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A note on Gabor frames

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Abstract. Wilson frames $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ in $L^2(\mathbb{R})$ have been defined and a characterization of Wilson frames in terms of Gabor frames is given when $w_0 = w_{-1}$. Also, under certain conditions a necessary condition for a Wilson system to be a Wilson Bessel sequence is given. We have also obtained sufficient conditions for a Wilson system to be a Wilson frame in terms of Gabor Bessel sequences. For $w_0 = w_{-1}$, stability of Wilson frames is discussed. Also, under the same assumption a necessary and sufficient condition is given for a Wilson system to be a Wilson Bessel sequence in terms of a Wilson frame.
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1. Introduction

Gabor [13] proposed a decomposition of a signal in terms of elementary signals, that displays simultaneously the local time and frequency content of the signal, as opposed to the classical Fourier transform which displays only the global frequency content for the entire signal. On the basis of this development, Duffin and Schaeffer [9] introduced frames for Hilbert spaces to study some deep problems in non-harmonic Fourier series. In fact, they abstracted the fundamental notion of Gabor for studying signal processing. Janssen [17] showed that while being complete in $L^2(\mathbb{R})$ the set suggested by Gabor is not a Riesz basis. This apparent failure of Gabor system was then rectified by resorting to the concept of frames. Since then the theory of Gabor systems has been intimately related to the theory of frames and many problems in frame theory find their origin in Gabor analysis. For example, the localized frames were first considered in the realm of Gabor frames [1, 2, 5, 16]. For more literature on Gabor frames one may refer to [6, 7, 11, 12, 19, 20, 22]. Gabor frames have found wide applications in signal and image processing. Balian-Low Theorem for Gabor frames on locally compact abelian groups is discussed in [14]. In view of Balian-Low Theorem [15], Gabor frames for $L^2(\mathbb{R})$ (which is a Riesz basis) have bad localization properties in either time or frequency. Thus, a system to replace Gabor systems which do not have bad localization properties in time and frequency was required. Wilson [21, 23] suggested a system of functions which are localized

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around the positive and negative frequency of the same order. The idea of Wilson was used by Daubechies, Jaffard and Journe [8] to construct orthonormal "Wilson bases" which consist of functions given by

$$\psi_j^k(x) = \begin{cases} \varepsilon_k \cos(2k\pi x) w(x - \frac{j}{2}), & \text{if } j \text{ is even}, \\ 2\sin(2(k+1)\pi x) w(x - \frac{j+1}{2}), & \text{if } j \text{ is odd}, \end{cases}$$

and

$$\varepsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0, \\ 2, & \text{if } k \in \mathbb{N}, \end{cases}$$

with a smooth well localized window function w. For such bases the disadvantage described in the Balian-Low Theorem is completely removed.

In [10], it has been proved that Wilson bases of exponential decay are not unconditional bases for all modulation spaces on $\mathbb{R}$ including the classical Bessel potential space and the Schwartz spaces. Also, it is shown in [10] that Wilson bases are not unconditional bases for the ordinary L^p spaces for $p \neq 2$. Approximation properties of Wilson bases are studied in [4]. Wilson bases for general time-frequency lattices are studied in [18]. Generalizations of Wilson bases to non-rectangular lattices are discussed in [21] with motivation from wireless communication and cosines modulated filter banks. Wojdyllo studied modified Wilson bases in [25] and discussed Wilson system for triple redundancy in [24]. Motivated by the fact that we have different trigonometric functions for odd and even indices of j, Bittner [3] considered Wilson bases introduced by Daubechies et al. with non symmetrical window functions for odd and even indices of j.

In the present, paper we consider the Wilson system defined by Bittner [3]. In this paper, Wilson frames $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ in $L^2(\mathbb{R})$ have been defined and a characterization of Wilson frames in terms of Gabor frames is given when $w_0 = w_{-1}$. Also, under certain conditions a necessary condition for a Wilson system to be a Wilson Bessel sequence is given. We have also obtained sufficient conditions for a Wilson system to be a Wilson frame in terms of Gabor Bessel sequences. For $w_0 = w_{-1}$, stability of Wilson frames is discussed. Also, under the same assumption a necessary and sufficient condition is given for a Wilson system to be a Wilson Bessel sequence in terms of a Wilson frame.

