equality (26), the right-hand side is equal to 0. This contradiction proves Theorem 5. □

Definition 1. *Let f be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that*

$$\left|f\left(\lambda\right)\right| \leq C_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}, \tag{27}$$

then f is called an entire function of order ≤ 1 of growth and minimal type [10].

Let $\varphi\left(x, \lambda\right)$ be a single linearly independent solution of the equation $l\left(y\right) = \lambda y$, and

$$\tau_1\left(\lambda\right) := [\varphi\left(x, \lambda\right), u\left(x\right)]_a,$$

$$\tau_2\left(\lambda\right) := [\varphi\left(x, \lambda\right), v\left(x\right)]_a,$$

$$\tau\left(\lambda\right) := \tau_1\left(\lambda\right) - h\tau_2\left(\lambda\right).$$

It is clear that

$$\sigma_p(L) = \{\lambda : \lambda \in \mathbb{C},\ \tau(\lambda) = 0\},$$

where $\sigma_p(L)$ denotes the set of all eigenvalues of L. Since $\varphi(a,\lambda)$ and $D_{q^{-1}}\varphi(a,\lambda)$ are entire functions of λ of order $\leq \frac{1}{2}$ (see [2]), consequently, $\tau(\lambda)$ have the same property. Then $\tau(\lambda)$ is entire functions of order ≤ 1 of growth, and of minimal type. It is clear that all zeros of $\tau(\lambda)$ (all eigenvalues of L) are discrete and possible limit points of these zeros (eigenvalues of L) can only occur at infinity.

For $y \in D(L)$, $Ly(x) = f(x)$ $\left(x \in (0,a),\ f(x) \in L_q^2(0,a)\right)$ is equivalent to the nonhomogeneous differential equation $l(y) = f(x)$ $(x \in (0,a))$, subject to the conditions

$$\begin{aligned} \cos\alpha y(0) + \sin\alpha D_{q^{-1}} y(0) &= 0, \\ [y,u]_a - h[y,v]_a &= 0, \quad Imh > 0. \end{aligned}$$

By Theorem 5, there exists an inverse operator L^{-1}. In order to describe the operator L^{-1}, we use the Green's function method. We consider the functions $v(x)$ and $\theta(x) = u(x) - hv(x)$. These functions belong to the space L_q^2 $(0,a)$. Their Wronskian $W_q(v,\theta) = -1$.

Let

$$G(x,t) = \begin{cases} v(x)\,\theta(t), & 0 \leq x \leq t \leq a \\ v(t)\,\theta(x), & 0 \leq t \leq x \leq a \end{cases}. \tag{28}$$

Let us consider the integral operator K defined by the formula

$$Kf = \int_0^a G(x,t)\,\overline{f(t)}\,d_q t, \quad \left(f \in L_q^2(0,a)\right). \tag{29}$$

It is evident that $K = L^{-1}$. Consequently, the root lineals of the operators L and K coincide and, therefore, the completeness in $L_q^2(0,a)$ of the system of all eigenfunctions and associated functions of L is equivalent to the completeness of those for K.

We obtain that $G(x,t)$ is a Hilbert-Schmidth kernel since $v, \theta \in L_q^2$ $(0,a)$. Furthermore, $G(x,t)$ is measurable and integrable on $(0,a)$. Hence K is of trace class. Since L is a dissipative operator, $-K$ is a dissipative operator by Theorem 1. Thus all conditions are satisfied for the Lidskii's theorem. Hence we have;

Theorem 6. *The system of all root functions of $-K$ (also K) is complete in L_q^2 $(0,a)$.*

From all above conclusions, we have;

Theorem 7. *All eigenvalues of the operator L lie in the open upper half-plane and they are purely discrete. The limit points of these eigenvalues can only occur at infinity. The system of all eigenfunctions and associated functions of the L is complete in the space L_q^2 $(0,a)$.*

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