2. Preliminaries

In this section, we give some standard definitions which will be used throughout the paper.

Definition 1. *Let $\mathbb{H}$ denote a Hilbert space and let I be a countable index set. A family of vectors $\{f_i\}_{i\in I}$ is called a frame for $\mathbb{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \textit{for all } f \in \mathbb{H}. \tag{1}$$

Positive constants A and B are called lower frame bound and upper frame bound for the frame $\{f_i\}_{i\in I}$, respectively. The inequality (1) is called the frame inequality. If $A = B$, then the frame is called a tight frame with frame bound A. If in (1) only the upper inequality holds, then $\{f_i\}_{i\in I}$ is called a Bessel sequence.

If removal of even one f_n leaves the remaining set $\{f_i\}_{i\in I, i\neq n}$ no longer a frame, then the frame $\{f_i\}_{i\in I}$ is called an exact frame or a Riesz basis.

If $\{f_i\}_{i\in I}$ is a frame for H, then the bounded linear operator $T : l^2(\mathbb{N}) \to \mathbb{H}$, given by $T\{\alpha_n\}_{n\in I} = \sum_{n\in I} \alpha_n f_n$ is called the preframe operator.

The adjoint operator of T or the analysis operator $T^* : \mathbb{H} \to l^2(\mathbb{N})$ is given by $T^*\{x\} = \{\langle x, f_n\rangle\}_{n\in I}$. By composing T and T^*, we obtain the frame operator $S : \mathbb{H} \to \mathbb{H}$ defined as $Sx = \sum_{n\in I} \langle x, f_n\rangle f_n, x \in \mathbb{H}$.

For $a \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$, the translation operator T_a on $L^2(\mathbb{R})$ is defined as $T_a g(x) = g(x - a), x \in \mathbb{R}$, and the modulation operator E_a on $L^2(\mathbb{R})$ is defined as $E_a g(x) = e^{2\pi iax} g(x), x \in \mathbb{R}$.

Definition 2 (see [7])**.** *Let $g \in L^2(\mathbb{R})$ and a, b be positive constants. The sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Gabor system for $L^2(\mathbb{R})$. Further,*

- *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, it is called a Gabor frame.*
- *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$, it is called a Gabor Bessel sequence.*

Definition 3 (see [3])**.** *The Wilson system associated with $w_0, w_{-1} \in L^2(\mathbb{R})$ is defined as a sequence of functions $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ in $L^2(\mathbb{R})$ given by*

$$\psi_j^k(x) = \begin{cases} \varepsilon_k \cos(2k\pi x) w_0\left(x - \frac{j}{2}\right), & \text{if } j \text{ is even},\\ 2\sin(2(k+1)\pi x) w_{-1}\left(x - \frac{j+1}{2}\right), & \text{if } j \text{ is odd}, \end{cases}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$\varepsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0,\\ 2, & \text{if } k \in \mathbb{N}. \end{cases}$$

Definition 4 (see [7])**.** *The Zak transform of $f \in L^2(\mathbb{R})$ is defined as a function of two variables given by $(Zf)(t, v) = \sum_{k\in\mathbb{Z}} f(t - k)\exp(2\pi ikv), t, v \in \mathbb{R}$.*

Definition 5 (see [7])**.** *Given a positive number a, the Wiener space is defined by $W = \{g : \mathbb{R} \to \mathbb{C} : g$ is measurable and $\sum_{k\in\mathbb{Z}} \|g\chi_{[ka,(k+1)a)}\| < \infty\}$.*

3. Main results

We begin this section with the definition of a Wilson frame.

Definition 6. *The Wilson system* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *for* $L^2(\mathbb{R})$ *associated with* $w_0, w_{-1} \in L^2(\mathbb{R})$ *is called a Wilson frame if there exist constants* A, B *with* $0 < A \le B < \infty$ *such that*

$$A\|f\|^2 \le \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}). \tag{2}$$

The constants A, B *are called lower frame bound and upper frame bound, respectively, for the Wilson frame* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$.

If $w_0 = w_{-1}$, *then the Wilson system associated with* $w_0 \in L^2(\mathbb{R})$ *is denoted as* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$.

If in (2) only the upper inequality holds for all $f \in L^2(\mathbb{R})$, then the Wilson system $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is called a Wilson Bessel sequence with Bessel bound B.

Example 1. *Let* $w_0, w_{-1} \in L^2(\mathbb{R})$ *be bounded and compactly supported. Then,* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence for* $L^2(\mathbb{R})$.

Example 2. *Let* W *denote the Wiener space. If* $w_0, w_{-1} \in W$, *then* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence for* $L^2(\mathbb{R})$.

Example 3. *Let* $w_0 = \chi_{[0,1)}$. *Then* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a tight Wilson frame for* $L^2(\mathbb{R})$ *with frame bound 2.*

Example 4. *Let* $w_0 \ne w_{-1}$ *be such that* $|w_{-1}(x)| \le C(1+|x|)^{-1-\epsilon}, |w_0(x)| \le C(1+|x|)^{-1-\epsilon}$ *for some constant* C *and* $\epsilon > 0$. *Let* $Q^+ = (0, \frac{1}{2}) \times [\frac{-1}{2}, \frac{1}{2}]$. *Consider the matrix*

$$M(x,\xi) = \begin{pmatrix} \overline{Zw_0(x,\xi)} & \overline{Zw_0(-x,\xi)} \\ \overline{-Zw_{-1}(x,\xi)} & \overline{Zw_{-1}(-x,\xi)} \end{pmatrix}.$$

Let $A_0 = ess \inf_{(x,\xi)\in Q^+} \|M^{-1}(x,\xi)\|_2^{-2}$ *and* $B_0 = ess \sup_{(x,\xi)\in Q^+} \|M(x,\xi)\|_2^2$. *If* $0 < A_0 \le B_0 < \infty$, *then the Wilson system* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$ *with bounds* A_0 *and* B_0.

Example 5. *Let* $w_0 = \chi_{[0,\frac{1}{2})}$. *Then,* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is not a Wilson frame for* $L^2(\mathbb{R})$.

Example 6. *Let*

$$w_0(x) = \begin{cases} \dfrac{\sin \pi x}{\pi x}, & \text{if } x \ne 0\ , \\ 1, & \text{otherwise.} \end{cases}$$

Then $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a tight Wilson frame for* $L^2(\mathbb{R})$ *with frame bound 2.*

Example 7. *Let* $w_0(x) = 2^{\frac{1}{2}} e^{-x}\chi_{[0,\infty)}(x)$. *Then* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$.

Example 8. *Let* $w_0(x) = \dfrac{2^{\frac{1}{2}}}{1+2\pi i x}$. *Then,* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$.

Example 9. *If* $w_0(x) = e^{-\xi(x-\frac{1}{4})^2}$ *and* $w_{-1}(x) = e^{-\xi(x+\frac{1}{4})^2}$, *where* $\xi > 0$ *and* $w_0 \neq w_{-1}$, *then* $\{\psi_j^k : w_0,\ w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$.

Example 10. *Let*

$$w_0(x) = \begin{cases} 1+x, & \text{if } x \in [0,1)\ , \\ \dfrac{x}{2}, & \text{if } x \in [1,2)\ , \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$.

Example 11. *Let*

$$w_0(x) = \begin{cases} 0, & \text{if } x \in (-\infty,-1]\ , \\ \sin(\frac{\pi}{2}(x+1)), & \text{if } x \in (-1,0]\ , \\ (-1)^n \cos^2(\frac{\pi}{2}(x-n))\sin^n(\frac{\pi}{2}(x-n)), & \text{if } x \in (n,n+1], n = 0,1,2,\dots. \end{cases}$$

Then $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a tight Wilson frame for* $L^2(\mathbb{R})$ *with frame bound 2.*

Next, we give Lemmas which will be used in the subsequent results.

Lemma 1. *If* f, g, h *are in* $L^2(\mathbb{R})$, *then*

$$|\langle f,g\rangle|^2 + |\langle f,h\rangle|^2 = |\langle f, g+ih\rangle|^2 + 2\ Im(\langle f,g\rangle\langle h,f\rangle).$$

Proof. Straightforward. □

Lemma 2. *Let* $f, g \in L^2(\mathbb{R})$. *Let* T_j *denote the translation operator on* $L^2(\mathbb{R})$ *defined by* $(T_j f)(x) = f(x-j), f \in L^2(\mathbb{R}),\ x \in \mathbb{R}$. *Then*

$$\sum_{j,k\in\mathbb{Z}} Im\{\langle f, \cos(2k\pi\cdot)T_j g(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_j g(\cdot)}\rangle\} = 0$$

where

$$(\cos(2k\pi\cdot)T_j g(\cdot))(x) = \cos(2k\pi x)g(x-j), \quad x \in \mathbb{R},$$
$$(\sin(2k\pi\cdot)T_j g(\cdot))(x) = \sin(2k\pi x)g(x-j), \quad x \in \mathbb{R}.$$

Proof. Straightforward. $\square$

Lemma 3. *Let* $\{\psi_j^k : w_0,\ w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *be the Wilson system associated with* $w_0, w_{-1} \in L^2(\mathbb{R})$*. Then for* $f \in L^2(\mathbb{R})$,

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2).$$

Proof. Let $f \in L^2(\mathbb{R})$. Then

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = \sum_{\substack{j:even\\ k\in\mathbb{N}_0}} \left| \int f(x)\varepsilon_k \cos(2k\pi x)\overline{w_0\left(x-\frac{j}{2}\right)}dx\right|^2 + \sum_{\substack{j:odd\\ k\in\mathbb{N}_0}} \left| \int 2f(x)\sin(2(k+1)\pi x)\overline{w_{-1}\left(x-\frac{j+1}{2}\right)}dx\right|^2.$$

This gives

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{w_0(x-j)}dx\right|^2 + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \left| \int f(x)\cos(2k\pi x)\overline{w_0(x-j)}dx\right|^2 + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \left| \int f(x)\sin(2k\pi x)\overline{w_{-1}(x-j)}dx\right|^2.$$

Thus

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{T_j w_0(x)}dx\right|^2 + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f, \sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2.$$

Using that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$ and $\langle f, -g\rangle = -\langle f, g\rangle$, $f, g \in L^2(\mathbb{R})$, we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2).$$

$\square$

Lemma 4. *Let* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *be the Wilson system associated with* $w_0, w_{-1} \in L^2(\mathbb{R})$*. Then for* $f \in L^2(\mathbb{R})$*,*

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_{-1}\rangle|^2 \\ - 2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot) T_j w_{-1}(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot) T_j w_0(\cdot)\rangle|^2).$$

Proof. Let $f \in L^2(\mathbb{R})$. Then

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = \sum_{\substack{j:even\\ k\in\mathbb{N}_0}} \left| \int f(x)\varepsilon_k \cos(2k\pi x)\overline{w_0\left(x - \frac{j}{2}\right)}dx\right|^2 \\ + \sum_{\substack{j:odd\\ k\in\mathbb{N}_0}} \left| \int f(x)\sin(2(k+1)\pi x)\overline{w_{-1}\left(x - \frac{j+1}{2}\right)}dx\right|^2.$$

This gives

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{w_0(x-j)}dx\right|^2 \\ + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \left| \int f(x)\cos(2k\pi x)\overline{w_0(x-j)}dx\right|^2 \\ + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \left| \int f(x)\sin(2k\pi x)\overline{w_{-1}(x-j)}dx\right|^2.$$

Thus

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{T_j w_0(x)}dx\right|^2 + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 \\ + 4\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f, \sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2.$$

Using that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$ and $\langle f, -g\rangle = -\langle f, g\rangle$, $f, g \in L^2(\mathbb{R})$, we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{T_j w_0(x)}dx\right|^2 + 2\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f\cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 \\ + 2\sum_{\substack{j\in\mathbb{Z}\\ -k\in\mathbb{N}}} |\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 + 2\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} |\langle f\sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2 \\ + 2\sum_{\substack{j\in\mathbb{Z}\\ -k\in\mathbb{N}}} |\langle f, \sin(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2.$$

Therefore, using Lemma 1, we obtain

$$
\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 &= 2\sum_{j\in\mathbb{Z}} \left| \int f(x)\overline{T_j w_0(x)}dx \right|^2 \\
&+ 2\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \Big\{ |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_0(\cdot)\rangle|^2 - |\langle f, \sin(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 \\
&+ 2\ Im(\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_j w_0(\cdot)}\rangle) \Big\} \\
&+ 2\sum_{\substack{j\in\mathbb{Z}\\ -k\in\mathbb{N}}} \Big\{ |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_0(\cdot)\rangle|^2 - |\langle f, \sin(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2. \\
&+ 2\ Im(\langle f, \cos(2k\pi\cdot)T_j w_0(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_j w_0(\cdot)}\rangle) \Big\} \\
&+ 2\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}}} \Big\{ |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_{-1}(\cdot)\rangle|^2 - |\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2 \\
&+ 2\ Im(\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_j w_{-1}(\cdot)}\rangle) \Big\} \\
&+ 2\sum_{\substack{j\in\mathbb{Z}\\ -k\in\mathbb{N}}} \Big\{ |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_{-1}(\cdot)\rangle|^2 - |\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2 \\
&+ 2\ Im(\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_j w_{-1}(\cdot)}\rangle) \Big\}.
\end{aligned}
$$

Using Lemma 2, we get

$$
\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_0(\cdot)\rangle|^2 \\
&- 2\sum_{j,k\in\mathbb{Z}} |\langle f, \sin(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2 \\
&+ 2\sum_{\substack{j,k\in\mathbb{Z}\\ k\neq 0}} |\langle f, (\cos(2k\pi\cdot) + i\sin(2k\pi\cdot))T_j w_{-1}(\cdot)\rangle|^2 \\
&- 2\sum_{\substack{j,k\in\mathbb{Z}\\ k\neq 0}} |\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2. \qquad (3)
\end{aligned}
$$

Since $E_k = e^{2\pi i k(\cdot)} = \cos(2k\pi\cdot) + i\sin(2k\pi\cdot)$, equation (3) gives

$$
\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_{-1}\rangle|^2 \\
&- 2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_j w_{-1}(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_j w_0(\cdot)\rangle|^2).
\end{aligned}
$$

□

Remark 1. *Combining Lemma 3 and Lemma 4, we obtain*

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 + \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_{-1}\rangle|^2 &= \sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_jw_{-1}(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_jw_0(\cdot)\rangle|^2)\\ &\quad + \sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_jw_0(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_jw_{-1}(\cdot)\rangle|^2)\end{aligned}$$

Remark 2. *If we choose $w_0 = w_{-1}$ in Lemma 4, then*

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2, \quad \textit{for all } f \in L^2(\mathbb{R}).$$

Proof. In view of Lemmas 3 and 4, we have

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 &= 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 - 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2\\ &\quad - 4\sum_{j,k\in\mathbb{Z}} Im\{\langle f, \cos(2k\pi\cdot)T_jw_0(\cdot)\rangle\langle \bar{f}, \sin(2k\pi\cdot)\overline{T_jw_0}(\cdot)\rangle\}.\end{aligned}$$

Using Lemma 2, we obtain

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2, \quad \text{for all } f \in L^2(\mathbb{R}).$$

□

In the following result, we give a characterization of Wilson frames for $w_0 = w_{-1}$ in terms of Gabor frames.

Theorem 1. *Let $w_0, w_{-1} \in L^2(\mathbb{R})$ be such that $w_0 = w_{-1}$. Then $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ with frame bounds A and B if and only if the Gabor system $\{E_kT_jw_0\}_{k,j\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{2}$ and $\frac{B}{2}$. Also, if $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ K\in\mathbb{N}_0}}$ is a Wilson frame with frame bounds A and B, then*

$$\frac{A}{2} \le \sum_{j\in\mathbb{Z}} |w_0(x-j)|^2 \le \frac{B}{2}, \quad \textit{a.e. } x \in \mathbb{R}.$$

Proof. By Remark 2, we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2, \quad \text{for all } f\in L^2(\mathbb{R}).$$

Also, we have that $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ K\in\mathbb{N}_0}}$ is a Wilson frame with frame bounds A and B

$$\Leftrightarrow \quad \frac{A}{2}\|f\|^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \leq \frac{B}{2}\|f\|^2, \quad \text{for all } f\in L^2(\mathbb{R})$$

$\Leftrightarrow$ The Gabor system $\{E_kT_jw_0\}_{j,k\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{2}, \frac{B}{2}$.

Further, if $\{\psi_j^k\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame with frame bounds A and B, then $\{E_kT_j w_0\}_{j,k\in\mathbb{Z}}$ is a Gabor frame with frame bounds $\frac{A}{2}$ and $\frac{B}{2}$. Hence, by Proposition 9.1.2 in [7], we have

$$1\cdot\frac{A}{2} \leq \sum_{j\in\mathbb{Z}} |w_0(x-j)|^2 \leq 1\cdot\frac{B}{2} \text{ a.e } x\in\mathbb{R}.$$

□

Next, we give a necessary condition for a Wilson system to be a Wilson Bessel sequence in terms of a Gabor Bessel sequence under certain conditions.

Theorem 2. *Let $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ be a Wilson Bessel sequence for $L^2(\mathbb{R})$ associated with the functions $w_0, w_{-1} \in L^2(\mathbb{R})$ with Wilson Bessel bound B. Also, let the Wilson system obtained by interchanging w_0 and w_{-1} in the given Wilson system be a Wilson Bessel sequence with Bessel bound B'. Then $\{E_kT_jw_0\}_{j,k\in\mathbb{Z}}$ and $\{E_kT_jw_{-1}\}_{j,k\in\mathbb{Z}}$ are Gabor Bessel sequences with Bessel bound $\frac{B+B'}{2}$.*

Proof. Let $f\in L^2(\mathbb{R})$. Using Lemma 3, we have

$$2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_jw_0(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_jw_{-1}(\cdot)\rangle|^2) \leq B\|f\|^2,$$

$$2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot)T_jw_{-1}(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot)T_jw_0(\cdot)\rangle|^2) \leq B'\|f\|^2.$$

Using Remark 1, we obtain

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \leq (\frac{B+B'}{2})\|f\|^2,$$

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_{-1}\rangle|^2 \leq (\frac{B+B'}{2})\|f\|^2, \quad \text{for all } f\in L^2(\mathbb{R}).$$

□

Finally, we give sufficient conditions for a Wilson system to be a Wilson Bessel sequence in terms of Gabor Bessel sequences.

Theorem 3. *Let* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *be a Wilson system. Let* $\{E_k T_j w_0\}_{j,k\in\mathbb{Z}}$ *and* $\{E_k T_j w_{-1}\}_{j,k\in\mathbb{Z}}$ *be Gabor Besel sequences with Bessel bounds* B_1 *and* B_2*, respectively. Then* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson Besel sequence with Wilson Bessel bound* $2(B_1 + B_2)$*.*

Proof. Let $f \in L^2(\mathbb{R})$. Then, by Lemma 4

$$\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_{-1}\rangle|^2 \\
&\quad - 2\sum_{j,k\in\mathbb{Z}} (|\langle f, \cos(2k\pi\cdot) T_j w_{-1}(\cdot)\rangle|^2 + |\langle f, \sin(2k\pi\cdot) T_j w_0(\cdot)\rangle|^2) \\
&\le 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_{-1}\rangle|^2 \\
&\le 2(B_1 + B_2)\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R})\,.
\end{aligned}$$

Hence $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence for $L^2(\mathbb{R})$ with Wilson Bessel bound $2(B_1 + B_2)$. □

4. Stability of Wilson frames with $w_0 = w_{-1}$

Let $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ be a Wilson frame associated with $w_0 \in L^2(\mathbb{R})$. Let $h \in L^2(\mathbb{R})$ be such that $\{\psi_j^{'k} : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence. Then, the Wilson system associated with $h \in L^2(\mathbb{R})$ may or may not be a Wilson frame for $L^2(\mathbb{R})$.

Example 12. *Let* $w_0 = w_{-1} = h = \chi_{[0,1)}$*. Then* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$ *and* $\{\psi_j^{'k} : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence for* $L^2(\mathbb{R})$*. Also,* $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$*.*

Example 13. *Let* $w_0 = w_{-1} = \chi_{[0,1)}$ *and* $h = \chi_{[0,\frac{1}{2})}$*. Then* $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$ *and* $\{\psi_j^{'k} : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence for* $L^2(\mathbb{R})$*. But* $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is not a Wilson frame for* $L^2(\mathbb{R})$*.*

In the following result, we give sufficient conditions for the stability of Wilson frames with $w_0 = w_{-1}$.

Theorem 4. *Let* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *be a Wilson frame for* $L^2(\mathbb{R})$ *having* A *and* B *as its lower and upper frame bound, respectively. Let* S *be its frame operator and let* $h \in L^2(\mathbb{R})$ *be any function such that the Wilson system* $\{\psi_j^{'k} : w_0+h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence with Wilson Bessel bound* M*. Then* $\{\psi_j^{''k}; h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence for* $L^2(\mathbb{R})$*. Also, if* $(A^2\|S\|^{-1} - 2M) > 0$*, then* $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson frame for* $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$. Consider the Wilson system $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$. Then, by Remark 2, we have

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^{''k}\rangle|^2 &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2\\ &\le 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2 + 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.\end{aligned}$$

Since $\{\psi_j^{'k} : w_0+h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence with Bessel bound M, we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2 \le \left(\frac{M}{2}\right)\|f\|^2.$$

Since $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ with upper frame bound B, we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \le \left(\frac{B}{2}\right)\|f\|^2.$$

Hence, we obtain

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^{''k}\rangle|^2 \le 2\{M+B\}\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R})$$

Also, for $f \in L^2(\mathbb{R})$

$$\begin{aligned}(A^2\|S\|^{-1} - 2M)\|f\|^2 &\le (A^2\frac{1}{A}\|f\|^2 - 2M\|f\|^2)\\ &= A\|f\|^2 - 2M\|f\|^2\\ &\le \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 - 2\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^{'k}\rangle|^2\\ &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 - 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2.\end{aligned}$$

Therefore, we get

$$(A^2\|S\|^{-1} - 2M)\|f\|^2 \leq 2(\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 - 2\sum_{j,w\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2)$$

Thus

$$(A^2\|S\|^{-1} - 2M)\|f\|^2 \leq 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2.$$

Hence $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame with frame bounds $\frac{1}{4}(A^2\|S\|^{-1} - 2M)$ and $2(M+B)$. □

Remark 3. *The condition that* $(A^2\|S\|^{-1} - 2M) > 0$ *is not necessary as seen in Example 12.*

Finally, we give a necessary and sufficient condition for the Wilson system $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ to be a Wilson Bessel sequence in terms of a Wilson frame.

Theorem 5. *Let* $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *be a Wilson frame for* $L^2(\mathbb{R})$ *having* A *and* B *as its lower and upper frame bound respectively. Let* $h \in L^2(\mathbb{R})$ *be any function. Then, the Wilson system* $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ *is a Wilson Bessel sequence in* $L^2(\mathbb{R})$ *if and only if there exist* $C > 0$ *such that*

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0-h)\rangle|^2 \leq C\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_j^k\rangle|^2, \quad \textit{for all } f \in L^2(\mathbb{R}).$$

Proof. Let $f \in L^2(\mathbb{R})$. Suppose $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Bessel sequence with Bessel bound M. Then

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0-h)\rangle|^2 \leq 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2.$$

Using a hypothesis, we have

$$\frac{1}{M}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2 \leq \frac{1}{A}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.$$

Hence

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0-h)\rangle|^2 &\leq 2(\frac{M}{A}+1)\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \\ &\leq (\frac{M}{A}+1)\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_j^k\rangle|^2,\end{aligned}$$

where $C = \frac{M}{A} + 1$.

Conversely, for $f \in L^2(\mathbb{R})$, we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2 \le 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(h-w_0)\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.$$

Hence

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} (|\langle f, E_kT_jh\rangle|^2 &\le 2(C+1)\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \\ &\le (C+1)B\|f\|^2.\end{aligned}$$

□

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References

[1] R. Balan, P. G. Casazza, C. Heil, Z. Landau, *Density, overcompleteness, and localization of frames I*, J. Fourier Anal. Appl. **12**(2006), 105–143.

[2] R. Balan, P. G. Casazza, C. Heil, Z. Landau, *Density, overcompleteness, and localization of frames II*, J. Fourier Anal. Appl. **12**(2006), 307–344.

[3] K. Bittner, *Biorthogonal Wilson Bases*, Proc. SPIE Wavelet Applications in Signal and Image Processing VII **3813**(1999), 410–421.

[4] K. Bittner, *Linear approximation and reproduction of polynomials by Wilson bases*, J. Fourier Anal. Appl. **8**(2002), 85–108.

[5] P. Casazza, G. E. Pfander, *Infinite dimensional restricted invertibility*, J. Funct. Anal. **263**(2012), 3784–3803.

[6] O. Christensen, *Frames, Reisz Bases and Discrete Gabor/Wavelet expansions*, Bull. Amer. Math. Soc. (N.S.) **38**(2001), 273–291.

[7] O. Christensen, *Frames and Bases*, An Introductory Course, Birkhausher, Boston, 2008.

[8] I. Daubechies, S. Jaffard, J. L. Journe, *A simple Wilson orthonormal basis with exponential decay*, SIAM J. Math. Anal. **22**(1991), 554–573.

[9] R. J. Duffin, A. C. Schaeffer, *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc. **72**(1952), 341–366.

[10] H. G. Feichtinger, K. Gröchenig, D. Walnut, *Wilson bases and Modulation spaces*, Math. Nachr. **155**(1992), 7–17.

[11] H. G. Feichtinger, T. Strohmer, *Gabor Analysis and Algorithms*, Theory and Applications, Birkhäuser, Boston, 1998.

[12] H. G. Feichtinger, T. Strohmer, *Advances in Gabor Analysis*, Birkhäuser Boston Inc., Boston, 2003.

[13] D. Gabor, *Theory of communication*, J. Inst. Elec. Eng. (London) **93**(1946), 429–457.

[14] K. Gröchenig, *Aspects of Gabor analysis on locally compact abelian groups*, in: *Gabor Analysis and algorithms*, (H. G. Feichtinger and Th. Strohmer, Eds.), Birkhäuser Boston, 1998, 211–231.

[15] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[16] K. Gröchenig, *Localization of frames, Banach frames, and the invertibility of the frame operator*, J. Fourier Anal. Appl. **10**(2004),105–132.

[17] A. J. E. M. Janssen, *Gabor representation of generalized functions*, J. Math. Anal. Appl. **83**(1981), 377–394.

[18] G. Kutyniok, T. Strohmer, *Wilson bases for general time-frequency lattices*, SIAM J. Math. Anal. **37**(2005), 685–711.

[19] T. Strohmer, *Numerical algorithms for discrete Gabor expansions*, in: *Gabor Analysis and Algorithms, Theory and Applications*, (H.G. Feichtinger and T.Strohmer, Eds.), Birkhauser, Boston, 1998, 267–294.

[20] T. Strohmer, *Approximation of dual Gabor frames, window decay, and wireless communications*, Appl. Comp. Harm. Anal. **11**(2011), 243–262.

[21] D. J. Sullivan, J. J. Rehr, J. W. Wilkins, K. G. Wilson, *Phase space Warnier functions in electronic structure calculations*, Cornell University, 1987, preprint.

[22] D. F. Walnut, *Continuity properties of Gabor frame operator*, J. Math. Anal. Appl. **165**(1992), 479–504.

[23] K. G. Wilson, *Generalised Warnier functions*, Cornell University, 1987, preprint.

[24] P. Wojdyllo, *Wilson system for triple redundancy*, Int. J. Wavelets Multiresolut. Inf. Process. **9**(2011), 151–167.

[25] P. Wojdyllo, *Modified Wilson Orthonormal Bases*, Sampl. Theory Signal Image Process. **6**(2007), 223–235.